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Weighted stability in hypergraphs and weighted domination in split graphs

by

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ABSTRACT

We define a family of hereditary subclasses of split graphs where the weighted domination problem is polynomially solvable. Our approach is similar to that of Balas and Yu [2] for maximal stable sets in graphs.
We apply the result to the subproblem of finding the weighted stability number of a hypergraph.

1 Introduction

Let $H = (V, E)$ be a hypergraph. A set $S \subseteq V$ is a *stable set* in H if S does not contain any hyperedge of H. The complement $V \backslash S$ of a stable set is called a *transversal*. Suppose that H is a vertex-weighted hypergraph, i.e., there is a non-negative weight function $w : V \to [0, +\infty)$. As usual, the *weight* of a subset $X \subseteq V$ is $w(X) = \sum\limits_{x \in X} w(x)$. The *weighted stability number* of (H, w), denoted by $\alpha_w(H)$, is the maximum weight of a stable set in H.

We shall refer to the following problem as WSPH.

Problem 1 (Weighted Stability Problem For Hypergraphs). *Given a weighted hypergraph (H, w), find $\alpha_w(H)$.*

There are only few cases where WSPH can be solved in polynomial time. The most known among them is the class of matroids.

A *split graph* is a graph $G = (V, E)$ such that there exists a partition $A \cup B = V$, where A induces a complete subgraph and B is a stable set. We shall refer to A and B as the *complete part* and the *stable part* of a split graph G with a given *split partition* $A \cup B = V(G)$.

A set $D \subseteq V$ is a *dominating set* in a graph $G = (V, E)$ if every vertex of $V \backslash D$ is adjacent to some vertex of G. If we have a weight function $w : V \to [0, +\infty)$ then the *weighted domination number* $\gamma_w(G)$ is the minimum weight of a dominating set in G.

The following problem will be abbreviated as WDPSG.

Problem 2 (Weighted Domination Problem For Split Graphs). *Given a weighted split graph (G, w), find $\gamma_w(G)$.*

Although this problem seems not very general, it contains WSPH as a particular case. For a proof, we modify a reduction due to Bertossi [3].

Proposition 1. *WSPH is polynomially reducible to WDPSG.*

Proof. Let (H, w) be a vertex-weighted hypergraph. We construct a weighted split graph (G, w') with a split partition $A \cup B$ as follows:

- $A = V(H)$ and $B = E(H)$,
- A is the complete part and B is the stable part,
- a vertex $a \in A$ is adjacent to a vertex $b \in B$ if and only if the vertex a of H is contained in the hyperedge b of H,
- $w'(a) = w(a)$ for each vertex $a \in A$, and $w'(b) = 1 + \max\{w(a) : a \in A\}$ for each vertex $b \in B$.

Without loss of generality we may assume that there are no isolated vertices in G. Let D be a minimum weight dominating set of G. Suppose that D contains some vertex b of B. Since b is not an isolated vertex, b is adjacent to some vertex $a \in A$. It follows from the definition of w' that $w'(b) > w'(a)$ (recall that the weight function is non-negative). Also, each neighbor of b is either a or a neighbor of a. Therefore $D' = (D \backslash \{b\}) \cup \{a\}$ is a dominating set in G and $w'(D') < w'(D)$, a contradiction. Thus, every minimum weight dominating set of G is contained in A.

Now, a set $D \subseteq A$ is a dominating set of G if and only if D is a transversal of H. Since $w'(a) = w(a)$ for all $a \in A = V(H)$, $\gamma_{w'}(G)$ is equal to the minimum weight $\tau_w(H)$ of a transversal in H. Clearly, $\tau_w(H) = w(A) - \alpha_w(H)$. Thus, to find $\alpha_w(H)$ it is sufficient to solve WDPSP, i.e., to calculate $\gamma_{w'}(G)$. □

We shall refer to the graph G constructed in the proof as the *split graph of* H.

Open Problem 1. *Construct a polynomial-time reduction of WDPSP to WSPH.*

2 Polynomially bounded number of minimal dominating sets

A dominating set in a graph G is called *minimal* if it does not contain a smaller dominating set. We denote by $md(G)$ the number of minimal dominating sets of a graph G.

Definition 1. *A class of graphs $\mathcal{P}$ is called* dom-polynomially bounded *if there exists a polynomial $p(n)$ such that $md(G) \leq p(|V(G)|)$ for each graph $G \in \mathcal{P}$.*

A class of graphs is *hereditary* if it is closed under taking induced subgraphs. Our main result is a criterion for a hereditary class of split graphs to be a dom-polynomially bounded class. It is similar to a well-known result of Balas and Yu [2] (rediscovered by Alekseev [1] and Prisner [4]) that a hereditary class $\mathcal{P}$ has a polynomial bounded number of maximal stable sets is and only if the size of an induced matching in graphs of $\mathcal{P}$ is bounded above.

Definition 2. *We define a* comb *as a graph G having a partition $X \cup Y = V(G)$ such that*

- *$|X| = |Y|$,*
- *Y is a stable set,*
- *the edges between X and Y constitute a perfect matching in G.*

If X induces a complete graph in a comb G than G is called a complete comb.

In other words, a (complete) comb is obtained from a (complete) graph H by attaching a new pendant vertex to each vertex of H. Clearly, every comb has an even number of vertices.

Proposition 2. *A comb G of order $2n$ has exactly 2^n minimal dominating sets.*

Proof. Let X and Y be such as in the definition of a comb. There exists a perfect matching $M = \{x_iy_i : x_i \in X, y_i \in Y, i = 1, 2, \dots, n\}$ in G. Let D be an arbitrary minimal dominating set of G. Since y_i is adjacent to x_i only, $D \cap \{x_i, y_i\} \neq \emptyset$ for every $i = 1, 2, \dots, n$. Moreover, $|D \cap \{x_i, y_i\}| = 1$. Indeed, if D contains both x_i and y_i then it is not minimal (we can delete y_i from D).

Conversely, every set $C \subseteq V(G)$ with $|C \cap \{x_i, y_i\}| = 1$ is a minimal dominating set of G. We obtain that $md(G) = 2^n$. □

Theorem 1. (i) *A hereditary class of split graphs $\mathcal{P}$ is dom-polynomially bounded if and only if there exists a complete comb that does not belong to $\mathcal{P}$.*

(ii) *All minimal dominating sets of a dom-polynomially bounded hereditary class of split graphs can be generated in polynomial time.*

Proof. (i) Necessity. If all complete combs belong to $\mathcal{P}$ then we have a contradiction between Proposition 2 and the condition that $\mathcal{P}$ is dom-polynomially bounded class.

Sufficiency. Now a complete comb of order $2n$ does not belong to $\mathcal{P}$. We consider a split graph $G \in \mathcal{P}$ with a split partition $A \cup B$. Let D be a minimal dominating set in G.

Suppose that $|D \cap A| > n$. Since A induces a complete subgraph, and D is a minimal dominating set, each vertex of $u \in D \cap A$ must have a private neighbor inside D, i.e., a vertex $v_u \notin D$ that is adjacent to u and non-adjacent to all vertices of $D \backslash \{u\}$. Clearly, v_u belongs to B. The set $(D \cap A) \cup \{v_u : u \in D \cap A\}$ induces a complete comb H of order at least $2(n + 1)$, a contradiction. Thus, $|D \cap A| \leq n$ for every minimal dominating set D in G.

We denote by B' the set of all vertices in B that are not dominated by the set $D \cap A$. Since D is a minimal dominating set, $D \cap B \subseteq B'$. Indeed, if a vertex x of $B \backslash B'$ belongs to D, we can delete it from D, since its neighbor $y \in D \cap A$ dominates x and all its neighborhood. Since the set B' is stable, we must include the whole B' into D to dominate B'.

We have proved that $D = (D \cap A) \cup B'$ and $|D \cap A| \leq n$. In other words, each minimum dominating set D in G is uniquely determined by a subset $D' \subseteq A$ of cardinality at most n. For a fixed n, the total number of such subsets D' is polynomially bounded. Therefore, $\mathcal{P}$ is a dom-polynomially bounded class.

(ii) The statement follows from the proof of (i), since we essentially need to generate all bounded subsets of A only. Note that some of generated subsets may be redundant (they do not correspond to a minimal dominating set), and we reject such subsets. □

Let CC_n denote a complete comb of order n.

Corollary 1. *For a CC_n-free split graphs WDPSG can be solved in polynomial time.*

Proof. By Theorem 1 we can generate in polynomial time all minimal dominating sets of a CC_n-free split graph. To solve WDPSG it is sufficient to calculate their weights, and choose the minimum. $\square$

For a fixed n, we define the class of hypergraphs $\mathcal{H}_n = \{H$: the split graph of H is CC_n-free$\}$.

Corollary 2. *WSPH can be solved in polynomial time for each class $\mathcal{H}_n$.*

Proof. Follows from Proposition 1 and Corollary 1. $\square$

We can propose a generalization of our results.

Conjecture 1. *A hereditary class of graphs $\mathcal{P}$ is dom-polynomially bounded if and only if there exists n such that all combs of order $2n$ do not belong to $\mathcal{P}$.*

Conjecture 2. *There exists a polynomial time algorithm for generating all minimal dominating sets of every dom-polynomially bounded class.*

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