DIMACS Technical Report 2001-47
December 2001

Integral Complete Split Graphs

by

Pierre Hansen[1]
GERAD and École des Hautes Études Commerciales,
Montréal, Québec H3T 2A7, Canada

Hadrien Mélot[2]
Dept of Computer Science, University of Mons-Hainaut, Belgium

Dragan Stevanović[3,4,5]
DIMACS Center, Rutgers University,
Piscataway, New Jersey 08854-8018, USA

[1]`pierreh@crt.umontreal.ca`
[2]`hadrien.melot@umh.ac.be`
[3]Supported in part by Special Focus on Data Analysis and Mining under grant NSF CCR 00 87022
[4]On leave from Faculty of Science, University of Niš, Yugoslavia
[5]`dragance@pmf.pmf.ni.ac.yu`

ABSTRACT

We give characterizations of integral graphs in the family of complete split graphs and a few related families of graphs.

1 Introduction

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be undirected graphs without loops or multiple edges. The *union* $G_1 \cup G_2$ of graphs G_1 and G_2 is the graph $G = (V, E)$ for which $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The notation nG is short for $\underbrace{G \cup G \cup \ldots \cup G}_{n}$. The *complete product* $G_1 \nabla G_2$ of graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2. The *sum* $G_1 + G_2$ of graphs G_1 and G_2 is the graph with the vertex set $V(G_1) \times V(G_2)$ in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1 = v_1$ and $(u_2, v_2) \in E_2$ or $u_2 = v_2$ and $(u_1, v_1) \in E_1$. Further, let K_n denote the *complete graph* on n vertices, and let $\overline{K}_n$ denote the graph with n vertices and no edges.

For $a, b, n \in N$ we define the following classes of graphs:

- the *complete split graph* $CS_b^a \cong \overline{K}_a \nabla K_b$;
- the *multiple complete split-like graph* $MCS_{b,n}^a \cong \overline{K}_a \nabla (nK_b)$;
- the *extended complete split-like graph* $ECS_b^a \cong \overline{K}_a \nabla (K_b + K_2)$;
- the *multiple extended complete split-like graph*

$$MECS_{b,n}^a \cong \overline{K}_a \nabla \Big(n(K_b + K_2) \Big).$$

In Fig. 1 we have shown examples of graphs from all of these classes.

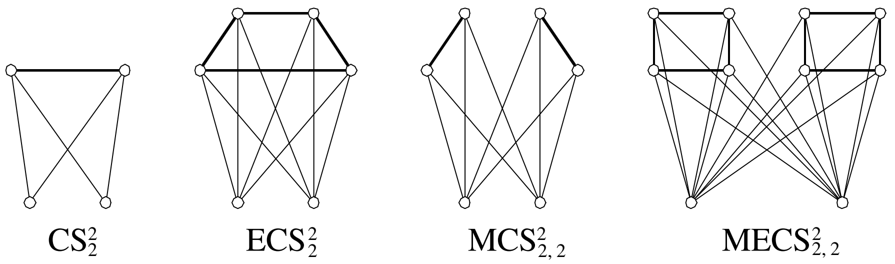

Figure 1: Examples of graphs.

A graph is called *integral* if all the eigenvalues of its adjacency matrix are integers. The search for integral graphs, initiated by F. Harary and A. Schwenk in [5] and continued in many papers thereafter, revealed that not only the number of integral graphs is infinite, but that one can find them in almost all classes of graphs. For a recent survey on integral graphs, see [2].

All 263 integral graphs with up to 11 vertices were enumerated in [1][2]. A close look at graphical representations of these graphs showed that among several known families, some of them were complete split graphs and some had split graphs as induced subgraphs (see Fig. 1).

However, not all complete split graphs are integral. Computer investigations with Matlab [6] on complete split graphs with up to 500 vertices led to the conjecture that the following two families of complete split graphs are integral:

(i) Complete split graphs CS_b^a satisfying

$$a = \left\lceil \frac{i}{2} \right\rceil^2 + (b-1) \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{i+2}{2} \right\rfloor \quad (i \in N). \tag{1}$$

Moreover, if b is the power of a prime, there are no other integral complete split graphs.

(ii) Complete split graphs CS_b^a satisfying $b = 4k+2$ for $k \in N$ and

$$a = -\left\lfloor \frac{b}{4} \right\rfloor + (b-1) \left\lceil \frac{i}{2} \right\rceil^2 + \left\lfloor \frac{i}{2} \right\rfloor \left\lfloor \frac{i+2}{2} \right\rfloor \quad (i \in N). \tag{2}$$

In the next section we give a necessary and sufficient condition for the complete product of two regular graphs to be integral. Using this condition in Section 3 we characterize integral graphs in the families of complete split graphs and multiple complete split-like graphs, by giving the explicit formulae for their parameters a, b and n. We also show that the above families (1) and (2) satisfy the corresponding formula. At the end, in Section 4 we characterize integral graphs in the families of extended complete split-like graphs and multiple extended complete split-like graphs.

2 Integrality condition for the complete product of graphs

The following theorem found in [3] is proven in [4].

Theorem 1 (Finck, Grohmann) *For $i = 1, 2$ let G_i be regular graphs of degree r_i with n_i vertices. The characteristic polynomial of the complete product of graphs G_1 and G_2 is given by the relation*

$$P_{G_1 \nabla G_2}(\lambda) = \frac{P_{G_1}(\lambda) P_{G_2}(\lambda)}{(\lambda - r_1)(\lambda - r_2)} [(\lambda - r_1)(\lambda - r_2) - n_1 n_2]. \quad \blacksquare$$

Since G_i is regular graph, its largest eigenvalue is equal to r_i (with the eigenvector equal to all-1 vector) and the fraction $\frac{P_{G_i}(\lambda)}{\lambda - r_i}$ is the polynomial with roots equal to the remaining eigenvalues of G_i with the same multiplicities. Thus in order that the complete product $G_1 \nabla G_2$ of two regular graphs is integral both G_1 and G_2 must be integral and the expression

$$Q(\lambda) = (\lambda - r_1)(\lambda - r_2) - n_1 n_2$$

must have integer roots. The roots of $Q(\lambda)$ are equal to

$$\lambda_{1,2} = \frac{r_1 + r_2 \pm \sqrt{(r_1 - r_2)^2 + 4n_1n_2}}{2},$$

and they are integers if and only if $r_1 + r_2$ and $\sqrt{(r_1 - r_2)^2 + 4n_1n_2}$ are integers of the same parity. The last fact means that there exists $k \in N$ such that $(r_1 - r_2)^2 + 4n_1n_2 = (|r_1 - r_2| + 2k)^2$, wherefrom we get the following *integrality condition*

$$n_1n_2 = k(k + |r_1 - r_2|). \tag{3}$$

Thus we have proved

Corollary 1 *For $i = 1, 2$ let G_i be regular graphs of degree r_i with n_i vertices. The complete product $G_1 \nabla G_2$ is integral graph if and only if both G_1 and G_2 are integral graphs and there exists $k \in N$ such that the integrality condition (3) holds.* ■

3 Integral complete split graphs

We have that $CS_b^a \cong \overline{K}_a \nabla K_b$. Graph $\overline{K}_a$ is regular with degree 0 ($n_1 = a, r_1 = 0$), while K_b is regular with degree $b-1$ ($n_2 = b, r_2 = b-1$). By Corollary 1, graph CS_b^a is integral if and only if there exists $k \in N$ such that $ab = k(k + b - 1)$, hence

$$a = k + \frac{k(k-1)}{b} \quad \text{and} \quad b \text{ divides } k(k-1). \tag{4}$$

Since the greatest common divisor $(k, k-1)$ of k and $k-1$ is equal to 1, we have that

$$b = (b, k(k-1)) = (b, k) \cdot (b, k-1).$$

Let $p = (b, k)$ and $q = (b, k-1)$. Thus $b = pq$, $(p, q) = 1$, $p|k$ and $q|k-1$. Let $\alpha, \beta \in Z$ be determined by the Euclidean algorithm such that $p\alpha - q\beta = 1$, and let $k' = k - p\alpha = k - 1 - q\beta$. From $p|k$ it follows that $p|k'$, while from $q|k-1$ it follows that $q|k'$. Since $(p, q) = 1$ we have that $pq|k'$ and

$$k = p\alpha + cpq \quad \text{for some } c \in Z,$$

while $k - 1 = q\beta + cpq$. Now from (4) it follows that

$$a = p\alpha + cpq + (\alpha + cq)(\beta + cp) = (\alpha + cq)(\beta + p + cp).$$

Thus we have proved

Theorem 2 *The complete split graph CS_b^a is integral if and only if there exist $p, q \in N$ with $(p, q) = 1$ and $c \in Z$ such that*

$$\alpha + cq > 0, \quad a = (\alpha + cq)(\beta + p + cp), \quad \textit{and} \quad b = pq,$$

where $\alpha, \beta \in Z$ are determined by the Euclidean algorithm such that $p\alpha - q\beta = 1$. ■

Let us return now to the conjectured families of integral complete split graphs.

The family (1) for odd i corresponds to the case $p=1$, $q=b$, $\alpha=1$, $\beta=0$ and $c=\left\lfloor\frac{i}{2}\right\rfloor$, while for even i it corresponds to the case $p=b$, $q=1$, $\alpha=0$, $\beta=-1$ and $c=\left\lfloor\frac{i}{2}\right\rfloor$.

Moreover, if b is a power of prime, we have only two possibilities. The first possibility is that $p=b$ and $q=1$, and thus $\alpha=1$ and $\beta=(b-1)$. In this case we have that

$$a=(1+c)(b-1+b+cb)=(c+1)^2+(c+1)(c+2)(b-1),$$

which corresponds to $i=2c+2$. The second possibility is that $p=1$ and $q=b$, wherefrom $\alpha=b+1$ and $\beta=1$. In this case we have that

$$a=(b+1+cb)(1+1+c)=(c+2)^2+(c+1)(c+2)(b-1),$$

which corresponds to $i=2c+3$.

The family (2) for odd i corresponds to the case $p=2k+1$, $q=2$, $\alpha=1$, $\beta=k$ and $c=\left\lfloor\frac{i}{2}\right\rfloor$, while for even i it corresponds to the case $p=2$, $q=2k+1$, $\alpha=k+1$, $\beta=1$ and $c=\left\lfloor\frac{i}{2}\right\rfloor-1$.

Multiple complete split-like graphs provide a simple generalization of complete split graphs. We have that $MCS_{b,n}^a\cong\overline{K}_a\nabla(nK_b)$. Graph $\overline{K}_a$ is regular with degree 0 ($n_1=a, r_1=0$), while nK_b is regular with degree $b-1$ ($n_2=nb, r_2=b-1$). By Corollary 1, graph $MCS_{b,n}^a$ is integral if and only if there exists $k\in N$ such that $anb=k(k+b-1)$, hence

$$an=k+\frac{k(k-1)}{b}\quad\text{and}\quad b \text{ divides } k(k-1).$$

Repeating the consideration from above we get the following

Theorem 3 *The multiple complete split graph $MCS_{b,n}^a$ is integral if and only if there exist $p,q\in N$ with $(p,q)=1$ and $c\in Z$ such that*

$$\alpha+cq>0,\quad an=(\alpha+cq)(\beta+p+cp),\quad \textit{and}\quad b=pq,$$

where $\alpha,\beta\in Z$ are determined by the Euclidean algorithm such that $p\alpha-q\beta=1$. ■

4 Integral extended complete split-like graphs

We have that $ECS_b^a\cong\overline{K}_a\nabla(K_b+K_2)$. Graph $\overline{K}_a$ is regular with degree 0 ($n_1=a, r_1=0$), while K_b+K_2 is regular with degree b ($n_2=2b, r_2=b$). By Corollary 1, graph ECS_b^a is integral if and only if there exists $k\in N$ such that $2ab=k(k+b)$. From this it follows that

$$a=\frac{1}{2}\left(k+\frac{k^2}{b}\right)\quad\text{and}\quad 2b \text{ divides } k^2. \tag{5}$$

Let
$$b = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$$
be a prime factorization of b, such that $p_1 \le p_2 \le \dots \le p_r$. From $b|k^2$ it follows that $p_i^{\lceil \frac{\alpha_i}{2} \rceil}|k$ for each $i = 1, 2, \dots, r$, and we can write
$$k = p_1^{\lceil \frac{\alpha_1}{2} \rceil} p_2^{\lceil \frac{\alpha_2}{2} \rceil} \cdots p_r^{\lceil \frac{\alpha_r}{2} \rceil} \cdot c, \quad c \in N.$$

Let $\mathrm{odd}\,(n)$, $n \in N$, be the characteristic function of the set of odd numbers, i.e.
$$\mathrm{odd}\,(n) = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases} = 2\left\lceil \frac{n}{2} \right\rceil - n.$$
From (5) it follows that
$$a = \frac{1}{2} \cdot c \cdot \left(p_1^{\lceil \frac{\alpha_1}{2} \rceil} p_2^{\lceil \frac{\alpha_2}{2} \rceil} \cdots p_r^{\lceil \frac{\alpha_r}{2} \rceil} + p_1^{\mathrm{odd}\,(\alpha_1)} p_2^{\mathrm{odd}\,(\alpha_2)} \cdots p_r^{\mathrm{odd}\,(\alpha_r)} \cdot c \right). \tag{6}$$

We see that $a \in N$ if either c is even or b is odd. In this second case, all primes p_1, p_2, $\dots$, p_r are odd and the right hand side of (6) contains the product of two numbers of distinct parities, which is always even.

In the remaining case when both c is odd and b is even, it must hold that
$$2 \,|\, p_1^{\lceil \frac{\alpha_1}{2} \rceil} p_2^{\lceil \frac{\alpha_2}{2} \rceil} \cdots p_r^{\lceil \frac{\alpha_r}{2} \rceil} + p_1^{\mathrm{odd}\,(\alpha_1)} p_2^{\mathrm{odd}\,(\alpha_2)} \cdots p_r^{\mathrm{odd}\,(\alpha_r)} \cdot c.$$
Since b is even, we have that $p_1 = 2$ and $\lceil \frac{\alpha_1}{2} \rceil \ge 1$, so that the above condition, taking into account that p_2, $\dots$, p_r are all odd, becomes $2 \,|\, p_1^{\mathrm{odd}\,(\alpha_1)}$, which is equivalent to the condition that α_1 is odd.

We summarize this in the following

Theorem 4 *Let $b = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ be a prime factorization of b. The extended complete split graph ECS_b^a is integral if and only if there exists $c \in N$ such that*
$$a = \frac{1}{2} \cdot c \cdot \left(p_1^{\lceil \frac{\alpha_1}{2} \rceil} p_2^{\lceil \frac{\alpha_2}{2} \rceil} \cdots p_r^{\lceil \frac{\alpha_r}{2} \rceil} + p_1^{\mathrm{odd}\,(\alpha_1)} p_2^{\mathrm{odd}\,(\alpha_2)} \cdots p_r^{\mathrm{odd}\,(\alpha_r)} \cdot c \right).$$
and either c is even or b is odd or the highest power of 2 which divides b is odd. ■

Similar as with complete split graphs, multiple extended complete split-like graphs provide a simple generalization of extended complete split-like graphs. We have that $MECS_{b,n}^a \cong \overline{K}_a \nabla \big(n(K_b + K_2)\big)$. Graph $\overline{K}_a$ is regular with degree 0 ($n_1 = a, r_1 = 0$), while $n(K_b + K_2)$ is regular with degree b ($n_2 = 2nb, r_2 = b$). By Corollary 1, graph $MECS_{b,n}^a$ is integral if and only if there exists $k \in N$ such that $2anb = k(k + b)$. From this it follows that
$$an = \frac{1}{2}\left(k + \frac{k^2}{b}\right) \quad \text{and} \quad 2b \text{ divides } k^2.$$
Repeating the consideration from above we get the following

Theorem 5 *Let $b = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ be a prime factorization of b. The multiple extended complete split graph $MECS_{b,n}^a$ is integral if and only if there exists $c \in N$ such that*

$$an = \frac{1}{2} \cdot c \cdot \left(p_1^{\lceil \frac{\alpha_1}{2} \rceil} p_2^{\lceil \frac{\alpha_2}{2} \rceil} \cdots p_r^{\lceil \frac{\alpha_r}{2} \rceil} + p_1^{\mathrm{odd}\,(\alpha_1)} p_2^{\mathrm{odd}\,(\alpha_2)} \cdots p_r^{\mathrm{odd}\,(\alpha_r)} \cdot c \right).$$

and either c is even or b is odd or the highest power of 2 which divides b is odd. ■

Remark Consider the even more general case of graph $\overline{K}_a \nabla (G + K_n)$, where G is an integral, regular graph with m vertices and degree r. Here we have that $n_1 = a, r_1 = 0$ and $n_2 = mn, r_2 = r + n - 1$, and the integrality condition (3) says that $\overline{K}_a \nabla (G + K_n)$ is an integral graph if and only if there exists $k \in N$ such that

$$amn = k(k + r + n - 1).$$

If $a = mn + r + n - 1$ then $k = mn$ satisfies the above condition, but the task of characterization of set of parameters a, m, r, n for which the graph $\overline{K}_a \nabla (G + K_n)$ is integral appears to be difficult.

References

[1] K.T. Balińska, M. Kupczyk, S. Simić, K.T. Zwierzyński, *On generating all integral graphs on 11 vertices*, Tech. Rep. 469, Computer Science Center, Techn. Univ. of Poznań, 1999/2000.

[2] K.T. Balińska, D. Cvetković, Z. Radosavljević, S. Simić, D. Stevanović, *A survey on integral graphs*, accepted for publication in *Univ. Beograd. Publ. Elektr. Fak, Ser. Mat.*

[3] D. Cvetković, M. Doob, H. Sachs, *Spectra of graphs—Theory and application*, Deutscher Verlag der Wissenschaften—Academic Press, Berlin—New York, 1980; second edition 1982; third edition, Johann Ambrosius Barth Verlag, Heidelberg-Leipzig, 1995

[4] H.-J. Finck, G. Grohmann, *Vollständiges Produkt, chromatische Zahl und charakteristisches Polynom regulärer Graphen,* I, Wiss. Z. TH Ilmenau 11 (1965), 1–3.

[5] F. Harary, A.J. Schwenk, *Which graphs have integral spectra?*, Graphs and Combinatorics (eds. R. Bari and F. Harary), Springer-Verlag, Berlin (1974), 45–51.

[6] MathWorks, Inc., *Matlab: The language of Technical Computing*, The Mathworks, Inc.