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Additive Bases and Extremal Problems in Groups, Graphs and Networks

by

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ABSTRACT

Bases in sets and groups and their extremal problems have been studied in additive number theory such as the postage stamp problem. On the other hand, Cayley graphs based on specific finite groups have been studied in algebraic graph theory and applied to construct efficient communication networks such as circulant networks with minimum diameter (or transmission delay). In this paper we establish a framework which defines and unifies additive bases in groups, graphs and networks and survey results on the bases and their extremal problems. Some well known and well studied problems such as harmonious graphs and perfect addition sets are also shown to be special cases of the framework.

Contents

1 Introduction

In order to improve the performance of an interconnection network, one needs to construct good topological models (graphs) for the underlying communication structure. Among many technical considerations, one would like to

1. maximize the size of network;
2. minimize the total number of links; and
3. minimize the transmission delay within the network.

Many networks' underlining topologies have been studied. Because of their connectivity, symmetry, expandability and many other good properties, n-cubes are often used in the construction of communication networks. According to some recent studies, networks built on extremal Cayley graphs of finite cyclic groups have many advantages over that based on n-cubes. While preserving the connectivity, symmetry and expandability of n-cubes, extremal Cayley graphs have a much smaller diameter, which means a much smaller transmission delay in the network, thus a better performance network. In what follows, we shall give a few examples to compare n-cubes and Cayley graphs.

But before we begin to explain the examples, here is the definition of Cayley graphs. Let Γ be a finite group, and A a subset of Γ. The Cayley graph Cay (Γ, A) of Γ generated by A is the digraph with vertex set Γ and edge set $\{(x, y)| : \ x^{-1}y \in A\}$. The following are two examples of Cayley graphs.

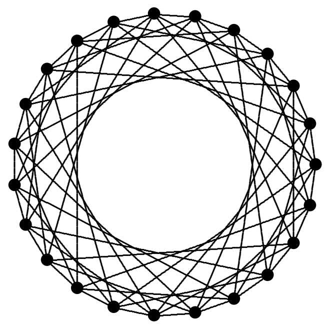

FIG. 1. Cayley graph of $\mathbb{Z}_{23}$ generated by $A = \{1, 4, 7\}$.

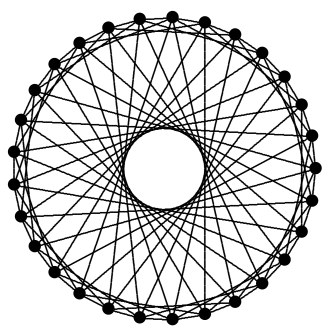

FIG. 2. Cayley graph of $\mathbb{Z}_{29}$ generated by $A = \{1, 4, 17\}$.

As shown in the following figures, a 3-cube has 8 vertices with degree 3 and diameter 3, while the Cayley graph of $\mathbb{Z}_{12}$ generated by $A = \{\pm 1, \pm 6\}$, as an undirected graph, has 12 vertices, degree 3 and diameter 3. A 4-cube has 16 vertices, degree 4 and diameter 4, while the Cayley graph of $\mathbb{Z}_{41}$ generated by $\{\pm 1, \pm 9\}$ has 41 vertices, degree 4 and diameter 4.

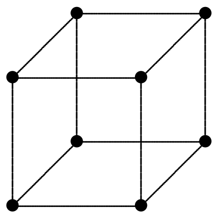

FIG. 3. The 3-cube.

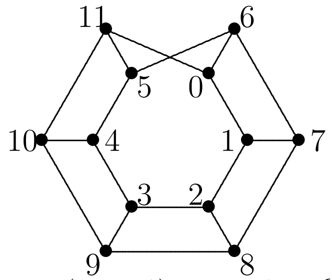

FIG. 4. Cay $(\mathbb{Z}_{12}, A)$ with $A = \{\pm 1, \pm 6\}$.

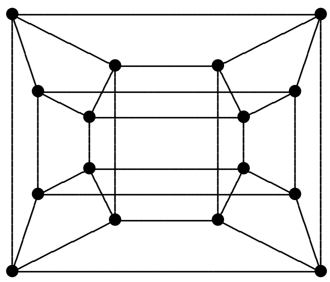

FIG. 5. The 4-cube.

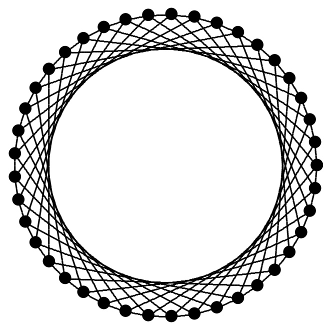

FIG. 6. Cay $(\mathbb{Z}_{41}, A)$ with $A = \{\pm 1, \pm 9\}$.

Cayley graphs are closely related to finite bases in number theory. In this paper, we discuss the relation between several variations of bases (such as bases used in the study of the postage stamp problem) and Cayley graphs. We discuss their applications in the construction of communication networks.

2 Definition and preliminary results

In this section we shall give the definition of various kinds of bases, and the extremal functions that we are going to discuss in this paper. We shall also discuss some of their preliminary properties in this section. Further results will be discussed in the following sections.

Let $|S|$ denote the cardinality of the set S. We use $[a, b]$ to denote the set of all integers n with $a \le n \le b$. Let $h \ge 1$ be any integer, and $A = \{a_1 < a_2 < \cdots < a_k\}$ be a set of k integers. Define

hA – the set of all sums of at most h not necessarily distinct elements of A;

$h * A$ – the set of all sums of at most h distinct elements of A.

One of the fundamental problems in additive number theory is to determine the structure of the sum sets hA and $h * A$. Most famous examples include Goldbach's Conjecture which states that every even integer ≥ 4 is a sum of two primes. In this paper, we only consider finite sum sets. We are mostly interested in whether or not hA and $h * A$ can cover $[0, n]$ or $\mathbb{Z}_n$. First, we have the following definition.

Definition 2.1 *Let A be a finite set of nonnegative integers. Let h be a positive integer.*

- (*a*) *A is a* restricted basis of order h *(or an (α, h)-basis) for n if $[0, n] \subseteq h * A$;*
- (*b*) *A is a* basis of order h *(or a (β, h)-basis) for n if $[0, n] \subseteq hA$;*
- (*c*) *A is a* restricted basis of order h *(or a (γ, h)-basis) for $\mathbb{Z}_n$ if $\mathbb{Z}_n = h * A$;*
- (*d*) *A is a* basis of order h *(or a (δ, h)-basis) for $\mathbb{Z}_n$ if $\mathbb{Z}_n = hA$.*

One question of particular interest is how big the integer n can be if we are given h and A so that A is a $(*, h)$-basis for n or $\mathbb{Z}_n$, where $* = \alpha, \beta, \gamma$, or δ. This leads to the following definitions.

Definition 2.2 *The* $(*, h)$-range of A, *denoted by $n_*(h, A)$, is the largest number n such that A is an $(*, h)$-basis for n or $\mathbb{Z}_n$.*

Definition 2.3 *The* extremal $(*, h)$-range, *denoted by $n_*(h, k)$, is the largest n such that there exists a k-element $(*, h)$-basis A for n (or $\mathbb{Z}_n$). This A is called an* extremal $(*, h)$-basis.

If A is a (β, h)-basis for n, then

$$n \leq |hA| \leq \binom{k+h}{h}.$$

Therefore, for any given $h \geq 2$, we have

$$n_\beta(h, k) \leq \frac{k^h}{h!} + O(k^{h-1}).$$

Similarly,

$$n_*(h, k) \leq \frac{k^h}{h!} + O(k^{h-1}), \qquad \text{where} \quad * = \alpha, \beta, \delta, \gamma. \tag{1}$$

Here are few examples to illustrate these extremal functions.

Example 1. Let $A = \{1, 2, 4, 8\}$. Since

$$\begin{aligned} 1 * A &= \{0, 1, 2, 4, 8\}, \\ 2 * A &= \{0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12\}, \\ 3 * A &= [0, 14]. \end{aligned}$$

we see that $n(3, A) = 14$. In fact, A is an extremal $(\alpha, 3)$-basis, thus $n_\alpha(3, 4) = 14$. Noting that

$$3A = [0, 14] \cup \{16, 17, 18, 20, 24\},$$

we see that A is a $(\beta, 3)$-basis for 14. However, for a given set A of integers and a positive integer n, A may not be a restricted basis of any order for n. Even when its restricted order exists, it is generally greater than its order. For instance,

$$A = \{1, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$$

is a $(\beta, 10)$-basis for 118, but it is not a restricted basis of any order for 118. It is easy to see that

$$B = \{1, 2, 3, 4\}$$

is a $(\beta, 2)$-basis for 8, while its restricted order of B for 8 is three.

Example 2. Let $A = \{1, 4, 5\}$. Then

$$\begin{aligned} 1A &= \{0, 1, 4, 5\}, \\ 2A &= \{0, 1, 2, 4, 5, 6, 8, 9, 10\}, \\ 3A &= [0, 15]. \end{aligned}$$

Hence $n_\beta(3, A) = 15$. In fact, it is easy to show that A is an extremal $(\beta, 3)$-basis for 15, thus $n_\beta(3, 3) = 15$.

Example 3. Consider

$$A = \{1, 4, 5, 6, 7, 9, 11, 16, 17\}.$$

It is easy to verify that $\mathbb{Z}_{19} = 2 * A$ and $n_\gamma(2, A) = 19$. However, A is not an extremal $(\gamma, 2)$-basis. In fact, $n_\gamma(2, 9) = 36$ and

$$B = \{1, 2, 3, 6, 12, 19, 20, 27, 33\}$$

is an extremal $(\gamma, 2)$-basis. However, we note that A has an extra property that every nonzero element k of $\mathbb{Z}_{19}$ has exactly two distinct ways to be written as a sum of two different elements of A.

Example 4. Let $A = \{1, 6, 15\}$. We have

$$\begin{aligned} 1A &= \{0, 1, 6, 15\}, \\ 2A &= \{0, 1, 2, 6, 7, 12, 15, 16, 21, 30\}, \\ 3A &= \{0, 1, 2, 3, 5, 6, 7, 8, 12, 13, 15, 16, 17, 18, \\ &\quad 21, 22, 27, 30, 31, 36\} \pmod{40}, \\ 4A &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, \\ &\quad 20, 21, 22, 23, 24, 27, 28, 30, 31, 32, 33, 36, 37\} \pmod{40} \\ &= \mathbb{Z}_{40} - \{10, 25, 26, 29, 34, 35, 38, 39\}, \\ 5A &= \mathbb{Z}_{40}. \end{aligned}$$

Hence $A = \{1, 6, 15\}$ is a $(\delta, 5)$-basis for 40. It follows from a rather tedious computation that $n_\delta(5, A) = 40$. Furthermore, $n_\delta(5, 3) = 40$ and thus $A = \{1, 6, 15\}$ is an extremal $(\delta, 3)$-basis.

In what follows we shall show some obvious bounds for the extremal functions that we defined above.

Theorem 2.1 *For $h \geq 1$ and $k \geq 1$, we have*

(i) $n_\alpha(h, k) + 1 \leq n_\gamma(h, k)$;

(ii) $n_\beta(h, k) + 1 \leq n_\delta(h, k)$;

(iii) $n_\beta(h, k) \geq n_\alpha(h, k)$;

(iv) $n_\delta(h, k) \geq n_\gamma(h, k)$.

Theorem 2.2 *Let h be a positive integer. Then, as $k \to \infty$,*

$$\frac{1}{h^h}k^h + O(k^{h-1}) \leq n_*(h, k) \leq \frac{1}{h!}k^h + O(k^{h-1}), \tag{2}$$

where $* = \alpha, \beta, \gamma$, or δ.

Proof. The upper bounds follow from (1) and Theorem 2.1. To see the lower bound, we only need to construct an (α, h)-basis for $\frac{1}{h^h}k^h + O(k^{h-1})$.

Assume that $k \geq h$. Let $u = \lfloor k/h \rfloor$. Define

$$A = \bigcup_{i=0}^{h-1} \left\{ju^i \ : \ j = 1, 2, \ldots, u - 1\right\}.$$

Then A is a restricted basis of order h for $u^h - 1$. Furthermore,

$$|A| = hu \leq k.$$

Therefore,

$$n_\alpha(h, k) \geq u^h - 1 = \frac{1}{h^h}k^h + O(k^{h-1}).$$

Other lower bounds follow from this and Theorem 2.1. The proof is complete. ■

Remark. In the non-restricted cases, one could consider the extremal functions in terms of h for any fixed k. We shall discuss this in a later section.

Graham and Sloane [13] initiated the study of these extremal functions in the special case where $h = 2$ and discussed applications in graph theory. In this paper, we are mainly interested in two directions. First, bases which cover each element the same number of times are often referred as (v, k, λ)-addition set or (v, k, λ, μ)-perfect addition sets, which will be studied in a later section. Second, the functions $n_\delta(2, k)$ and $n_\gamma(2, k)$ have been shown to be related to graph labeling problems, which will be briefly discussed later.

3 The postage stamp problem

Suppose that there is a sufficient supply of all stamp types: 1¢, 5¢ and 7¢, but the envelopes do not have enough room for more than 4 stamps. What is the smallest postage that one cannot stamp? It is easy to see that the answer is 23¢. This is an example of the *Postage Stamp Problem*, an old problem in number theory.

In general, the postage stamp problem is to compute the (β, h)-range $(N_\beta(h, A))$ for a given set A of positive integers, as well as the extremal (β, h)-range ($n_\beta(h, k)$where $k = |A|$) and extremal (β, h)-bases (A so that $n_\beta(h, A) = n_\beta(h, k)$) . It has been studied by many people, including Rohrbach [51], Stöhr [53], Selmer [52], and many others.

Let $A = \{a_1 < a_2 < \cdots < a_k\}$ be a (β, h)-basis for n. In other words, A is an h-basis for n, as we usually say in the postage stamp problem. Obviously, a_1 has to be 1.

If $k = 1$, then the only possible (β, h)-basis is $A = \{1\}$. Thus $n_\beta(h, 1) = h$ for every $h \geq 1$. For $k = 2$, Stöhr [53] proved that, for any set $A = \{1, a\}$,

$$n_\beta(h, A) = (h - a + 3)a - 2.$$

Hence, $n_\beta(h, A)$ reaches its maximum if

$$a = \begin{cases} \frac{1}{2}(h+3), & \text{if } h \text{ is odd,} \\ \frac{1}{2}(h+3 \pm 1), & \text{otherwise.} \end{cases}$$

Therefore,

$$n_\beta(h, 2) = n_\beta(h, A) = \left\lfloor \frac{h^2 + 6h + 1}{4} \right\rfloor, \tag{3}$$

and the corresponding extremal (β, h)-basis is $A = \{1, a\}$ with a defined as above.

In 1968, Hofmeister [20] constructed the following extremal (β, h)-basis $A = \{1, a_2, a_3)$ for h sufficiently large

$$\begin{aligned} a_2 &= 2\left\lfloor \frac{4h+4}{9} \right\rfloor - \left\lfloor \frac{2h}{9} \right\rfloor + 3, \\ a_3 &= \left(\left\lfloor \frac{2h}{9} \right\rfloor + 2 \right) a_2 - \left\lfloor \frac{4h+4}{9} \right\rfloor - 2. \end{aligned}$$

This provides the precise value of $n_\beta(h, 3)$:

$$\begin{aligned} n_\beta(h, 3) &= \left(h - \left\lfloor \frac{4h+4}{9} \right\rfloor - \left\lfloor \frac{2h}{9} \right\rfloor \right) a_3 + \left\lfloor \frac{2h}{9} \right\rfloor a_2 + \left\lfloor \frac{4h+4}{9} \right\rfloor \\ &= \frac{4}{81}h^3 + \frac{2}{3}h^2 + Ah + B, \end{aligned} \tag{4}$$

where A and B are constants depending on $h \pmod 9$. Later, Hertsch [18] showed that (4) holds for $h \geq 500$, and Hofmeister [21] showed that for $h \geq 200$. Mossige [43] verified the

Table 1: Extremal (β, h)-bases for $1 \leq h \leq 22$

h	A	$n_\beta(h, A)$	h	A	$n_\beta(h, A)$
1	$\{1, 2, 3\}$	3	12	$\{1, 11, 37\}$	212
2	$\{1, 3, 4\}$	8	13	$\{1, 13, 34\}$	259
3	$\{1, 4, 5\}$	15	14	$\{1, 12, 52\}$	302
4	$\{1, 5, 8\}$	26	15	$\{1, 12, 52\}$	354
5	$\{1, 6, 7\}$	35	16	$\{1, 15, 54\}$	418
6	$\{1, 7, 12\}$	52	17	$\{1, 14, 61\}$	476
7	$\{1, 8, 13\}$	69	18	$\{1, 15, 80\}$	548
8	$\{1, 9, 14\}$	89	19	$\{1, 18, 65\}$	633
9	$\{1, 9, 20\}$	112	20	$\{1, 17, 91\}$	714
10	$\{1, 10, 26\}$	146	21	$\{1, 17, 91\}$	805
11	$\{1, 9, 30\}$, $\{1, 10, 26\}$	172	22	$\{1, 19, 102\}$, $\{1, 29, 92\}$	902

formula for $23 \leq h \leq 200$ on a computer. And he also computed all the exact values of $n_\beta(h, 3)$ for $1 \leq h \leq 22$ and their corresponding extremal (β, h)-basis of order h, which are listed in Table 1.

In spite of the great effort of many researchers, there is still no exact formula available for $n_\beta(h, 4)$. However, some (β, h)-bases have been constructed to provide good lower bound. Hofmeister and Schell [22] showed that

$$n_\beta(h, 4) \geq 2\left(\frac{h}{4}\right)^4 + O(h^3). \tag{5}$$

For a long time the main term here was thought to be the right main term for $n_\beta(h, 4)$, until Mossige [44] proved in 1986 that

$$n_\beta(h, 4) \geq 2.0080397\left(\frac{h}{4}\right)^4 + O(h^3).$$

In the general case where $k \geq 5$, Klotz [35] proved that

$$n_\alpha(h, k) \geq \frac{(k-1)^{k-1}}{k^{k-2}}\left(\frac{h}{k}\right)^k + O\left(h^{k-1}\right). \tag{6}$$

For $k \geq 4$, Mrose [45] proved an *addition theorem* for $n_\alpha(h, k)$ as follows:

$$n_\beta(h_1 + h_2, k_1 + k_2) \geq (n_\beta(h_1, k_1) + 1)(n_\beta(h_2, k_2) + 1).$$

By using this addition theorem, he showed that for $k \geq 4$:

$$n_\beta(h, k) \geq \beta_k \cdot 2^{\lfloor k/4 \rfloor} \cdot \left(\frac{h}{k}\right)^k + O\left(h^{k-1}\right), \tag{7}$$

where $\beta_k = 1,\ 1.024,\ 1.205$ or 1.388 according as $k \equiv 0,\ 1,\ 2$ or $3 \pmod 4$.

The only nontrivial upper bound available in the general case is due to Rödseth [50], who proved that

$$n_\beta(h,k) \leq \frac{(k-1)^{k-2}}{(k-2)!}\left(\frac{h}{k}\right)^k + O\left(h^{k-1}\right), \tag{8}$$

When $k = 4$, Kirfel [33, 34] has a better upper bound:

$$n_\beta(h,4) \leq 2.35\left(\frac{h}{4}\right)^4 + O(h^3).$$

In the same paper, Kirfel also discussed upper bound of $n_\beta(h, A)$ for some special 4-element sets A. Kirfel [34] proved that the following limit

$$\lim_{h\to\infty} \frac{n_\beta(h,k)}{\left(\frac{h}{k}\right)^k}$$

exits for every $k \geq 1$.

On the other hand, one could consider $n_\beta(h,k)$ for any given $h \geq 1$ as k tends to infinity. If $h = 1$, it is easy to see that $A = \{1, 2, \ldots, k\}$ is the extremal $(\beta, 1)$-basis, which gives h-range k. That is,

$$n_\beta(1,k) = k.$$

However, it is a very difficult open problem to obtain an exact formula for $n_\beta(h,k)$ for any given $h > 1$ as k tends to infinity, even in the simplest case where $h = 2$.

It follows from Theorem 2.2 that

$$\frac{1}{h^h}k^h + O(k^{h-1}) \leq n_\beta(h,k) \leq \frac{1}{h!}k^h + O(k^{h-1}) \tag{9}$$

as k tends to infinity. Moser and Ridell [41] proved the following improved upper bound:

$$n_\beta(h,k) \leq \left\{1 - \left(\frac{\cos\frac{\pi}{h}}{2+\cos\frac{\pi}{h}}\right)^h\right\} \cdot \frac{k^h}{h!} + O(k^{h-1}).$$

The case where $h = 2$ has been particularly in focus. (9) implies

$$\frac{k^2}{4} + O(k) \leq n_\beta(2,k) \leq \frac{k^2}{2} + O(k).$$

Rohrbach [51] conjectured in 1937 that

$$n_\beta(2,k) \sim \frac{k^2}{4}, \tag{10}$$

and proved that

$$n_\beta(2,k) < (1-0.0016)\frac{k^2}{2}.$$

This was improved by Moser [42] to

$$n_\beta(2,k) < (1-0.0269)\frac{k^2}{2}.$$

The best known upperbound is due to Riddell [49], who proved that

$$n_\beta(2,k) < (1-0.1329)\frac{k^2}{2}.$$

As for the lower bound, Hämmerer and Hofmeister [17] proved by an explicit construction of a $(\beta,2)$-basis that

$$n_\beta(2,k) \geq \frac{10}{9}\cdot\frac{k^2}{4},$$

which disproves the conjecture of Rohrbach (10). Mrose [45] improved the construction and proved that

$$n_\beta(2,k) \geq \frac{8}{7}\cdot\frac{k^2}{4}.$$

A simpler proof of this lower bound was given by Klöve and Mossige [43]. It is still an open problem to find the exact formula for $n_\beta(2,k)$. In the case $h=3$, Windecker [55] proved that

$$n_\beta(3,k) \geq \frac{4}{81}k^3 + O(k^2). \tag{11}$$

Mrose [45] proved with his addition theorem that, for every $h \geq 1$ as $k \to \infty$,

$$n_\beta(h,k) \geq \gamma_h \left(\frac{4}{3}\right)^{\lfloor h/3 \rfloor}\left(\frac{k}{h}\right)^h + O(k^{h-1}),$$

where $\gamma_h = 1$ if $h \equiv 0,1 \pmod 3$, $8/7$ if $h \equiv 2 \pmod 3$.

Let $A = \{a_1 < a_2 < \cdots < a_k\}$. If $n_\beta(h,A) \geq a_k$, it is easy to see that

$$n_\beta(h+1,\ A) \geq a_k + n_\beta(h,A).$$

Furthermore, the above equality holds if h is sufficiently large. Hofmeister [19] call a representation

$$\sum_{i=1}^k x_i a_i, \quad \text{where} \quad x_i \geq 0, \quad \sum_{i=1}^k x_i \leq h$$

a *regular h-representation* if

$$\sum_{i=1}^j x_i a_i < a_{j+1} \quad \text{for all } j.$$

Table 2: $n_\beta(2,k)$ with corresponding extremal bases.

k	$n_\beta(2,k)$	Extremal (β,h)-bases A
1	2	$\{1\}$
2	4	$\{1,2\}$
3	8	$\{1,3,4\}$
4	12	$\{1,3,5,6\}$
5	16	$\{1,3,5,7,8\}$
6	20	$\{1,2,5,8,9,10\}$
7	26	$\{1,2,5,8,11,12,13\}$
8	32	$\{1,2,5,8,11,14,15,16\}$
9	40	$\{1,3,4,9,11,16,17,19,20\}$
10	46	$\{1,2,3,7,11,15,19,21,22,24\}$
11	54	$\{1,2,5,7,11,15,19,23,25,26,28\}$
12	64	$\{1,3,4,9,11,16,21,23,28,29,31,32\}$
13	72	$\{1,3,4,9,11,16,20,25,27,32,33,35,36\}$
14	80	$\{1,2,5,8,11,14,17,20,23,24,25,51,53,55\}$
15	92	$\{1,3,4,5,8,14,20,26,32,38,41,42,43,45,46\}$

A representation is said to be minimal if it uses the minimal number of elements. A (β,h)-basis A for n is said to be *pleasant* if every regular representation is minimal. Similar extremal functions can be defined for regular and pleasant bases. For further information, see, for instance, Selmer [52].

Let A be a set of k integers. For each integer x, let $r(A,i)$ denote the least number of summands that add up to i. The average order of A is defined as

$$\bar{h}(A,n) = \frac{1}{n}\sum_{i=1}^{n} r(A,i).$$

For every $k \geq 1$, define

$$\bar{h}(k,n) = \min_{\substack{A \\ |A|\leq k}} \bar{h}(A,n).$$

Dix [8] proved that

$$\bar{h}(2,n) = \sqrt{n} - \frac{9}{8} + O(n^{-1/2}).$$

For any given small values h, k, $n_\beta(h,k)$ and the corresponding extremal (β,h)-bases can be found by a simple computer search. Here are some examples. For a more complete list of known values, see Selmer [52] and Alter and Barnett [2].

Table 3: $n_\beta(h,k)$ with corresponding extremal (β,h)-bases A

$n_\beta(h,k)$	Extremal (β,h)-bases A
$n_\beta(3,3)=14$	$\{1,4,6\}$
$n_\beta(3,4)=24$	$\{1,4,7,8\}$
$n_\beta(3,5)=36$	$\{1,4,6,14,15\}$
$n_\beta(3,6)=52$	$\{1,3,7,9,19,24\},\{1,4,6,14,17,29\}$
$n_\beta(3,7)=70$	$\{1,4,5,15,18,27,34\}$
$n_\beta(3,8)=93$	$\{1,3,6,10,24,26,39,41\}$
$n_\beta(3,9)=121$	$\{1,3,8,9,14,32,36,51,53\}$
$n_\beta(3,10)=154$	$\{1,2,6,8,19,28,40,43,91,103\}$
$n_\beta(4,3)=26$	$\{1,5,8\}$
$n_\beta(4,4)=44$	$\{1,3,11,18\}$
$n_\beta(4,5)=70$	$\{1,3,11,15,32\}$
$n_\beta(4,6)=108$	$\{1,4,9,16,38,49\},\{1,5,8,27,29,44\}$
$n_\beta(4,7)=162$	$\{1,4,9,24,35,49,51\}$, $\{1,4,8,25,31,52,71\}$ $\{1,4,10,15,37,50,71\}$
$n_\beta(4,8)=228$	$\{1,3,8,19,33,39,92,102\}$

4 Cayley graphs of finite cyclic groups

We recall that a (δ,h)-basis for n is a basis of order h for the cyclic group $\mathbb{Z}_n$. Because of their applications in the construction of communication networks and their connection to problems in number theory, it has been of great interest in recent years to study extremal (δ,h)-bases (i.e. given h and k, find A such that $n_\delta(h,A)=n_\delta(h,k)$).

It follows from Theorem 2 that

$$\frac{1}{h^h}k^h+O(k^{h-1})\le n_\delta(h,k)\le \frac{1}{h!}k^{h-1}+O(k^{h-1}),$$

Noting that

$$n_\beta(h,k)\le n_\delta(h,k)+1,\quad h\ge 1,k\ge 1, \tag{12}$$

we see that every lower bound for $n_\beta(h,k)-1$ is also a lower bound for $n_\delta(h,k)$.

It is easy to see that

$$n_\delta(h,1)=h+1,\qquad n_\delta(1,k)=k+1.$$

In the case $k=2$, Hsu and Jia [23] proved that

$$n_\delta(h,2)=\left\lfloor\frac{h(h+4)}{3}\right\rfloor+1\ \text{ for all } h\ge 2.$$

The lower bound was proved by constructing the following extremal (δ, h)-basis $A = \{1, a\}$ where

$$a = \begin{cases} h+3, & \text{if } h \equiv 0 \pmod 3, \\ h+1, & \text{if } h \equiv 1 \pmod 3, \\ h+2, & \text{if } h \equiv 2 \pmod 3, \end{cases}$$

The upper bound was proved by utilizing a *lattice point visitation method* developed by Wong and Coppersmith [56] in 1972. This method will be discussed later in this section. Hsu and Jia [23] also proved that

$$\begin{aligned} n_\delta(h,3) &\geq \frac{1}{16}h^3 + O(h^2) \approx 0.0625h^3 + O(h^2), \\ n_\delta(h,3) &\leq \frac{1}{14 - 3\sqrt{3}}(d+3)^3 \approx 0.1136h^3 + O(h^2). \end{aligned}$$

Chen and Gu [7] improved this to

$$n_\delta(h,3) \geq \frac{5}{64}h^3 + O(h^2) = 0.078125h^3 + O(h^2).$$

The best known lower bound for $n_\delta(h,3)$ is due to Jia and Su [32], who proved that

$$n_\delta(h,3) \geq \frac{176}{2197}h^3 + O(h^2) \approx 0.080109h^3 + O(h^2).$$

There is still a big gap between the lower and upper bounds of $n_\delta(h,3)$:

$$0.080109h^3 + O(h^2) \leq n_\delta(h,3) \leq 0.1135867h^3 + O(h^2),$$

and we have no idea what the main term might be. The exact values of $n_\delta(h,3)$ $(2 \leq h \leq 20)$ together with corresponding extremal $(\delta,3)$-bases are listed in Table 4.

Jia [29] proved that

$$n_\delta(h,4) \geq \frac{1}{125}h^4 + O(h^3).$$

by constructing the following (δ, h)-basis $A = \{1, a_2, a_3, a_4\}$:

$$\begin{aligned} t &= \lfloor h/5 \rfloor, \quad a_1 = 1, \\ a_2 &= 4h - 15t + 7, \\ a_3 &= a_2 t + h - 4t + 2, \\ a_4 &= a_3 t + 2h - 4t + 4, \quad \text{and} \\ m &= a_4 t + 3h - 12t + 5 \end{aligned}$$

Jia [30] proved a similar addition theorem for $n_\delta(h,k)$, with which he could prove that, for fixed $k \geq 4$ as $h \to \infty$,

$$n_\delta(h,k) \geq \epsilon_k \left(\frac{256}{125}\right)^{\lfloor k/4 \rfloor} \left(\frac{h}{k}\right)^k + O(h^{k-1}),$$

Table 4: $n_\delta(h,3)$ for $4 \le d \le 20$ with all corresponding extremal $(\delta,3)$-bases

h	$n_\delta(h,3)$	Extremal $(\delta,3)$-bases A
2	9	{1, 3, 4}, {1, 4, 6}
3	16	{1, 4, 5}, {1, 5, 12}
4	27	{1, 4, 17}, {1, 5, 12}, {1, 6, 8}, {1, 16, 23}
5	40	{1, 6, 15}, {1, 6, 25}, {1, 16, 35}, {1, 26, 35}
6	57	{1, 13, 33}, {1, 16, 36}
7	78	{1, 6, 49}, {1, 7, 48}, {1, 12, 61}, {1, 30, 73}
8	111	{1, 31, 69}
9	138	{1, 11, 78}, {1, 17, 96}, {1, 19, 26}, {1, 43, 122}
10	176	{1, 17, 56}, {1, 24, 33}, {1, 32, 153}, {1, 41, 64}, {1, 81, 104}, {1, 121, 160}
11	217	{1, 13, 119}, {1, 18, 46}, {1, 34, 161}, {1, 51, 92}
12	273	{1, 14, 153}, {1, 49, 104}, {1, 53, 186}, {1, 88, 221}
13	340	{1, 90, 191}
14	395	{1, 35, 271}, {1, 125, 361}
15	462	{1, 29, 97}, {1, 33, 254}, {1, 44, 56}, {1, 44, 408}, {1, 55, 419}, {1, 89, 121}, {1, 110, 254}, {1, 122, 165}, {1, 165, 188}, {1, 209, 430}, {1, 224, 380}, {1, 275, 298}, {1, 282, 296}, {1, 298, 341}, {1, 342, 374}, {1, 366, 434}
16	560	{1, 215, 326}, {1, 235, 346}
17	648	{1, 76, 237}, {1, 412, 573}
18	748	{1, 41, 147}, {1, 174, 362}, {1, 490, 676}, {1, 602, 708}
19	861	{1, 27, 463}, {1, 84, 298}, {1, 84, 319}, {1, 543, 778}
20	979	{1, 22, 351}, {1, 138, 787}, {1, 193, 842}, {1, 374, 637}

where

$$\epsilon_k = \begin{cases} 1 & \text{if } k \equiv 0 \text{ or } 1 \pmod 4 \\ \dfrac{4}{3} & \text{if } k \equiv 2 \pmod 4 \\ \dfrac{27}{16} & \text{if } k \equiv 3 \pmod 4 \end{cases}.$$

Chen and Gu [7] improved the lower bound of $n_\delta(h,4)$, thus by using the addition theorem for $n_\delta(h,k)$, they proved

$$n_\delta(h,k) \geq \sigma_k \left(\frac{2048}{625}\right)^{\lfloor k/4 \rfloor} \cdot \left(\frac{h}{k}\right)^k + O(h^{k-1}), \tag{13}$$

where $\sigma_k = 1, 1, 4/3$ or $135/64$ according as $k \equiv 0, 1, 2$ or $3 \pmod 4$. Recently, Su [54] constructed a new 5-element (δ, h)-basis which provides a new lower bound for $n_\delta(h,5)$, and hence a better lower bound in the general case:

$$\begin{aligned} n_\delta(h,k) &\geq \tau_k \left(\frac{5^5 \cdot 7^4}{17^5}\right)^{\lfloor k/5 \rfloor} \left(\frac{h}{k}\right)^k + O(h^{k-1}) \\ &\approx \tau_k (5.2844)^{\lfloor k/5 \rfloor} \left(\frac{h}{k}\right)^k + O(h^{k-1}), \end{aligned} \tag{14}$$

where

$$\tau_k = \begin{cases} 1 & \text{if } k \equiv 0, 1 \pmod 5 \\ 4/3 & \text{if } k \equiv 2 \pmod 5 \\ \dfrac{4752}{2197} \approx 2.163 & \text{if } k \equiv 3 \pmod 5 \\ \dfrac{165888}{50625} = 3.2768 & \text{if } k \equiv 4 \pmod 5 \end{cases}$$

As in the postage stamp problem, we may consider $n_\delta(h,k)$ for any fixed $h \geq 1$ as $k \to \infty$. It follows from (12) and the lower bounds of $n_\beta(h,k)$ that

$$\begin{aligned} n_\delta(2,k) &\geq \frac{2}{7}k^2 + O(k), \quad \text{and} \\ n_\delta(3,k) &\geq \frac{4}{81}k^3 + O(k^2), \end{aligned}$$

and

$$n_\delta(h,k) \geq \gamma_h \left(\frac{8}{7}\right)^{\lfloor h/3 \rfloor} \left(\frac{k}{h}\right)^h + O(k^{h-1}).$$

It would be very interesting to see a $(\delta, 2)$-basis which gives a better lower bound for $n_\delta(2,k)$. We list in Table 5 some of the known values of $n_\delta(2,k)$ together with the corresponding extremal $(\delta, 2)$-bases.

Table 5: $n_\delta(2,k)$ with corresponding extremal $(\delta,2)$-bases

k	$n_\delta(2,k)$	A	k	$n_\delta(2,k)$	A
1	3	$\{1\}$	5	19	$\{1,3,12,14,15\}$
2	5	$\{1,2\}$	6	21	$\{1,2,3,4,10,15\}$
3	9	$\{1,3,4\}$	7	30	$\{1,3,9,11,12,16,26\}$
4	13	$\{1,2,6,9\}$	8	35	$\{1,2,7,8,11,26,29,30\}$

Now we explain the lattice point visitation method of Wong and Coppersmith. Let $A = \{1, a\}$ be a (δ, h)-basis for n. We need to find its order. We proceed to fill the lattice point (x, y) ($x \geq 0, y \geq 0$ are integers) of the Euclidean plane with an integer m $(0 \leq m < n)$ if

$$x \cdot 1 + y \cdot s \equiv m \pmod{n}.$$

We start from the origin $(0,0)$, then the points on the line $(1,0)$, $(0,1)$, and then the points on the line $(2,0)$, $(1,1)$, $(0,2)$, and so on. At each point (x,y) if the value n has not appeared so far, we write it down, otherwise we just leave a blank. The process ends when all values of n in $\mathbb{Z}_n$ have been used. Wong and Coppersmith [56] proved that the filled pattern is always of the form shown below in FIG. 3, where $s \geq 0$, $t \geq 0$, $p > 0$ and $q > 0$. Clearly, the order of $A = \{1, s\}$ as a (δ, h)-basis for n is equal to $s + q + \max\{t, p\} - 2$.

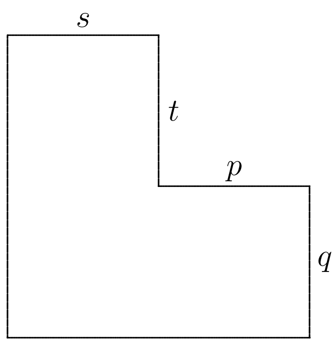

FIG. 3: Visitation method of Wong and Coppersmith

The following picture is the resulting pattern from the extremal basis $A = \{1, 11\}$ for $\mathbb{Z}_{47}$:

7	18	29	40					
6	17	28	39					
5	16	27	38					
4	15	26	37					
3	14	25	36					
2	13	24	35	46	10	21	32	43
1	12	23	34	45	9	20	31	42
0	11	22	33	44	8	19	30	41

FIG. 4: Resulting pattern from $A = \{1, 11\}$ for $\mathbb{Z}_{47}$.

This method can be generalized to handle (δ, h)-bases with k elements. All the upper bounds of $n_\delta(h, k)$ were obtained by using this method.

Let n be a positive integer, and let $A \subseteq \mathbb{Z}_n$. For any $x \in \mathbb{Z}_n$, let $f(A, x)$ denote the least number of elements in A that have sum x. Define

$$r(n, A) = \frac{1}{n} \sum_{x \in \mathbb{Z}_n} f(x, A),$$

then $r(n, A)$ is called the *average order of* A as a (δ, h)-basis for n. For any given real number r, let $\overline{n}_\delta(r, A)$ denote the greatest positive integer n such that $r(n, A) \leq r$. Define

$$\overline{n}_\delta(r, k) = \max_{\substack{A \\ |A|=k}} \overline{n}_\delta(r, A).$$

Wong and Coppersmith [56] proved that

$$\left(\frac{2r}{k} + 1\right)^k \leq \overline{n}_\delta(r, k) \leq \left(1 + \frac{1}{k}\right)^k \frac{r^k}{k!}. \tag{15}$$

In particular, they proved that

$$\begin{aligned} (r+1)^2 \leq \quad & \overline{n}_\delta(r, 2) \quad \leq \frac{9}{8} r^2, \\ 8\left(\frac{r}{3}\right)^3 + O(r^2) \leq \quad & \overline{n}_\delta(r, 3) \quad \leq 10.6667 \left(\frac{r}{3}\right)^3 + O(r^2). \end{aligned}$$

Jia [31] proved that

$$\overline{n}_\delta(r, 2) = \frac{27}{25} r^2 + O(r) \quad \text{as} \quad r \to \infty,$$

and that, for any fixed $k \geq 3$ as r tends to infinity,

$$\overline{n}_\delta(r, k) \geq \beta_k \left(\frac{4\sqrt[3]{4}}{3}\right)^k \left(\frac{r}{k}\right)^k + O(r^{k-1}),$$

where β_k is a constant close to 1. He improved his lower bound again to

$$\overline{n}_\delta(r,k) \geq \alpha_k \left(\frac{10\sqrt[4]{5}}{7}\right)^k \left(\frac{r}{k}\right)^k + O(r^{k-1}).$$

In particular, it was proved that

$$\overline{n}_\delta(r,3) \geq 9.4824 \left(\frac{r}{3}\right)^3 + O(r^2).$$

Recently, Jia and Su [32] proved that

$$\overline{n}_\delta(r,3) \geq \frac{176^4}{1391^3} r^3 + O(r^2) \approx 9.6257 \left(\frac{r}{3}\right)^3 + O(r^2).$$

For any given positive integers n and k, define

$$r(n,k) = \min_{\substack{A \\ |A|=k}} r(n,A).$$

In other words, $r(n,k)$ is the minimal average order of a k-element (δ,h)-basis for n. Table 6 lists the values (up to 4 decimal places) of $r(n,3)$ with a corresponding (δ,h)-basis for $4 \leq n \leq 195$.

As we can see from the table, $r(n,3)$ is not a monotonic increasing function of n, for instance,

$$\begin{aligned} r(33,3) = 3.0909 &> r(34,3) = 3.0882, \\ r(72,3) = 4.3889 &> r(73,3) = 4.3836. \end{aligned}$$

Is it true that this "zig-zag" behavior occurs infinitely many times? An extremal (δ,h)-basis A may not be extremal with respect to its average order. For example, $A = \{1,13,33\}$ is an extremal $(\delta,1)$-basis, and $n_\delta(6,A) = n_\delta(6,3) = 57$. However,

$$r(57,\{1,13,33\}) = 4.0526 > r(57,3) = r(57,\{1,5,22\}) = 3.9649.$$

Extremal bases for general finite groups (not necessarily abelian) have been recently studied by many people. Let δ be a finite group with $|\delta| = n$. Babai, Kantor and Lubotzky [4] proved that every non-simple group of order n contains a basis A with at most seven elements whose order is $O(\log n)$. It is probably true the number seven can be reduced to two. Note that if A is a basis for n with seven elements then the order of A is at least $O(\sqrt[7]{n})$ (in the cyclic group case). For more information on bases for finite groups, see a survey paper by Babai and others [3].

Table 6: $r(n, 3)$ for $4 \leq n \leq 195$ with a corresponding $A = \{1, a, b, \}$.

n	$r(n,3)$	a, b	n	$r(n,3)$	a, b	n	$r(n,3)$	a, b	n	$r(n,3)$	a, b
4	0.7500	2, 3	52	3.7500	6, 31	100	5.0400	7, 59	148	5.9595	7, 107
5	1.0000	2, 3	53	3.8113	5, 22	101	5.0693	6, 39	149	5.9732	7, 58
6	1.1667	2, 3	54	3.8333	6, 32	102	5.0882	28, 63	150	5.9800	8, 57
7	1.2857	2, 3	55	3.8727	5, 34	103	5.1068	8, 61	151	6.0199	7, 44
8	1.3750	2, 5	56	3.8929	6, 33	104	5.1346	11, 28	*152*	6.0132	7, 59
9	1.4444	3, 4	57	3.9649	5, 22	105	5.1714	6, 77	153	6.0261	8, 109
10	1.6000	3, 4	58	3.9828	6, 34	106	5.1792	7, 62	154	6.0325	10, 90
11	1.7273	2, 5	59	4.0169	5, 24	107	5.2056	6, 41	155	6.0710	7, 60
12	1.8333	2, 5	60	4.0333	6, 35	108	5.2222	6, 79	156	6.0769	8, 59
13	1.8462	3, 9	61	4.0656	5, 45	109	5.2477	14, 60	157	6.1147	7, 113
14	1.9286	4, 6	62	4.0968	7, 36	110	5.2636	8, 41	*158*	6.1076	7, 61
15	2.0000	3, 10	63	4.1429	5, 40	111	5.2703	31, 69	159	6.1195	13, 90
16	2.1250	3, 7	64	4.1563	5, 47	112	5.2857	16, 22	160	6.1500	9, 114
17	2.1765	3, 11	*65*	4.1538	6, 25	113	5.3186	8, 42	161	6.1615	7, 62
18	2.2778	3, 7	66	4.2273	7, 39	114	5.3246	34, 62	162	6.1667	11, 94
19	2.3158	3, 12	67	4.2537	5, 28	115	5.3652	7, 44	163	6.2025	7, 69
20	2.3500	4, 9	68	4.2647	5, 27	116	5.3793	11, 67	*164*	6.1951	10, 95
21	2.4762	3, 8	69	4.2899	6, 20	117	5.3846	16, 22	165	6.2182	17, 27
22	2.5000	3, 10	70	4.3285	5, 29	118	5.3983	16, 22	166	6.2349	13, 135
23	2.5652	3, 15	71	4.3521	6, 27	119	5.4286	9, 71	167	6.2695	13, 99
24	2.5833	4, 15	72	4.3889	6, 31	120	5.4500	8, 87	*168*	6.2440	16, 129
25	2.6400	4, 16	*73*	4.3836	6, 52	121	5.4711	8, 74	169	6.2899	10, 62
26	2.6923	5, 8	74	4.4189	12, 28	122	5.4836	16, 22	170	6.3000	22, 92
27	2.7407	4, 17	75	4.4667	7, 29	123	5.5122	15, 24	171	6.3216	13, 96
28	2.8214	4, 17	76	4.5000	5, 55	124	5.5323	7, 77	172	6.3488	7, 125
29	2.8966	4, 9	77	4.5065	11, 15	125	5.5520	8, 73	173	6.3699	7, 66
30	2.9000	5, 19	*78*	4.5000	6, 49	126	5.5635	8, 35	174	6.3736	8, 65
31	3.0000	3, 13	79	4.5443	7, 30	127	5.5827	9, 75	*175*	6.3600	14, 99
32	3.0000	4, 13	80	4.6000	11, 15	128	5.6250	7, 49	176	6.3920	41, 64
33	3.0909	4, 20	81	4.6049	13, 18	129	5.6279	8, 75	177	6.4237	8, 68
34	3.0882	4, 22	82	4.6220	23, 51	*130*	5.6231	11, 74	*178*	6.4213	48, 124
35	3.1143	7, 11	83	4.6627	6, 52	131	5.6718	7, 50	179	6.4413	13, 148
36	3.1667	4, 23	84	4.6905	6, 25	132	5.6515	11, 75	180	6.4556	56, 94
37	3.2162	4, 27	85	4.6941	8, 49	133	5.6992	7, 97	181	6.4641	22, 28
38	3.2368	7, 11	86	4.7326	9, 31	134	5.7090	10, 80	182	6.4780	14, 148
39	3.3077	5, 12	87	4.7586	6, 34	135	5.7259	9, 79	183	6.4973	21, 29
40	3.3250	6, 15	88	4.7955	6, 54	136	5.7500	7, 99	184	6.5217	8, 71
41	3.3902	4, 17	89	4.7978	6, 35	137	5.7737	13, 79	185	6.5243	11, 107
42	3.4524	4, 26	90	4.8111	12, 28	*138*	5.7609	11, 78	*186*	6.5215	15, 83
43	3.4884	4, 18	91	4.8352	6, 66	139	5.7986	16, 22	187	6.5722	20, 45
44	3.5000	5, 28	92	4.8696	6, 36	140	5.8143	9, 39	188	6.5745	11, 69
45	3.5333	5, 29	93	4.8925	12, 55	141	5.8298	8, 54	*189*	6.5608	16, 86
46	3.5652	8, 11	94	4.9043	6, 68	142	5.8662	8, 88	190	6.5842	14, 46
47	3.6170	5, 30	95	4.9263	6, 37	143	5.8811	8, 85	191	6.6230	11, 70
48	3.6250	5, 30	96	4.9375	7, 60	*144*	5.8750	8, 55	192	6.6250	9, 136
49	3.6939	4, 21	97	4.9794	6, 41	145	5.8966	8, 90	*193*	6.6218	15, 157
50	3.7000	5, 21	98	4.9898	7, 58	146	5.9247	9, 56	194	6.6546	20, 148
51	3.7451	5, 32	99	5.0202	9, 61	*147*	5.9184	28, 78	*195*	6.6513	12, 113

Table 7: Some perfect addition sets for $3 \leq k \leq 20$

(n,k,λ,μ)	(n,k,λ,μ)-perfect addition sets A
(7,4,2,0)	$\{0,1,2,4\}$
(6,4,2,2)	$\{0,1,3,5\}$
(11,5,2,0)	$1,3,4,5,9\}$
(9,5,2,4)	$\{0,1,3,6,8\}$
(8,6,4,2)	$\{0,1,2,3,4,6\}$
(13,6,2,6)	$\{1,3,4,9,10,12\}$
(10,7,4,6)	$\{1,2,3,5,7,8,9\}$
(13,8,4,8)	$\{1,2,3,5,8,10,11,12\}$
(19,9,4,0)	$\{1,4,5,6,7,9,11,16,27\}$
(17,9,4,8)	$\{0,1,2,4,8,9,13,15,16\}$
(14,11,8,6)	$\{0,1,2,3,4,5,7,8,9,11,13\}$
(23,12,60)	$\{0,1,2,3,4,6,8,9,12,13,16,18\}$
(21,12,6,12)	$\{1,3,4,5,6,8,9,12,15,16,17,18,20\}$
(16,12,8,12)	$\{1,3,4,5,6,7,9,10,11,12,13,15\}$
(37,13,4,12)	$\{0,1,6,8,10,11,14,23,26,27,29,31,36\}$
(19,13,8,12)	$\{0,1,2,3,5,6,7,9,11,12,13,15,16,17\}$
(18,14,10,12)	$\{0,1,2,3,5,6,7,9,11,12,13,15,16,17\}$
(29,14,6,14)	$\{1,4,5,6,7,9,13,16,20,22,23,24,25,28\}$
(31,16,8,0)	$\{0,1,2,4,5,7,8,9,10,14,16,18,19,20,25,28\}$
(37,18,8,18)	$\{1,3,4,7,9,10,11,12,16,21,25,26,27,28,30,33,34,36\}$

5 Some special bases

Because of their special properties, bases of order two are of great interest to many researchers. For instance, Ablow and Brenner [1] studied the perfect addition sets, which are defined as follows.

A k-element subset $A \subseteq \mathbb{Z}_n$ is called an (n,k,λ,μ)-*perfect addition set* if the equation

$$a_i + a_j \equiv m, \qquad (i \leq j)$$

has exactly λ solutions for every nonzero $m \in \mathbb{Z}_n$, and has exactly μ solutions when $m = 0$.

Table 7 lists some examples of perfect addition sets. For more results and problems on perfect addition sets, please see also Baumer [6], Lam [36, 37, 38, 39, 40], Hsu and Jia [24] and Isbell [26].

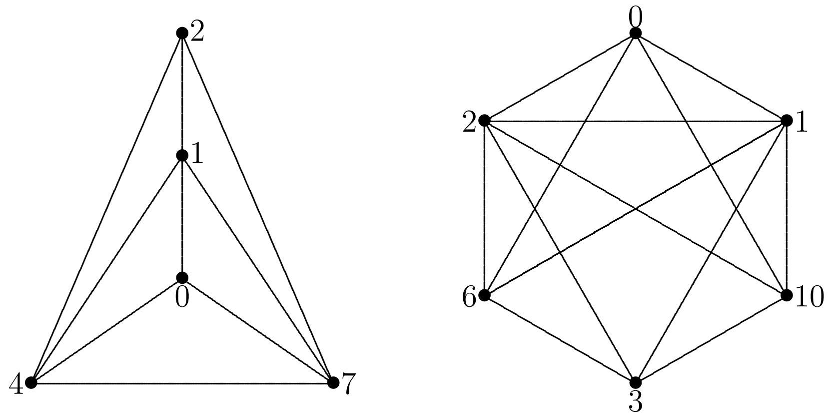

FIG. 5: Maximal harmonious graphs.

Graham and Sloane call a connected graph with v vertices and e ($e \geq v$) edges a *harmonious graph* if there is a labeling of the vertices i with distinct labels a_i so that the set of vertex labels $A = \{a_i \ : \ i = 1, \ldots, v\}$ form a (δ, v)-basis for $\mathbb{Z}_e$. A tree is also called harmonious if just one vertex label is repeated. An open conjecture of Graham and Sloane is that all trees are harmonious. The following are two examples of harmonious graphs.

A set A of nonnegative integers is called an *exact basis of order* h (or exact (β, h)-basis) for n if there exists an integer N such that every integer $m : N \leq m \leq N + n$ is a sum of exactly h not necessarily distinct elements of A. If A is a (β, h)-basis for n then $A \cup \{0\}$ is always an exact (β, h)-basis for n, However, A may not be an exact (β, h)-basis. The following theorem provides a sufficient condition.

Theorem 5.1 *Let $A = \{a_1 < a_2 < \cdots < a_k\}$ be any set of positive integers. If $n \geq 2$, then A is an exact (β, h)-basis for n if and only if*

$$\text{g.c.d.}(a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}) = 1.$$

Proof. ($\Rightarrow$) If the g.c.d. equals $d > 1$, then for every h, the difference between any two elements of $h * A$ is a multiple of d. Therefore, $h * A$ does not contain any two consecutive integers. Thus A is a not an exact basis of any order for n.

($\Leftarrow$) Suppose that the g.c.d.= 1. Then there exist integers u_i such that

$$\sum_{i=1}^{k-1} u_i(a_{i+1} - a_i) = 1.$$

Let $t = |u_1| + \cdots + |u_{k-1}|$. Then the sum on the left hand side can be written as the difference of two elements of $t * A$. In other words, there exists a positive integer λ such that

$$\sum_{i=1}^{k} x_i a_i \quad = \quad \lambda, \tag{16}$$

$$\sum_{i=1}^{k} y_i a_i \quad = \quad \lambda + 1 \tag{17}$$

for some nonnegative integers x_i's and y_i's with

$$\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i = t. \tag{18}$$

It follows from (16) and (17) that

$$\begin{aligned} \sum_{k=1}^{k}((n-j)x_i + jy_i)a_i &= (n-j)\sum_{k=1}^{k} x_i a_i + \sum_{k=1}^{k} jy_i \\ &= (n-j)\lambda + j(\lambda+1) \\ &= n\lambda + j \end{aligned}$$

for $j = 0, 1, 2, \ldots, n-1$. Noting (18), we see that

$$[n\lambda, n\lambda + n - 1] \subseteq (nt)_* A.$$

Therefore, A is an exact basis of order nt for n. The proof is complete. ■

In the case when a basis is also an exact basis, it is an interesting question to study how large its exact order can be in terms of the original order. For any given $h \geq 1$, define

$$G(h, n) = \max_{\substack{A \\ g(A,n)=h}} g_*(A, n),$$

where $g(A, n)$ denotes the order of A as an basis for n, and $g_*(A, n)$ denotes the exact order of A as an exact basis for n. Define

$$\hat{G}(h) = \sup_n G(h, n).$$

Clearly, $\hat{G}(1) = 1$. It is proved in the above theorem that $G(h, n) \leq nh$. Probably this is still very far away from the truth. We even do not know if $\hat{G}(h)$ exists for $h \geq 2$. The infinite version of this problem has been studied by quite few people.

A set A of positive integers is called an asymptotic basis of order h if hA contains all large positive integers. Let $G(h)$ denote the largest exact order of an asymptotic basis of order h. In 1979, Erdős and Graham [10] showed that

$$\frac{1}{4}h^2 + o(h^2) \leq G(h) \leq \frac{5}{4}h^2 + o(h^2).$$

The lower bound of Grekos [15] and the upper bound of Nash [46] are the best known estimate for $G(h)$:

$$\frac{1}{3}h^2 + O(h) \leq G(h) \leq \frac{1}{2}h^2 + O(h).$$

Nathanson [48], Jia [27, 28] and others studied the generalized version of the problem. Jia [29] discovered a connection between this problem and the extremal function $n_\delta(h, k)$. Similar functions can be defined for restricted bases.

6 Circulant networks

Let Cay (m, A) denote the Cayley digraph of $\mathbb{Z}_n$ generated by A.These diagraphs are also called Circulant networks. For more information, the readers are referred to the book by Gammakikakir, Hsu and Kraetzl [14]. Let $d(n, A)$ denote the diameter of the Cayley digraph Cay (n, A). Define

$$d(n, k) = \min_{\substack{A \\ |A|=k}} d(n, A).$$

It is easy to see that A is a (δ, h)-basis for n if and only if the diameter of the Cayley digraph Cay (n, A) is $\leq h$. It is obvious that $d(n, 1) = n - 1$. Wong and Coppersmith [56] proved that, for $k \geq 2$,

$$\sqrt[k]{k!n} - \frac{1}{2}(k+1) \leq d(n, k) \leq k\sqrt[k]{n} - k,$$

and an improved lower bound for $k = 2$: $d(n, 2) \geq \lfloor \sqrt{3n} \rfloor - 2$. The case where $k = 2$ has received extensive study in recent years mainly because Cayley digraphs Cay (n, A) with $|A| = 2$ are natural generalization of the popular ring network. Large infinite families of Cayley digraphs with two generators have been constructed that have diameter equal to the above lower bound: $\lfloor \sqrt{3n} \rfloor - 2$. For interested readers, please see, for example, Erdős and Hsu [9], Fiol et al [12], Hwang and Xu [25], and Bernard, Camellas and Hsu [5].

In the case where $k = 3$, the best known estimates for $d(n, 3)$ are as follows:

$$\begin{aligned} d(n, 3) &\leq 13\sqrt[3]{\frac{n}{13}} + o(\sqrt[3]{n}) \approx 2.31974\sqrt[3]{n} + o(\sqrt[3]{n}). \\ d(n, 3) &\geq \sqrt[3]{14 - 3\sqrt{3}}\sqrt[3]{n} - 3 \approx 2.06486\sqrt[3]{n} - 3. \end{aligned}$$

The upper bound is due to Hsu and Jia [23], and the lower bound Jia and Su [32]. There is still a big gap in our knowledge about $d(n, 3)$.

In the general case where $k \geq 4$, the best know estimate for $d(n, k)$ follows immediately from (14):

$$d(n, k) \leq \eta_k \frac{17\sqrt[5]{7}}{35} n^{1/k} + o(n^{1/k}),$$

where

$$\eta_k = \left\{ \frac{1}{\tau_k} \left(\frac{1419857}{7503125} \right)^{k-5\lfloor k/5 \rfloor} \right\}^{1/k}.$$

In what follows, we list few pictures of optimal Cayley digraphs.

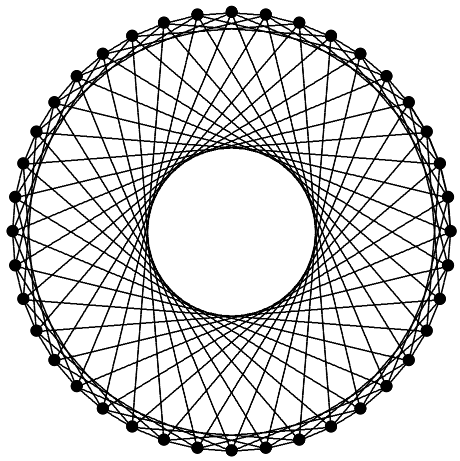

FIG. 7: Cayley graph of $\mathbb{Z}_{40}$ generated by $\{1, 6, 15\}$.

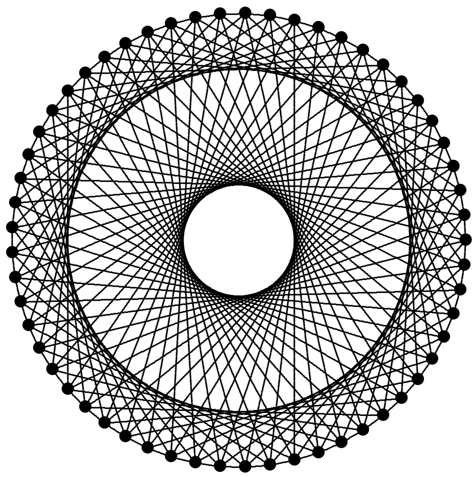

FIG. 8: Cayley graph of $\mathbb{Z}_{57}$ generated by $\{1, 13, 33\}$.

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