DIMACS Technical Report 2001-53
December 2001

Proof of a Tiling Conjecture of Komlós

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DIMACS was founded as an NSF Science and Technology Center, and also receives support from the New Jersey Commission on Science and Technology.

ABSTRACT

Given two graphs G and H, an H-*matching* of G (or a *tiling* of G with H) is a subgraph of G consisting of vertex-disjoint copies of H. For an r-chromatic graph H on h vertices, we write $u = u(H)$ for the smallest possible color-class size in any r-coloring of H. The *critical chromatic number* of H is the number $\chi_{cr}(H) = (r-1)h/(h-u)$. A conjecture of Komlós states that for every graph H, there is a constant K such that if G is any n-vertex graph of minimum degree at least $(1 - (1/\chi_{cr}(H)))\, n$, then G contains an H-matching that covers all but K vertices of G. In this paper we prove that the conjecture holds for all sufficiently large values of n.

1 Introduction

All graphs considered in this paper are finite, undirected and simple. If H is a graph on h vertices and G is a graph on n vertices, in *tiling* problems the objective is to find many vertex disjoint copies of H in G, or even a complete tiling (or H-factor) of G with $\lfloor n/h \rfloor$ copies of H. Perhaps, one of the earliest tiling results in extremal graph theory is Dirac's theorem on Hamilton paths [4] that solves the 1-factor problem. The case of triangle-factors is due to Corrádi-Hajnal [3], and finally the celebrated result of Hajnal-Szemerédi settles the K_r-factor problem for all r:

Theorem 1 (Hajnal-Szemerédi 1970 [5]). *Let G be a graph on n vertices with minimum degree*

$$\delta(G) \geq \left(1 - \frac{1}{r}\right) n$$

then G has a K_r-factor.

During the 90's Alon and Yuster extended the Hajnal-Szemerédi theorem in various ways:

Theorem 2 (Alon-Yuster 1992 [1]). *For every $\varepsilon > 0$ and for every integer h there exists an $n_0 = n_0(\varepsilon, h)$ such that for every graph H on h vertices with chromatic number $\chi(H)$, any graph G with $n > n_0$ vertices and with minimum degree*

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) n, \tag{1}$$

contains at least $(1 - \varepsilon)n/h$ vertex disjoint copies of H.

Theorem 3 (Alon-Yuster 1996 [2]). *For every $\varepsilon > 0$ and for every integer h there exists an $n_0 = n_0(\varepsilon, h)$ such that for every graph H on h vertices and for every $n > n_0$, any graph G with n vertices and minimum degree*

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)} + \varepsilon\right) n, \tag{2}$$

has an H-factor.

They conjectured that the error terms εn in Theorem 2 and in Theorem 3 could be released to a constant. In [1] they also remarked that this is essentially best possible. These conjectures have been recently proven by Komlós, Sárközy and Szemerédi :

Theorem 4 (Komlós, Sárközy and Szemerédi 1995 [9]). *For every graph H there is a constant K such that, if G is an n-graph satisfying*

$$\delta(G_n) \geq \left(1 - \frac{1}{\chi(H)}\right) n, \tag{3}$$

then it has an H-matching that covers all but at most K vertices.

Theorem 5 (Komlós, Sárközy and Szemerédi 1995 [9]). *Given the condition of Theorem 4, if*

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) n + K, \tag{4}$$

then G has an H-factor.

Let us use the notation $TT(n,H) = \min\{t : \delta(G) \geq t$ implies that n-graph G has an H-factor$\}$ and define $TT(n,H,M)$ to be the minimum number m such that, if G is an n-graph with minimum degree $\delta(G) \geq m$, then there is an H-matching covering at least M vertices in G. Then, the asymptotic behavior of $TT(n,H)$ seems to be completely understood: the limit of $TT(n,H)/n$ seems to be $1 - 1/\chi(H)$. This suggests that the chromatic number of H probably is the relevant quantity for tiling problems. Of course, it is still possible to fine-tune the known estimates for $TT(n,H)$, e.g. by characterizing those graphs H for which the error term K in Theorem 5 is actually 0, that is, $TT(n,H) \leq (1 - 1/\chi(H))n$. Dirac's theorem and the Hajnal-Szemerédi theorem show that paths and cliques are like that. In fact, for cliques the inequality is an equality. But a full characterization of these graphs is probably hard.
However, in [6] Komlós presented an improved form of Theorem 4 (but not of Theorem 5) in terms of *critical chromatic number* of H. For an r-chromatic graph H on h vertices, we write $u = u(H)$ for the smallest possible color-class size in any r-coloring of H. The critical chromatic number of H is the number $\chi_{cr}(H) = (r-1)h/(h-u)$. It is easy to see that $\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$, and $\chi_{cr}(H) = \chi(H) = r$ if and only if every r-coloring of H has equal color-class sizes.

Theorem 6 (Komlós 2000 [6]). *Let H have parameters $\chi = \chi(H)$ and $\chi_{cr} = \chi_{cr}(H)$. Then, for all $0 < M \leq n$,*

$$TT(n,H,M) \geq M\left(1 - \frac{1}{\chi_{cr}}\right) + (n-M)\left(1 - \frac{1}{\chi - 1}\right). \tag{5}$$

in particular, $TT(n,H) \geq \left(1 - \frac{1}{\chi_{cr}}\right) n$.

He also proved the following result for existence of an H-matching in an n-graph G, with minimum degree of G depending only on $\chi_{cr}(H)$

Theorem 7 (Komlós [6]). *For every graph H and $\varepsilon > 0$ there is a threshold $n_0 = n_0(H, \varepsilon)$ such that, if $n \geq n_0$ and an n-graph G satisfies the degree condition*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)}\right) n, \tag{6}$$

then G contains an H-matching that covers all but at most εn vertices.

He also posed the following conjecture:

Conjecture 1 (Komlós [6]). *For every graph H there is a constant $K = K(H)$ such that, if G is an n-graph satisfying* (6)*, then G contains an H-matching that covers all but at most K vertices.*
This is best possible for every H. Hence,

$$TT(n, H, n-K) = \left(1 - \frac{1}{\chi_{cr}(H)}\right) n.$$

In this paper we will show that the conjecture holds for all sufficiently large n.

Theorem 8. *For any k-chromatic graph H of order h with smallest color-class of order u, there exists an n_0 such that for all $n \geq n_0$, if G is any n vertex graph with*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)}\right) n, \tag{7}$$

then G contains an H-matching that covers all but at most $\frac{5(k-2)}{\alpha}(h-u)$ vertices of G, where $\alpha = \frac{(k-1)u}{h-u}$.

In the proof we will use the concept of *bottle-graphs.* A bottle-graph of chromatic number r is a complete r-partite graph with color-class sizes $(u, w, w, \ldots, w)$, where $u = \alpha w$, for some $\alpha \leq 1$. Clearly, the critical chromatic number of this graph is $r - 1 + \alpha$. The vector $(\frac{\alpha}{r-1+\alpha}, \frac{1}{r-1+\alpha}, \ldots, \frac{1}{r-1+\alpha})$ is called the *color-vector* of the bottle-graph. The parameters u and w are the *neck* and the *width* of the bottle-graph, respectively.
Given an r-chromatic graph H of order h with $u = u(H)$, we say that a graph $\mathcal{B} = \mathcal{B}(H)$ is the bottle-graph of H if $\mathcal{B}$ is the smallest bottle-graph that contains an H-factor, with the color-vector $\underline{\beta} = (s, t, \ldots, t)$, where $s = u/h$, and $t = (1-s)/(r-1)$. Observe that, $\chi_{cr}(\mathcal{B}) = \chi_{cr}(H) = (r-1)/(1-s)$. Moreover, let $u, u_1, u_2, \ldots, u_{r-1}$ denote the color-class sizes in an r-coloring of H, we can always construct a bottle-graph using $r-1$ vertex disjoint copies of H, with i-th copy of H having $u, u_i, u_{i+1}, \ldots, u_{r-1}, u_1, \ldots, u_{i-1}$ vertices in color class $1, 2, \ldots, r$ of bottle-graph. Consequently, the order of $\mathcal{B}(H)$ is at most $(r-1)h$. Therefore, it is sufficient to prove Theorem 8 for bottle-graphs:

Theorem 9 (Main Theorem). *For every bottle-graph H with chromatic number k, neck u, and width w, there exists an n_0 such that for all $n \geq n_0$ if G is any n vertex graph such that*

$$\delta(G) \geq \frac{\alpha + k - 2}{\alpha + k - 1} n = \left(1 - \frac{1}{k-1} + \gamma\right) n, \tag{8}$$

where $\alpha = \frac{u}{w}$ and $\gamma = \frac{\alpha}{(k-1)(k-1+\alpha)}$, then G contains an H-matching that covers all but at most $\frac{5k}{(k-1)^2\gamma} w$ vertices.

2 Notations

$V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G, and we write $v(G) = |V(G)|$ (order of G) and $e(G) = |E(G)|$ (size of G). G_n denotes n-graphs. $N(v, X)$ is the set of neighbors of $v \in V$ in the set $X \subset V$. We use $N(v)$ to denote $N(v, V)$. Hence $|N(v, X)| = \deg(v, X) = \deg_G(v, X)$ is the degree of v in set X, and $\deg(v) = \deg(v, V)$. In a directed graph D, we use $N_{out}(v)$ to denote $\{u \in V(D)| \ (v, u) \in E(D)\}$ (the *out* neighborhood of v), and $\deg_{out}(v) = |N_{out}(v)|$. $\delta(G)$ stands for the minimum, and $\Delta(G)$ for the maximum degree in G. $\nu_i(G)$ denotes the size of a maximum set of vertex disjoint i-stars (stars with i leaves) in G. In particular, $\nu_1(G) = \nu(G)$ is the size of a maximum matching in G. We write $\chi(G)$ and $\chi_{cr}(G)$ for the chromatic number and critical chromatic number of G, respectively. For an r-chromatic graph H of order h, we write $u = u(H)$ for the smallest possible color-class size in any r-coloring of H. When A and B are disjoint subsets of $V(G)$, we use $\deg(A, B)$ to denote the number of edges in $E(G)$ with one endpoint in A and the other in B. A bipartite graph G with color classes A and B and edge set E will be denoted by $G = (A, B, E)$, with $E \subseteq A \times B$. $K(n_1, n_2, \dots, n_r)$ is the complete r-partite graph with color classes of sizes $n_1, n_2, \dots, n_r$. The *density* between disjoint sets X and Y is defined as:

$$d(X, Y) = \frac{\deg(X, Y)}{|X||Y|}.$$

In the proof of Main Theorem Szemerédi's *Regularity Lemma* [11] plays a pivotal role. We will need the following definition to state the regularity lemma:

Definition 1 (Regularity Condition). *Let $\varepsilon > 0$. A pair (A, B) of disjoint vertex-sets in G is ε-**regular** if for every $X \subset A$ and $Y \subset B$, satisfying*

$$|X| > \varepsilon|A|, \ |Y| > \varepsilon|B|,$$

we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

This definition implies that regular pairs are highly uniform bipartite graphs; namely the density of any reasonably large subgraph is almost the same as the density of a regular pair. We will use the following form of the Regularity Lemma:

Lemma 10 (Degree Form). *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set V into $\ell+1$ clusters $V_0, V_1, \dots, V_\ell$, and there is a subgraph G' of G with the following properties:*

- *$\ell \le M$,*
- *$|V_0| \le \varepsilon|V|$,*
- *all clusters V_i, $i \ge 1$, are of the same size $L \le \lceil \varepsilon|V| \rceil$,*

- $\deg_{G'}(v) > \deg_G(v) - (d+\varepsilon)|V|$ *for all* $v \in V$*,*
- $G'|_{V_i} = \emptyset$ *(*V_i *is an independent set in* G'*), for all* i*,*
- *all pairs* (V_i, V_j)*,* $1 \le i < j \le \ell$*, are* ε*-regular, each with density either 0 or greater than* d*, in* G'*.*

A stronger one-sided property of regular pairs is the super-regularity:

Definition 2 (Super-Regularity condition). *Given a graph* G *and two disjoint subsets of its vertices* A *and* B*, the pair* (A, B) *is* (ε, d)-super-regular*, if it is* ε*-regular and,*

$$\deg(a) > d|B| \ \forall a \in A, \ \deg(b) > d|A| \ \forall b \in B.$$

We also use the Blow-up Lemma (see [8]):

Lemma 11. *Given a graph* R *of order* r *and positive parameters* δ *and* Δ*, there exists an* $\varepsilon = \varepsilon(\delta, \Delta, r) > 0$ *such that the following holds. Let* $n_1, n_2, \ldots, n_r$ *be arbitrary positive integers, and let us replace the vertices* $v_1, v_2, \ldots, v_r$ *of* R *with pairwise disjoint sets* $V_1, V_2, \ldots, V_r$ *of sizes* $n_1, n_2, \ldots, n_r$ *(blowing up. we construct two graphs on the same vertex-set* $V = \cup_i V_i$*. The first graph* R_b *is obtained by replacing each edge* $\{v_i, v_j\}$ *of* R *with the complete bipartite graph between the corresponding vertex-sets* V_i *and* V_j*. A sparser* G *is constructed by replacing each edge* $\{v_i, v_j\}$ *of* R *arbitrarily with some* (ε, δ)*-super-regular pair between* V_i *and* V_j*. If a graph* H *with* $\Delta(H) \le \Delta$ *is embeddable into* R_b *then it is already embeddable into* G*.*

3 Number of Left-over Vertices

The following example justifies the thightness on the number of leftover vertices in Theorem 9. Suppose the vertex set $V(G)$ can be partitioned to $A_0, A_1, \ldots, A_{k-2}$, with $|A_0| = \left(\frac{1}{k-1} + (k-2)\gamma\right) n$ and $|A_j| = \left(\frac{1}{k-1} - \gamma\right) n$, for $j \ge 1$, and $\deg(A_i, A_j) = |A_i||A_j|$ for $0 \le i < j \le k-2$, i.e. it is complete between any pair of $A_0, A_1, \ldots, A_{k-2}$. Also assume that all $A_j (j \ge 1)$ are independent sets, while A_0 consists of s connected components, where $s = \left\lceil \frac{|A_0|}{2(k-1)\gamma n + 2w - 2} \right\rceil$. Each component is a complete bipartite graph $\{(L_i, R_i)\}$, and $|L_i| = |R_i| = (k-1)\gamma n + w - 1$, when $i \in \{1, \ldots, s-1\}$. We also assume that $w + u$ divides $(k-1)\gamma n$. It is easy to see that G satisfies the degree condition in (8).
An H-matching of G is called the *best* matching if no other matching of G leaves less vertices uncovered. Naturally in any copy of H in G, two of its classes come from a component (L_i, R_i) of A_0, for some i, and the other $k-2$ color classes are from $A_1, \ldots, A_{k-2}$. We can further assume that in the best H-matching of G, no copy of H has its small color class in $A_j (j \ge 1)$. In fact, if there is a copy of H with its u-vertex class in A_1, then there must be at least $w - u$ vertices left uncovered in A_1. We can easily re-adjust its color classes such that A_1 contains one of its w-vertex classes.
Therefore, in the best matching of G, there will be at least $2w - 2$ uncovered vertices for

each component $(L_i, R_i) \in A_0, 1 \le i \le s-1$, thus totally at least $(s-1)(2w-2)$ uncovered vertices in A_0. In turn, there are at least $(s-1)(2w-2)\frac{|A_j|}{|A_0|}$ left in $A_j, j \ge 1$. Together, the number of uncovered vertices in the best H-matching of G is at least

$$(s-1)(2w-2)\frac{n}{|A_0|} = \left\lceil \frac{|A_0|}{2(k-1)\gamma n + 2w - 2} \right\rceil \frac{n}{|A_0|}(2w-2) \approx \frac{w}{(k-1)\gamma}.$$

4 Proof of Main Theorem

We will assume $u < w$ (that is, $\chi_{cr} < r$), since otherwise the theorem is already proven by Komlós, Sárközy and Szemerédi (Theorem 4). Throughout the paper, we will assume that n is sufficiently large and will use the following main parameters:

$$\varepsilon \ll d \ll \mu \ll \min(\alpha, 1-\alpha). \tag{9}$$

We apply Lemma 10 to G, with ε and d as in (9), and will get a partition of $V(G)$ into clusters $V_0, V_1, \ldots, V_\ell$. And we will assume that size of V_i, $i \ge 1$, L, is divisible by a few integers which will be determined later, otherwise we will move constant number of vertices from each V_i to V_0 to achieve this assumption. We define the following *reduced graph* G_r: The vertices of G_r are the clusters of V_i, $i \ge 1$, and we have an edge between two clusters if they form an ε-regular pair in G' with density exceeding d. Since in G', $\delta(G') \ge \left(1 - \frac{1}{k-1} + \gamma\right) n - (d+\varepsilon))n$, and $\varepsilon \ll d$, an easy computation shows, that in G_r, we have

$$\delta(G_r) \ge \left(1 - \frac{1}{k-1} + \gamma - 2d\right) \ell. \tag{10}$$

When we use c to denote a vertex of G_r, V_c stands for the vertex set of G which are reduced to c. For a set of clusters S, $V(S)$ denote the vertex set of G corresponding to all the clusters of S.
Similar to the complete r-partite graph $K(n_1, n_2, \ldots, n_r)$, a sparser r-partite graph $\mathcal{R}_\varepsilon(n_1, n_2, \ldots, n_r)$ has r color classes with sizes $n_1, \ldots, n_r$. However, it is an ε-regular pair that lies between any pair of color classes, and ε will be omitted if the value ε is clear from the context. For simplicity, we let $\mathcal{R}_r^a(t)$ to represent $\mathcal{R}_\varepsilon(n_1, n_2, \ldots, n_r)$ with $\forall 1 \le i \le r-1, n_i = t, n_r = at$. The parameters r and t might be omitted if they are clearly indicated from the context.

We will show that, except for a special case, the vertices of G'' can be tiled by $\mathcal{R}^{\alpha'}(L_1)$ with some $\alpha' > \alpha, L_1 = CL, 0 < C \le 1$. Next, the vertices in V_0 will be inserted into appropriate (cluster) cliques in a way that after we remove all copies of H containing new vertices, each of remaining triangles still has a H-tiling leaving at most constant vertices. The connection among clusters will finally reduce the number of left-over vertices to a constant depended only on w and u. In the special case, we will show that $V(G) = A \cup B$, where A is the union of several almost-independent vertex sets $U_1, \ldots, U_t$ with $|U_i| \approx \frac{n}{(k-1)+\alpha}$.

And B can be either partitioned into almost-independent vertex sets $U_{t+1}, \ldots, U_k$ such that $|U_i| \approx \frac{n}{(k-1)+\alpha}, i < k$ and $|U_k| \approx \frac{\alpha n}{k-1+\alpha}$, or tiled by $K(w, \ldots, w, u)$ except for constant many vertices. In either case, we can conclude that except for a constant number of vertices, G can be tiled by H.

4.1 The Maximal Clique Cover of the Reduced Graph

To obtain a tiling of G'' with copies of H^* and H^{**}, we need the following concept from graph theory.
Given a graph $\mathcal{G}$, a k-clique cover $\Phi = \{\Phi_k, \Phi_{k-1}, \ldots, \Phi_1\}$ is a collection of disjoint cliques, where Φ_i corresponds to a set of cliques of order i, for $i \in \{k, k-1, \ldots, 1\}$, and $V(\mathcal{G}) = \cup_{i=1}^{k} V(\Phi_i)$. A k-clique cover $\Phi = \{\Phi_k, \Phi_{k-1}, \ldots, \Phi_1\}$ is said to be *maximal* if for any other k-clique cover $\Phi' = \left\{\Phi'_k, \Phi'_{k-1}, \ldots, \Phi'_1\right\}$, if for some $1 \le i \le k$, $|\Phi'_i| > |\Phi_i|$ then there is a $i < j \le k$ such that $|\Phi_j| > |\Phi'_j|$.

Let $\Phi = (\Phi_k, \Phi_{k-1}, \ldots, \Phi_1)$ be an maximal decomposition of $\mathcal{G}$. We will estabish some properties of Φ. Let us assume that for $1 \le i \le j \le k$, K^i, K^j, K^k are any clique in $\Phi_i, \Phi_j, \Phi_{j'}, \Phi_k$, respectively. (Throughout the paper, K^i denote a clique of Φ_i.) For convenience, we use following notations:

- *Well-connected* (or $K^i \hookrightarrow K^j$), if $\deg(v, K^j) = j - 1$, $\forall v \in K^i$
- *Over-connected* (or $K^i \overset{>}{\hookrightarrow} K^j$), if $\deg(K^i, K^j) \ge i(j-1)$ and $K^i \not\hookrightarrow K^j$.
- *Under-connected* (or $K^i \overset{<}{\hookrightarrow} K^j$), if $\deg(K^i, K^j) < i(j-1)$.

Proposition 1.

1. For $i \le j < k$, $\deg(K^i, K^j) \le i(j-1)$. In other words, there are no K^i and K^j such that $K^i \overset{>}{\hookrightarrow} K^j$. Otherwise, there must exist $v_0 \in K^i$ such that $\deg(v_0, K^j) = j$. Thus, $\{v_0\} \cup K^j$ forms a new $j+1$-clique. Same reason causes that, if $\deg(K^i, K^j) = i(j-1)$ then $K^i \hookrightarrow K^j$.

2. If $K^i \overset{>}{\hookrightarrow} K^k$, then there is $v_0 \in K^i$, with $\deg(v_0, K^k) = k$. For another clique K^j, $\deg(K^j, K^k) \le k(j-1)$. In other words, any k-clique which is over-connected to some K^i will be under-connected to all other cliques of order smaller than k. Because, $\deg(K^j, K^k) > k(j-1)$ causes that there exists one vertex $v_1 \in K^k$ with $\deg(v_1, K^j) = j$. Then, $\{v_1\} \cup K^j$ makes a new $j+1$-clique, while $\{v_0\} \cup K^k \setminus \{v_1\}$ makes up a k-clique.
   Consequently for any K^k and arbitrary k i-cliques $K_1^i, K_2^i, \ldots, K_k^i \in \Phi_i$, we must have $\deg(\bigcup_{l=1}^{k} K_l^i, K^k) \le ik(k-1)$. Indeed, if none of $K_l^i, 1 \le l \le k$ satisfies $\deg(K_l^i, K^k) > i(k-1)$, we derive the conclusion immediately. When one of them, say, K_1^i is over-connected to K^k, the above paragraph verifies $\deg(\bigcup_{l=2}^{k} K_l^i, K^k) \le k(i-1)(k-1)$. Together $\deg(\bigcup_{l=1}^{t} K_l^i, K^k) \le ki + k(i-1)(k-1) \le ik(k-1)$.

3. Combine 1 and 2, if $K^j \in \Phi_j$ and two cliques $K^i, \bar{K}^i \in \Phi_i$ satisfy $\deg(K^i, K^j) = \deg(\bar{K}^i, K^j) = i(j-1)$, we have $K^i \hookrightarrow K^j, \bar{K}^i \hookrightarrow K^j$. Moreover, we can write $K^j = A_i(K^j) \cup B_i(K^j)$ such that $\forall v \in A_i(K^j), \deg(v, K^i) = \deg(v, \bar{K}^i) = i-1$, $\forall u \in B_i(K^j), \deg(u, K^i) = \deg(u, \bar{K}^i) = i$ and $|A_i(K^j)| = i, |B_i(K^j)| = j-i$.
   Let $S_i^j = \{K^j \in \Phi_j : \deg(K^i, K^j) = \deg(\bar{K}^i, K^j) = i(j-1)\}$ and $A^j = \{A(K^j) : K^j \in S_i^j\}$. Then $A_i^i \cup A_i^{i+1} \cup \cdots \cup A_i^k$ is a i-partite graph, whose color classes have the equal sizes.

Assume $e(\mathcal{G}) = \ell$. Let φ_i denote the normalized size of the set Φ_i, or $\varphi_i = \frac{|\Phi_i|}{\ell}$ for $i \geq 1$. That is,

$$k\varphi_k + (k-1)\varphi_{k-1} + \ldots + \varphi_1 = 1 \tag{11}$$

Let $i_0 = \min\{1 \leq i \leq k : |\Phi_i| \geq k\}$. Assume $i_0 < k$ and $\varphi_i = 0, \forall i < i_0$. The following proposition presents a lower bound on φ_k when $\delta(\mathcal{G})$ is given:

Proposition 2. *If* $\delta(\mathcal{G}) \geq \left(1 - \frac{1}{k-1} + \gamma - 2d\right)\ell$,

$$\varphi_k \geq \sum_{i=2}^{k-i_0} (i-1)\varphi_{k-i} + (k-1)\gamma - 2(k-1)d \tag{12}$$

Proof. Pick any k cliques $K_1, K_2, \ldots, K_k \in \Phi_{i_0}$, we use Proposition 1.1 and Proposition 1.2 to obtain:

$$\deg(\bigcup_{l=1}^{k} K_l, \Phi_j) \leq ki_0(j-1)\varphi_j\ell,\ \forall i_0 \leq j \leq k-1$$

$$\deg(\bigcup_{l=1}^{k} K_l, \Phi_k) \leq ki_0(k-1)\varphi_k\ell$$

With the minimum degree condition (10), we have

$$ki_0\left(1 - \frac{1}{k-1} + \gamma - 2d\right)\ell \leq \deg(\bigcup_{l=1}^{k} K_l, \mathcal{G}) \leq ki_0(k-1)\varphi_k\ell + ki_0(k-2)\varphi_{k-1}\ell + \cdots + ki_0(i_0-1)\varphi_{i_0}\ell$$

Or

$$1 - \frac{1}{k-1} + \gamma - 2d \leq (k-1)\varphi_k + (k-2)\varphi_{k-1} + \cdots + (i_0-1)\varphi_{i_0} \tag{13}$$

In fact, the right side $(k-1)\varphi_k + \cdots + (i_0-1)\varphi_{i_0}$ can be written as

$$\left(\frac{1}{k-1} + \frac{k-2}{k-1}k\right)\varphi_k + \frac{k-2}{k-1}(k-1)\varphi_{k-1} + \left(\frac{k-2}{k-1}(k-2) - \frac{1}{k-1}\right)\varphi_{k-2} + \cdots$$
$$+ \left(\frac{k-2}{k-1}i_0 - \frac{k-1-i_0}{k-1}\right)\varphi_{i_0}$$

$$= \frac{1}{k-1}\varphi_k + \frac{k-2}{k-1}\left(k\varphi_k + (k-1)\varphi_{k-1} + \cdots + i_0\varphi_{i_0}\right) - \frac{1}{k-1}\left(\varphi_{k-2} + 2\varphi_{k-3} + \cdots + (k-1-i_0)\varphi_{i_0}\right)$$

$$= \frac{1}{k-1}\varphi_k + \frac{k-2}{k-1} - \frac{1}{k-1}\sum_{i=2}^{k-i_0}\varphi_{k-i}(i-1)$$

That is

$$(k-1)\varphi_k + \cdots + (i_0-1)\varphi_{i_0} = \frac{1}{k-1}\left(\varphi_k + k - 2 - \sum_{i=2}^{k-i_0}\varphi_{k-i}(i-1)\right) \tag{14}$$

Apply (14) in (13), we have (12). $\square$

Then we can let

$$\varphi_k = \sum_{i=2}^{k-i_0}(i-1)\varphi_{k-i} + (k-1)\gamma + s, \text{ where } s \geq -2(k-1)d. \tag{15}$$

Consider a clique $K \in \Phi_{k-i}, 1 \leq i \leq k-i_0$, which is either well-connected or under-connected to all k-cliques. Let $\Lambda(K)$ be the set of k-cliques that are well-connected to K. And let λ_i be the minimum of $\Lambda(K)/\ell$ among all such $K \in \Phi_{k-i}$. We have:

Proposition 3.

$$\lambda_i \geq (k-1)\gamma + \frac{i-1}{k-1}s + \sum_{2\leq j\leq i}(j-1)\varphi_{k-j} + (i-1)\sum_{j>i}\varphi_{k-j} - 2(k-i)d \tag{16}$$

Proof. Let $m = \left|\left\{K^k \in \Phi_k : \deg(K, K^k) < (k-1)(k-i)\right\}\right|/\ell$. We have

$$(k-i)\left(1 - \frac{1}{k-1} + \gamma - 2d\right) \leq \frac{\deg(K, G_r)}{\ell} \leq (k-1)(k-i)\varphi_k - m + (k-2)(k-i)\varphi_{k-1} + \cdots + (k-i-1)(k-i)\varphi_{k-i} + \sum_{j>i}(k-i-1)(k-j)\varphi_{k-j}$$

$$\leq \quad (k-i)\left((k-1)\varphi_k + \cdots (k-i-1)\varphi_{k-i} + \textstyle\sum_{j>i}(k-j-1)\varphi_{k-j}\right) + \textstyle\sum_{j>i}(j-i)\varphi_{k-j} - m$$

Using (14) and (15), we have

$$m \leq s\frac{k-i}{k-1} + \sum_{j>i}(j-i)\varphi_{k-j} + 2kd,$$

thus (16) follows. $\square$

For example,

$$\begin{aligned}\lambda_1 &\geq (k-1)\gamma - 2kd \\ \lambda_2 &\geq (k-1)\gamma + \frac{s}{k-1} + \sum_{j\geq 2} \varphi_{k-j} - 2kd\end{aligned}$$

Now let us find an maximal decomposition on the reduced graph G_r $\Phi = (\Phi_k, \Phi_{k-1}, \ldots, \Phi_1)$. Define i_0 as before. We can assume that $i_0 < k$, otherwise we will remove all the vertices in the clusters of $\Phi_1, \ldots, \Phi_{k-1}$ to the exceptional set V_0 such that Φ becomes a $\mathcal{R}^1(L)$-covering of G_r and still $|V_0| \leq C\varepsilon n$ for some constant C. The same reason make us further assume that $\varphi_i = 0, \forall i < i_0$.

Because (10) holds, we can apply Proposition 2 to obtain (15). The assumption of non-extremal case is that $s \geq \mu$. The extremal case $s < \mu$ will be considered in Section 4.3.

4.2 The General Case

We have found a maximal clique cover $\Phi = \{\Phi_k, \Phi_{k-1}, \ldots, \Phi_1\}$ of the reduced graph G_r. Correspondingly, it defines a cluster clique cover $\Psi = \{\Psi_k, \Psi_{k-1}, \ldots, \Psi_1\}$ of G''. In each of such clique, vertices are clusters and edges are ε-regular pairs. We denote an element of Ψ_i by $\mathcal{K}^i$. All the terminologies we used in Section 4.1 apply to these cluster cliques as well. Let $s = \varphi_k - (k-1)\gamma - \sum_{i=2}^{k-i_0} \varphi_{k-i}(i-1) \leq \mu$. The assumption in the general case is that $s > \mu$.

4.2.1 The Decomposition Lemma

Consider a cluster clique $\mathcal{K} \in \Psi_{k-i}$, with corresponding $K \in \Phi_{k-i}$, for some $1 \leq i \leq k - i_0$. A k-cluster clique $\mathcal{K}^k$ is called *good* for $\mathcal{K}$ if its corrponding clique $K^k \in \Phi_k$ is well-connected or over-connected to K. Otherwise $\mathcal{K}^k$ is *bad* for $\mathcal{K}$. $\mathcal{K}$ is called *typical* if for a constant $c = \lceil \frac{k-1+\alpha'}{1-\alpha'} \rceil$, $\left|\left\{K^k \in \Phi_k : K \overset{>}{\hookrightarrow} K^k\right\}\right| < c$. Otherwise, we will refer to $\mathcal{K}$ as a *atypical* clique.

To state the decomposition algorithm for G'' we will use the following observation:

Proposition 4 *Let V_i and V_j , $1 \leq i, j \leq \ell$ be two clusters in G'' that correspond to endpoints of an edge e in the reduced graph R. If V_i and V_j are partitioned to p and q sub-clusters $\{V_i^1, \ldots, V_i^p\}$, and $\{V_j^1, \ldots, V_j^q\}$ respectively, and the size of the smallest sub-clusters is cL for some $\varepsilon \ll c \leq 1$, then the (V_i^r, V_j^t), $1 \leq r, t \leq p$ are ε'-regular pairs, with $\varepsilon' = \min(\frac{\varepsilon}{2}, \frac{\varepsilon}{c})$. In particular, if $p = q$, $(V_i^t, V_j^t), 1 \leq t \leq p$ are p disjoint ε'-regular edges.*

Evenly partitioning each cluster in a clique $\mathcal{K} \in \Psi$ into p parts thus replaces the old clique with p new (sub-cluster) cliques in the cluster graph. This procedure will be referred to as *p-partition* of a (cluster) clique. For simplicity, we will still use ε as the parameter in the new regular pairs. If a (sub-cluster) clique was derived from a good (or bad) clique of $\mathcal{K}$, we call the new clique good (or bad) as well.

α' is a parameter that will be determined later. Assume $\alpha' = p/q$.

Decomposition Algorithm of G'' is as follows:

1. *$(q-p)$-partition all the cliques in Ψ_k and denote the new clique family Ψ'_k:*
   We use the L' to denote the size of resulting sub-clusters, or $L = \frac{L'}{q-p}$. Note, since we can assume that L is divisible by q and $q-p$, $(1-\alpha')L'$, $(i+\alpha')L, i = 0, 1, \ldots, k-1$ are integers and $(1-\alpha')L'$ divides $(i+\alpha')L$.

2. *For $i = 1, \ldots, k-i_0$, convert typical cliques in Ψ_{k-i} to copies of $\mathcal{R}_k^{\alpha'}$.*
   For each typical clique $\mathcal{K} \in \Psi_{k-i}$ we arbitrarily pick a well-connected $\mathcal{K}^k$ in Ψ'_k. By Proposition 1.1, there are i clusters in $B(\mathcal{K}^k)$ and $k-i$ ones in $A(\mathcal{K}^k)$. Note, a special k-partite graph $\mathcal{R}^*(t) = \mathcal{R}(\underbrace{it, \ldots, it}_{i}, \underbrace{(i-1+\alpha')t, \ldots, (i-1+\alpha')t}_{k-i})$ is made up of i copies of $\mathcal{R}_k^{\alpha'}(t)$. We divide each cluster in $B(\mathcal{K}^k)$ into two parts: the large parts containing $\frac{i-1+\alpha'}{i}L'$ vertices and the small ones containing $\frac{1-\alpha'}{i}L'$ vertices. All these large sub-clusters are combined with clusters in $A(\mathcal{K}^k)$ to form a $\mathcal{R}^*(L'/i)$, while all the small sub-clusters are assotiated with clusters of $\mathcal{K}$ to make a copy of $\mathcal{R}^*(\frac{1-\alpha'}{i-1+\alpha'}\frac{L'}{i})$. Clearly when we repeat this procedure to $\frac{(i-1+\alpha')L}{(1-\alpha')L'}$ well-connected k-cliques in Ψ'_k, we elimanate $\mathcal{K}$ and obtain some k-partite graphs that are $\mathcal{R}_k^{\alpha'}(\frac{L'}{i})$- or $\mathcal{R}_k^{\alpha'}(\frac{1-\alpha'}{i-1+\alpha'}\frac{L'}{i})$-factors.

3. *For $i = 1, \ldots, k-i_0$, convert atypical cliques in Ψ_{k-i} to copies of $\mathcal{R}_k^{\alpha'}$.* We will follow the same procedure as typical cliques. The only difference is that over-connected k-cliques replace well-connected cliques in converting.
   Consider an atypical $k-i$-clique $\mathcal{K} \in \Phi_{k-i}$, any k-clique $\mathcal{K}^k$ over-connected to $\mathcal{K}$ has more than i clusters that are adjacents to all the clusters in $\mathcal{K}$. We can arbitrarily label i of them by $B(\mathcal{K}^k)$ and rest by $A(\mathcal{K}^k)$.

4. *Convert all existing k-partite graphs to $\mathcal{R}^{\alpha'}(L_1)$'s.* We finally partition the remaining k-cliques in Ψ'^k to $\mathcal{R}^{\alpha'}$'s and then refine all different-size copies of $\mathcal{R}^{\alpha'}$ to $\mathcal{R}^{\alpha'}(L_1)$'s, where $L_1 = c_l L$. Note, c_l is a constant depended on α'. Moreover, $c_l \geq c\mu$ for some constant c.

The correctness of the above algorithm is versified as follows:

In Step 3, under the Proposition 1.2, all k-cliques over-connected to $\mathcal{K}$ do not participate in the converting operation of any other (typical or atypical) cliques. While each $\mathcal{K} \in \Psi_{k-i}$ needs $\frac{(i-1+\alpha')L}{(1-\alpha')}L'$ good k-cliques in Ψ'_k, the total number of over-connected k-cliques is at least:

In Step 2, we consider typical cliques in $\Psi_{k-1}, \ldots, \Psi_{i_0}$ sequentially. Let us first estimate the number of well-connected k-cliques in Ψ_k for a typical clique in Ψ_{k-i}. Since $c \ll \ell$, ignoring the existence of over-connected k-cliques will not change the computation. Thus $\lambda_i \ell$, the quantity we found in Section 4.1 gives a lower bound. Thus in the new k-clique family Ψ'_k, there are al least $\lambda_i \ell L/L'$ well-connected k-cliques, we need to show that it is not smaller than $\sum_{l=1}^{k-i_0} \frac{(l-1+\alpha')L}{(1-\alpha')L'} \varphi_{k-l}\ell$, or

$$\lambda_i - \sum_{l\geq 1} \frac{l-1+\alpha'}{1-\alpha'} \varphi_{k-l} > 0. \tag{17}$$

Let us first compute $I_i = I_i = \lambda_i - \sum_{l\geq 1} \frac{l-1+\alpha}{1-\alpha}\varphi_{k-l}$. For example, $I_1 = (k-1)\gamma - \frac{\alpha}{1-\alpha}\varphi_{k-1}$:

$$\begin{aligned} I_1 &= (k-1)\gamma - \frac{\alpha}{1-\alpha}\frac{1}{k-1}\left(1 - k\varphi_k - \sum_{j\geq 2}(k-j)\varphi_{k-j}\right) \\ &= (k-1)\gamma - \frac{\alpha}{1-\alpha}\frac{1}{k-1}\left(1 - k\left((k-1)\gamma + s + \sum_{j\geq 2}(j-1)\varphi_{k-j}\right) - \sum_{j\geq 2}(k-j)\varphi_{k-j}\right) \\ &= (k-1)\gamma - \frac{\alpha}{1-\alpha}\left(\frac{1}{k-1} - k\gamma - \frac{k}{k-1}s - \sum_{j\geq 2} j\varphi_{k-j}\right) \end{aligned}$$

Since

$$(k-1)\gamma - \frac{\alpha}{1-\alpha}\frac{1}{k-1} = \frac{\alpha}{k-1+\alpha} - \frac{\alpha}{(1-\alpha)(k-1)} = -\frac{\alpha}{1-\alpha}k\gamma,$$

We have

$$I_1 = \frac{k}{k-1}\frac{\alpha}{1-\alpha}s + \frac{\alpha}{1-\alpha}\sum_{j\geq 2} j\varphi_{k-j}. \tag{18}$$

$$\begin{aligned} I_2 &= (k-1)\gamma + \frac{s}{k-1} + \sum_{j\geq 2}\varphi_{k-j} - \frac{\alpha}{1-\alpha}\varphi_{k-1} - \frac{1+\alpha}{1-\alpha}\varphi_{k-2} \\ &= \frac{s}{k-1} + \frac{k}{k-1}\frac{\alpha}{1-\alpha}s + \sum_{j\geq 3}\varphi_{k-j} + \frac{\alpha}{1-\alpha}\sum_{j\geq 3} j\varphi_{k-j} \end{aligned}$$

In general,

$$I_i = \left(\frac{i-1}{k-1} + \frac{k\alpha}{(k-1)(1-\alpha)}\right)s + \sum_{j>i}\left(i-1+\frac{\alpha}{1-\alpha}j\right)\varphi_{k-j} > k\mu_1, \tag{19}$$

where $\mu_1 = \frac{\alpha}{(1-\alpha)(k-1)}\mu$.

To make (17) holds, it is sufficient to let $\alpha' > \alpha$ satisfy

$$\frac{l-1+\alpha'}{1-\alpha'} - \frac{l-1+\alpha}{1-\alpha} \leq \mu_1, \text{ for } l = 1, 2, \ldots, k-i_0 \tag{20}$$

We let α' be the largest number such that (20) holds. Equivalently α' is the minimum solution of the equations if we replace "$\leq$" in (20) by "$=$". The solutions x_l satisfy:

$$x_l = \frac{\mu_1(1-\alpha) + l\alpha}{\mu_1(1-\alpha) + l}, \text{ and } x_l - \alpha \geq \frac{(1-\alpha)^2}{l+1}\mu_1, \text{ for } l = 1, \ldots, k-i_0$$

Therefore, the minimum solution α' satisfies:

$$\alpha' - \alpha = c_\alpha \mu. \tag{21}$$

where c_α is some constant only related to α.

Lemma 12 (Decomposition Lemma). *If the reduced graph G_r satisfies $\varphi_k \geq \sum_{i=2}^{k-i_0}(i-1)\varphi_{k-i} + (k-1)\gamma + \mu$, G_r can be decomposed into disjoint copies of $\mathcal{R}^{\alpha'}(L_1)$, for some $\alpha' = \alpha + c_\alpha\mu$, and $L_1 \geq c\mu L$* $\square$

4.2.2 Handling of Exceptional Vertices

The proof of Main Theorem in the general case is immediate from the following lemma:

Lemma 13 *Assume $\varepsilon \leq \theta \ll \rho \ll 1$. A n-vertex graph G satisfies (8), and contains a set of exceptional vertices V_0 with $|V_0| \leq \theta n$. Let $\mathcal{R}_k^{\alpha'}(L_1)$ be a k-partite graph on k clusters satisfying all but one clusters have sizes L_1 and all pairs of clusters are ε-regulars with densities at least $d(\varepsilon \ll d \ll \rho)$. Assume $G'' = G \setminus V_0$ contains a $\mathcal{R}_{\varepsilon,k}^{\alpha'}(L_1)$-factor, in which $\alpha' = \alpha + \rho$. Then G contains an H-matching that covers all but at most $\frac{5k}{(k-1)^2\gamma}w$ vertices.*

Note, there are two different sizes of clusters in the cluster (reduced) graph G_r: L_1 (large) and $\alpha' L_1$ (small). Let $\mathbf{R}$ denotes the $\mathcal{R}^{\alpha'}(L_1)$-cover, and $\mathcal{R}$ the element of the cover, or unbalanced cliques. We will get rid of the vertices of V_0 by sequentially removing disjoint copies of H with exactly one vertex of V_0 in each H copy. A corollary of Key Lemma in [10] is needed:

Lemma 14 *Let $\mathcal{R} = \{V_1, \ldots, V_k\}$ be a cluster clique in G'', i.e. $|V_i| = n_i$ and (V_i, V_j) is an ε-regular pair. Assume a sub-cluster clique $\mathcal{R}_0 = \{W_1, \ldots, W_k\}$ satisfies $W_i \subset V_i$, $|W_i| = d_1 L$, $i = 1, \ldots, k$, for $d_1 \gg \varepsilon$. Then $K_k(h) \subset \mathcal{R}_0$. In particular, $H \subset \mathcal{R}_0$, and we can let its small color-class be in any W_i.* $\square$

Proof of Lemma 13:
Let us define an edge between a vertex $v \in V_0$ and a cluster $C \in G_r$, $v \sim C$, whenever $\deg(v, C) \geq d\,|C|$. Suppose $\mathcal{R}_1 \in \mathbf{R}$ with clusters $\{U_1, U_2, \ldots, U_k\}$ (we always let U_k denote the small cluster) satisfies

$$v \sim U_i, \text{ for } i \neq j \ \text{ or for every } i \tag{22}$$

When $j = k$ or no j exists, we will remove a copy of H consisting of v, $u-1$ vertices of U_k, w vertices from each $U_i (i \neq k)$. When $j \neq k$, we will remove u vertices from U_k, w vertices from $U_i (i \neq j)$ and $w-1$ vertices from j to form a H with v. Note, this removal creates the same effect as adding one extra vertex to U_j, we may say that v is *inserted* to cluster U_j or clique $\mathcal{R}_1$ when an actual removal is processed. And we call $\mathcal{R}_1$ a host clique of v when (22) holds. While inserting all the vertices of V_0 to their host cliques, we allow at most $\theta_1 L_1 (\theta_1 = \sqrt{\theta})$ vertices to be added to each element of $\mathbf{R}$. Note that $\mathcal{R}^{\alpha'}(L_1) = \mathcal{R}^{\alpha}(L_2) \cup \mathcal{R}^1(L_0)$, where $L_2 = \frac{1-\alpha'}{1-\alpha} L_1$, $L_0 = \frac{\alpha'-\alpha}{1-\alpha} L_1$.

Proposition 5. *All exceptional vertices can be removed sequentially.*

Proof. We consider vertices of V_0 one by one. Suppose for the first t exceptional vertices, we have removed t disjoint copies of H each of which contains an exceptional vertex and $h-1$ vertices from some unbalanced clique. G'' still denotes the vertices of G that included in the cluster graph at present. Thus $|G''| = n - \theta n - t(h-1)$. Let v be the $t+1$st exceptional vertex of V_0. Let m denote the fraction of cliques in $\mathbf{R}$ that could be a host of v. We have:

$$\begin{aligned}\left(1-\frac{1}{k-1}+\gamma\right)n-\varepsilon n-(h-1)t &\le \deg_G(v,V(G''))\\ &\le \left((1-m)\frac{k-2+\alpha' d+d}{(k-1)+\alpha'}+m\right)(n-\varepsilon n-(h-1)t)\\ 1-\frac{1}{k-1}+\gamma-\varepsilon h &\le 1-\frac{1+\alpha'-\alpha' d-d}{(k-1)+\alpha'}+\frac{1+\alpha'-\alpha' d-d}{(k-1)+\alpha'}m\end{aligned}$$

$$\frac{1+\alpha'-\alpha' d-d}{(k-1)+\alpha'}-\frac{1}{k-1+\alpha}-\varepsilon h \le \frac{1+\alpha'-\alpha' d-d}{(k-1)+\alpha'}m$$

Therefore,

$$m > \frac{1}{1+\alpha'-\alpha' d-d}(\alpha-2d-\varepsilon kh) > \frac{\alpha}{2}$$

where we use

$$\frac{\alpha}{3} > kh\varepsilon \tag{23}$$

Because of the restriction that no cliques can host more than $\theta_1 L_1$ vertices, percentage of the disabled cliques is at most:

$$\frac{\theta n}{\sqrt{\theta}L_1}\Big/\frac{n}{(k-1+\alpha')L_1} < \sqrt{\theta}k$$

The proposition holds because

$$\frac{\alpha}{2} > \sqrt{\theta}k \tag{24}$$

$\square$

After V_0 becomes empty, we want all regular pairs inside the cliques to satisfy the super-regularity condition. Consider $\mathcal{R} = \{U_1,\ldots,U_k\}$. We will move a vertex v from cluster U_i in $\mathcal{R}$ to V_0, if there exists $j \ne i$ with $\deg(v,U_j) < (d-\varepsilon)|U_j|$. The ε-regularity of the cluster-pairs guarantees that there are at most $2\varepsilon|U_i|$ such vertices in each cluster. As the way we handle original vertices of V_0, We handle the new (at most $6\varepsilon n$) exceptional vertices. As a result, all cluster-edges satisfy $(\varepsilon, \frac{d}{2})$ super-regularity.

Let $\mathbf{R}'$ and $\mathcal{R}'$ denote the resulting clique family and element of the family, respectively. Note, the sizes of clusters varies even in the same clique. However, since $\alpha'-\alpha = \rho \gg \theta_1$, each copy of $\mathcal{R}'$ still contains $\mathcal{R}^1(\frac{L_0}{2})$. We observe that an unit of $w-u$ vertices can be moved among the clusters of $\mathcal{R}'$. Because, $\mathcal{R}^1(\frac{L_0}{2})$ contained in $\mathcal{R}'$ are H_k-factor, in other words, $\mathcal{R}'$ contains many copies of H whose small classes take place in every clusters of $\mathcal{R}'$. When we want to move $w-u$ vertices from U_1 to U_2 in $\mathcal{R}'$, we simply switch the color classes of one copy of H, such that the u-vertex class which was supposed to occur at U_1 take place at U_2. This results in that U_1 losses $w-u$ vertices while U_2 gains $w-u$ vertices.

Proposition 6. *Each element of* $\mathbf{R}'$ *has a* H*-tiling that covers all but at most* $(k-1)(2w-u)+w$ *vertices.*

In fact, consider an unbalanced clique $\mathcal{R}'$. Because of the super-regularity of all pairs in $\mathcal{R}'$, we can apply the Blow-up Lemma to it, i.e., the regular edges in $\mathcal{R}'$ can be treated as complete bipartite graphs. We pick the H-matching of $\mathcal{R}'$ (which leaves $a_1, a_2, \ldots, a_k$ vertices uncovered in cluster $U_1, U_2, \ldots, U_k$, respectively) satisfying (1) it is one of the best H-matchings, i.e. $\sum a_i$ is the minimum over all the H-matchings of $\mathcal{R}'$. (2) it is the most balanced, i.e. $\sum |a_i - w|$ is the smallest among the best matchings. Assume that $a_j = \min a_i$. Naturally, $a_j < w$, otherwise a copy of H could be found. And $a_j (j \neq j) < 2w-u$, otherwise we can move $w-u$ vertices from U_j to U_j to reduced $\sum |a_i - w|$. Therefore, $\sum a_i < w + (k-1)(2w-u)$. $\square$

If we directly apply Proposition 6 to all the elements of $\mathbf{R}'$, we will get an H-matching of G'' that leaves at most $((k-1)(2w-u)+w)\,|\mathbf{R}|$ vertices. Since $|\mathbf{R}| = \mathbf{O}(\ell)$ and ℓ is not larger than the constant $M(\varepsilon)$ (according to the Regularity Lemma), this already confirms Conjecture 1.

To further reduce the number of left-over, we need to consider the connectivity of clusters of G_r *before* inserting the exceptional vertices. First we define a directed graph $\mathcal{D}$ over clusters of G_r whose directed edges defined as follows: $U \xrightarrow{\mathcal{D}} U_1$, whenever $U_1 \in \mathcal{R} = \{U_1, U_2, \ldots, U_k\}$, and $(U, U_i) \in e(G_r), \forall i \neq 1$. Then we will have:

Proposition 7. *For any* $U \in V(\mathcal{D})$, $\deg_{out}(U) \geq \frac{4\gamma}{3}|V(\mathcal{D})|$.

In fact, for every $U \in V(\mathcal{D})$, let m_1, m_2, and m_3 denote the fraction of elements of $\mathbf{R}$, $\mathcal{R} = \{U_1, U_2, \ldots, U_k\}$ for which $(U, U_i) \in E(G_r)\ \forall i < k$, $(U, U_i) \in E(G_r)\ \forall i \neq j\ (j \neq k)$ and $(U, U_i) \in E(G_r)\ \forall i$, respectively. By degree condition in G_r, we have

$$\begin{aligned}
1 - \frac{1}{k-1} + \gamma - 2d &\leq \frac{k-2}{k-1+\alpha'} + \frac{1}{k-1+\alpha'}m_1 + \frac{\alpha'}{k-1+\alpha'}m_2 + \frac{1+\alpha'}{k-1+\alpha'}m_3 \\
&< \frac{k-2}{k-1+\alpha'} + \frac{1}{k-1+\alpha'}m_1 + \frac{1}{k-1+\alpha'}m_2 + \frac{2}{k-1+\alpha'}m_3 \\
1 - \frac{1}{k-1} + \gamma &< \frac{k-2}{k-1+\alpha} + \frac{1}{k-1+\alpha}(m_1 + m_2 + 2m_3) \\
&= \left(1 - \frac{1}{k-1} - (k-2)\gamma\right) + \left(\frac{1}{k-1} - \gamma\right)(m_1 + m_2 + 2m_3) \\
(k-1)\gamma &\leq \left(\frac{1}{k-1} - \gamma\right)(m_1 + m_2 + 2m_3)
\end{aligned}$$

Therefore in $\mathcal{D}$,

$$\begin{aligned}\deg_{out}(U) &= (m_1+m_2+km_3)\,|\mathbf{R}| \\ &\geq \frac{(k-1)\gamma}{\frac{1}{k-1}-\gamma}\,\frac{|V(\mathcal{D})|}{k} \\ &> \frac{(k-1)^2\gamma}{k}|V(\mathcal{D})|.\end{aligned}$$

□

In a directed graph D the *source set* of v is defined as $\mathcal{W}(v)=\{u\in V(D):\exists$ a directed path from u to $v\}$. It is easy to see that $|\mathcal{W}(u)|\geq|\mathcal{W}(v)|$, if $u\in N_{out}(v)$. A set $\mathcal{S}(D)\subseteq V(D)$ will be called a *sink* set if $V(D)=\bigcup\limits_{v\in\mathcal{S}(D)}\mathcal{W}(v)$.

Lemma 15. *In a directed graph D, with $\delta=\min\limits_{v\in D}\deg_{out}(v)$, there is a sink set $\mathcal{S}$ of size at most $\frac{|V(D)|}{\delta+1}$.*

Proof. Let x_1 be a vertex in which $|\mathcal{W}(x_1)|=\max\limits_{v\in V(D)}|\mathcal{W}(v)|$. Since $N_{out}(x_1)\in\mathcal{W}(x_1)$ (otherwise, some out-neighbor of x_1 produces a larger source set), we have $|\mathcal{W}(x_1)|\geq\delta+1$. Let $D'=D\setminus\mathcal{W}(x_1)$, then for every vertex $v\in D'$, the set $N_{out}(v)$ in D' is the same as $=N_{out}(v)$ in D, since, if $v'\in N_{out}(v)$ in D and $v'\in\mathcal{W}(x_1)$, v should also be in $\mathcal{W}(x_1)$. Now, let x_2 be a vertex with the largest source set in D'. Similar to x_1, we have $\mathcal{W}(x_2)\geq\delta+1$ in D'. The above procedure can be repeated at most $\frac{|V(D|)}{\delta+1}$ times and the proof will follow. □

Combining Proposition 7 and Lemma 15 in the directed graph $\mathcal{D}$, there is a cluster sink set $\mathcal{S}(\mathcal{D})$ of order at most $\frac{k}{(k-1)^2\gamma}$. The clusters of $\mathcal{S}(\mathcal{D})$ might be located in different elements of $\mathbf{R}$. Thus, if applying Proposition 6, we would obtain that number of left-over is at most

$$((k-1)(2w-u)+w)\,\frac{k}{(k-1)^2\gamma}\approx\frac{2w}{\gamma}.$$

However, recall the fact that an unit of $w-u$ vertices can be moved arbitrarily among any clique in $\mathbf{R}$. We can thus separate the left-over vertices from two categories: (I) less than $w-u$ vertices from each cluster, (II) number of $(w-u)$-vertex units from each cliques. Proposition 7 and Lemma 15 have reduced the number of clusters which carry the extra vertices to $\frac{k}{(k-1)^2\gamma}$. We will follow the same technique to control the number of cliques of (II). That is, we will define a directed graph $\mathcal{D}^*$ whose vertices are elements of $\mathbf{R}$ and the directed edge from $\mathcal{R}_1$ to $\mathcal{R}_2$ exists whenever $\exists U\in\mathcal{R}_1, U'\in\mathcal{H}_2, U\xrightarrow{\mathcal{D}}U'$. Similar as Proposition 7, we know that

$$\min_{\mathcal{R}\in\mathcal{D}^*}\deg_{out}(\mathcal{R})\geq\frac{k(k-1)\gamma}{2}\,V(\mathcal{D}^*).$$

When Lemma 15 applied, we obtain a clique sink set $\mathcal{S}(\mathcal{D})$ of order at most $\frac{2}{k(k-1)\gamma}$. Then applying Proposition 6, the number of left-over vertices in (II) is at most

$$((k-1)(2w-u)+w)\,\frac{2}{k(k-1)\gamma}<\frac{4w}{(k-1)\gamma}$$

Therefore, total left-over from (I) and (II) is at most

$$(w-u)\frac{k}{(k-1)^2\gamma}+\frac{4w}{(k-1)\gamma}<\frac{5k}{(k-1)^2\gamma}w$$

$\square$

4.3 The Extremal Cases

Recall that $\Phi = \{\Phi_k, \dots, \Phi_{i_0}\}$ denotes the maximum clique-cover of G_r. In this section we will assume

$$s=\varphi_k-(k-1)\gamma-\sum_{i=2}^{k-i_0}\varphi_{k-i}(i-1)\leq \mu. \tag{25}$$

Let $\psi_i = \sum_{j\leq i}(k-j)\varphi_j$ for $i<k$. By (11) and (15), we have

$$(k-1)k\gamma+ks+(k-1)\sum_{j\geq 1} i\varphi_{k-i}=1$$

$$\psi_{k-1}=\frac{1-\alpha}{k-1+\alpha}-\frac{k}{k-1}s\geq\alpha_0$$

here we define $\alpha_0=\frac{1-\alpha}{k-1+\alpha}-\frac{k}{k-1}\mu$.

Let $C=\left(\frac{\alpha_0}{\varepsilon}\right)^{\frac{1}{k}}$. Since $\psi_{k-1}\geq\alpha_0=C^k\varepsilon$, there is an integer t ($i_0\leq t\leq k-1$) such that $\psi_t\geq C^{j_0}\varepsilon$ and $\psi_{t-1}\leq C^{j_0-1}\varepsilon$ for some positive integer $j_0\leq k$. Set $\varepsilon'=k(C^{t-1}\varepsilon+2\mu)$, and $\mu_1=C^{j_0}\varepsilon$.

We will first move all the vertices of G in $\Phi_j (j<t)$ to V_0. The size of resulting exceptional set (still denoted by V_0) is less than $\sum_{j<t}j\varphi_j n<k\sum_{j<t}\varphi_j n<k\psi_{t-1}n\leq kC^{j_0-1}\varepsilon n<\varepsilon' n$.

It is not hard to see that number of atypical elements of Φ_t is small from the proof of Proposition 2 and (25). Let us fix two typical elements $K_1, K_2\in\Phi_t$. Define $\Phi^0_{k-j}=\{K\in\Phi_{k-j}:\deg(\{K_1,K_2\},K)<2t(k-j-1)\}$ and $m_j=|\Phi^0_{k-j}|/\ell$ for $j=0,1,\dots,k-t$. Repeat the calculation in the proof of Proposition 3, we obtain:

$$m_0+m_1+\cdots+m_{k-t}<\mu+\psi_{t-1}+2kd<\frac{\varepsilon'}{k}$$

Recall in Proposition 1.3, we divide each clique $K^j (j\geq t)$ which is well connected to K_1, K_2 into $A(K^j)$ and $B(K^j)$, where $A(K^j)$ consists of the clusters which are adjacent to all but one of K_1 and also K_2, $B(K^j)$ are those which are adjacent to all of K_1, K_2. And combining $A(K)$ for all clusters $K\notin\Phi^0_t\cup\cdots\cup\Phi^0_k$, we obtain a cluster set A of size at least $t(\varphi_k+\cdots+\varphi_t-\frac{\varepsilon'}{k})\ell=t(\frac{1}{k-1}-\gamma-\frac{\varepsilon'}{k})\ell$. On one hand, A is covered by a family of t-cliques. On the other hand, by Proposition 1.3, A is made up of t independent set $U_1,\dots,U_t$, with $|U_j|\geq(\frac{1}{k-1}-\gamma-\frac{\varepsilon'}{k})\ell$. The degree condition on G_r (10) requires

$$\deg(c,G_r\setminus U_i)\geq|G_r\setminus U_i|-\frac{\varepsilon'}{k}\ell,\forall c\in U_i \tag{26}$$

i.e., elements of U_i are almost adjacent to all the clusters outside U_i.

Based one the value of t, we consider two cases:

4.3.1 Extremal Case (I): $t = k-1$.

Let $U_k = V(G_r) \setminus \{U_1, \dots, U_{k-1}\}$. From now on, $U_i (1 \le i \le k)$ will refer to their underlying G-vertex sets. We will move all exceptional vertices (set V_0) to U_k. Then remove a vertex $v \in U_k$ to $U_i (i < k)$ if $\deg(v, U_i) < \varepsilon_1 |U_i|$ where $\varepsilon' \ll \varepsilon_1 \ll 1$. Let us still denote the resulting sets by $U_1, \dots, U_k$. In the ideal case, $|U_i| = \left(\frac{1}{k-1} - \gamma\right) n, \forall i < k$ and $|U_k| = (k-1)\gamma n$. Because of the super-regularity between every pair of $U_1, \dots, U_k$, we apply the Blow-up Lemma to find the desired H-factor in G.
If $|U_i| < \left(\frac{1}{k-1} - \gamma\right) n, \forall i < k$, we can create some copies of H whose w-vertex classes are located at U_k. Then an argument similar to Proposition 6 shows that all but $2kw$ vertices of G will be covered by a H-tiling. Consequently, we can assume that $|U_1| > \left(\frac{1}{k-1} - \gamma\right) n$, also $|U_i| < \left(\frac{1}{k-1} - \gamma\right) n (1 < i < k), |U_k| < (k-1)\gamma n$ (the other cases are similar).
¿From U_1 we will move a vertex v for which $\deg(v, U_1) > \varepsilon_1 |U_1|$ to other classes U_i with fewer vertices, without hurting super-regularity. Let $U_1, \dots, U_k$ still denote the resulting sets. After this step, either we can achieve the ideal case, or for all $v \in U_1, \deg(v, U_1) \le \varepsilon_1 |U_1|$. Let us assume that the latter is the case, and let $x = |U_1| - \left(\frac{1}{k-1} - \gamma\right) n$, we will use the following fact:

Proposition 8.
$$\nu_i(G) \ge (\delta(G) - i + 1) \frac{n}{2(i+1)\Delta(G)}$$

To see this take a maximal set of i-stars, let m denote its size, and $\mathbf{E}$ be the number of edges between the stars and the remaining vertices of G, we have the following chain of inequalities which in turn proves this proposition:

$$(n - m(i+1))(\delta(G) - (i-1)) \le \mathbf{E} \le m(i+1)\Delta(G).$$

Proposition 8 implies that we can find x vertex disjoint w-stars in U_1 and after moving their centers to other U_i with fewer vertices we immediately remove x copies of H. The remaining sets $U_1, \dots, U_k$ and C have size ratio $(1, \dots, 1, \alpha)$ and also satisfy super-regularity condition. Applying the Blow-up Lemma will complete the proof.

4.3.2 Extremal Case (II): $t < k-1$.

Let $B = V(G_r) \setminus A$. We let V_b to be represent corresponding vertices of G in the clusters of B and V_0. V_a denotes those in the clusters of A. Let H_0 be the $k-t$-partite bottle graph with with width w and neck u. Our goal is to find a almost perfect H_0-matching in V_b such that each copy of H_0 have a big neighborhood (defined as common neighbors of all the vertices in H_0) in V_a. Thus we can apply the matching lemma to match each copy of H_0 with a copy of $K_t(w)$ from V_a. This will create the desired H-tiling of G that leaves only constant number of vertices.

Finding the H_0-matching of V_b is almost same as what we did in non-extremal case: we first tile B with $\mathcal{R}_{k-t}^{\alpha'}(L_1)$, the $k-t$-partite graph whose color classes have size L_1 except for one class (size $\alpha' L$), and every two classes form an ε-regular pair. Then we insert V_0 to B. The only difference is that we have to take special care of certain vertices in V_b, whose degrees to V_a are small.

In fact, to tile B with $\mathcal{R}^{\alpha'}_{k-t}(L_1)$, for some $\alpha' > \alpha$, $L_1 \mid L$, we take advantage of the existing clique cover on B: by definition, $B = \{\Psi_1, \Psi_2, \dots, \Psi_{k-t}, \Phi^0\}$, where Ψ_i denote family of i-cliques generated from $i+t$-cliques $(\Phi_{i+t} \setminus \Phi^0_{i+t})$, for $i = 1, \dots, k-t$, while $\Phi^0 = \Phi^0_k \cup \dots \cup \Phi^0_t$, denotes all j-cliques ($j \geq t$) which did not participate in the creation of A. We will repeat the algorithm in Section 4: each $K \in \Psi_1, \Psi_2, \dots, \Psi_{k-t-1}$ will be converted to a copy of $\mathcal{R}^{\alpha'}_{k-t}(L_1)$ with the help of certain elements of Ψ_{k-t}. The rest of Ψ_{k-t} and Φ^0 will naturally be divided into copies of $\mathcal{R}^{\alpha'}_{k-t}$. To explain why the coverting algorithm is feasible, we will re-use the calculations in Section 4.4.2.1. It is easy to see that if $K' \in \Psi_{j-t}, K" \in \Psi_{k-t} (t < j < k)$ came from $K^j \in \Phi_j, K^k \in \Phi_k$, respectively, and $K^j \hookrightarrow K^k$, then $K' \hookrightarrow K"$. Thus, the number of well-connected $k-t$-cliques of $K \in \Psi_{i-t}$, is at least $\Lambda(K^i) \geq \lambda_i$, if K was generated from K^i. In the key expression (18) and (19), although this time s is not bigger than μ, we use $\forall i \leq t, \psi_i = \sum_{j \geq i} j\varphi_{k-j} \geq C^{j_0}\varepsilon$. Let $\mu' = C^{j_0}\varepsilon$, then μ' play the same role as μ. That is, the right side of (20) were replaced by $\frac{\alpha}{1-\alpha}(k-t)\mu'$. Thus, $\alpha' - \alpha = c_1\mu'$. Eventually, we find a $\mathcal{R}^{\alpha'}_{k-t}(L_1)$-factor Φ_B in B.

Before inserting V_0 to B, we remove (at most $2\varepsilon'|V_b|$) vertices v from V_b to U_i if

$$\deg(v, U_i) < \beta_1 |U_i|, \tag{27}$$

with $(\varepsilon' \ll \beta_1 \ll \mu')$.

We might move extra vertices from $\mathcal{R}^{\alpha'}_{k-t}(L_1)$ to V_0 in order to maintain the ratio of cluster sizes as $(1, \dots, 1, \alpha')$. If any cluster lost more than one half of its vertices because of (27), the entire copy of $\mathcal{R}^{\alpha'}_{k-t}(L_1)$ will be removed to V_0. It is easy to see that the resulting V_0 satisfies: $|V_0| < 2\varepsilon' n/\alpha'$.

To finish the proof of the extremal cases, we seperate cases according to sizes of U_i:
The Ideal Case: $|U_i| = \left(\frac{1}{k-1} - \gamma\right) n$, for $i = 1, \dots, t$.
Since densities bwteen pairs of U_i are amlost one, we first apply the Blow-up Lemma to $U_1 \cup \dots \cup U_t$, and get a $K_t(w)$-factor. Note that the neighborhood of any copy of $K_t(w)$ in V_b is amost entire V_b.
Define a subset V_{bad} of V_b, where $\forall v \in V_{bad}$,

$$\deg(v, A) \leq (1 - \beta_2)|A|, \tag{28}$$

with $\varepsilon' \ll \beta_2 \ll \beta_1$. It is easy to see that $|V_{bad}| < \frac{2\varepsilon'}{\beta_2} n < \beta_2 n$.
We first assume that after inserting V_0 to B, we obtain a H_0-matching such that each copy of H_0 contains at most one vertex of V_{bad}. This assumption leads to an easy matching between H_0 copies from V_b and $K_t(w)$ from V_a: we can first use the greedy algorithm to match those of H_0 containing vertices of V_{bad}, then use the matching lemma to handle the rest. The number of uncovered vertices by H-matching of G will propotional to the number of left-over by H_0-matching over V_b.
To force each copy of H_{k-t} contain at most one V_{bad} vertex, we move all of V_{bad} (and some

extra to maintain sizes of clusters in $\mathcal{R}^{\alpha'}_{k-t}(L_1)$ under the same ratio) to V_0. Since $\beta_2 \ll \mu$, we can still insert vertices in the new V_0 back to B.

The Other Cases:

When some $1 \leq i_0 \leq t$ causes $|U_{i_0}| < \left(\frac{1}{k-1} - \gamma\right) n$, instead of $K_t(w)$, we find some copies of H_t from V_a whose u-vertex class comes from U_{i_0}. To match them, we need to find the same amount of $K_{k-t}(w)$ from V_b. This is possible because $\alpha' - \alpha = c_1\mu' \gg \varepsilon'$.

When $|U_{i_0}| > \left(\frac{1}{k-1} - \gamma\right) n$ for some i_0, we find $|U_{i_0}| - \left(\frac{1}{k-1} - \gamma\right) n$ copies of w-star in A then move therr centers of to other U_i or V_b which has less vertices (than the ideal level).

The rest is exactly same as in the Ideal Case.

□

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