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On the components of NEPS of connected bipartite graphs

by

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ABSTRACT

Back in 1983, D. Cvetković posed the conjecture that the components of NEPS of connected bipartite graphs are almost cospectral. In 2000, we showed that this conjecture does not hold for infinitely many bases of NEPS, and we posed a necessary condition on the base of NEPS for NEPS to have almost cospectral components. At the same time, D. Cvetković posed weaker version of his original conjecture which claims that each eigenvalue of NEPS is also the eigenvalue of each component of NEPS.

Here we prove this weaker conjecture, give an upper bound on the multiplicity of an eigenvalue of NEPS as an eigenvalue of its component, give new sufficient condition for the almost cospectrality of components of NEPS of connected bipartite graphs, and characterize the bases of NEPS which satisfy this condition.

1 Introduction

Let $G_1, G_2, \ldots, G_n$ be simple graphs whose non-empty vertex sets are denoted by $V(G_1)$, $V(G_2)$, $\ldots$, $V(G_n)$, respectively. Let $\mathcal{B}$ be a non-empty set of non-zero binary n-tuples such that for each $i = 1, 2, \ldots, n$, there exists an n-tuple in $\mathcal{B}$ whose ith component equals 1. The *non-complete, extended p-sum*, abbreviated NEPS, of $G_1, G_2, \ldots, G_n$ with *basis* $\mathcal{B}$ is the graph (see e.g. [3])

$$G = \mathrm{NEPS}\,(G_1, G_2, \ldots, G_n; \mathcal{B})$$

defined as follows:

(i) the vertex set of G is $V(G_1) \times V(G_2) \times \ldots \times V(G_n)$;

(ii) two vertices $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ are adjacent if and only if there exists $(\beta_1, \beta_2, \ldots, \beta_n)$ in $\mathcal{B}$ such that u_i is adjacent to v_i in G_i whenever $\beta_i = 1$ and $u_i = v_i$ whenever $\beta_i = 0$.

The graphs $G_1, G_2, \ldots, G_n$ are called the *factors* of NEPS.

Examples of NEPS when $n = 2$ are the product $G_1 \times G_2$ for $\mathcal{B} = \{(1, 1)\}$, the sum $G_1 + G_2$ for $\mathcal{B} = \{(1, 0), (0, 1)\}$, the strong sum $G_1 \oplus G_2$ for $\mathcal{B} = \{(1, 1), (0, 1)\}$ and the strong product $G_1 \star G_2$ for $\mathcal{B} = \{(1, 1), (1, 0), (0, 1)\}$.

One of the most important properties of NEPS of graphs is that its eigenvalues and eigenvectors are easily expressed using eigenvalues and eigenvectors of its factors.

Theorem 1 ([3]) *Spectrum of* $\mathrm{NEPS}\,(G_1, G_2, \ldots, G_n; \mathcal{B})$ *consists of all possible values* Λ *given by*

$$\Lambda = \sum_{\beta \in \mathcal{B}} \lambda_1^{\beta_1} \lambda_2^{\beta_2} \cdots \lambda_n^{\beta_n},$$

where λ_i *is an arbitrary eigenvalue of* G_i*,* $i = 1, 2, \ldots, n$*. The eigenvectors corresponding to the eigenvalue* Λ *are given by*

$$x = x_1 \otimes x_2 \otimes \cdots \otimes x_n,$$

where x_i *is an arbitrary eigenvector of* G_i *corresponding to the eigenvalue* λ_i*,* $i = 1, 2, \ldots, n$*, and* $\otimes$ *denotes the Kronecker product of matrices.*

Another important question is that of connectedness of NEPS. Notice that we can consider the base $\mathcal{B}$ of NEPS also as a set of vectors in GF_2^n, or as a matrix over the field GF_2. From now on, let $r(\mathcal{B})$ denotes the rank of matrix $\mathcal{B}$ over the field GF_2. The following theorem settles the question of connectedness by giving the number of components of NEPS.

Theorem 2 ([5]) *Let* B_1*,* B_2*,* $\ldots$*,* B_n *be the connected bipartite graphs. The number of components of* $\mathrm{NEPS}\,(B_1, B_2, \ldots, B_n; \mathcal{B})$ *is* $2^{n-r(\mathcal{B})}$.

Let $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ be the vertices of

$$G = \mathrm{NEPS}\,(B_1, B_2, \ldots, B_n; \mathcal{B}).$$

Define the *binary distance vector* $\overrightarrow{bd}(u, v)$ by

$$(\overrightarrow{bd}(u, v))_i = \begin{cases} 0 & d_{B_i}(u_i, v_i) \text{ is even} \\ 1 & d_{B_i}(u_i, v_i) \text{ is odd} \end{cases}$$

where $d_{B_i}(u_i, v_i)$ is the distance between vertices u_i and v_i in B_i, $i = 1, 2, \ldots, n$. Theorem 2 is proved by showing that the vertices u and v are in the same component of G if and only if $\overrightarrow{bd}(u, v) \in \mathcal{L}(\mathcal{B})$, where $\mathcal{L}(\mathcal{B})$ denotes the lineal of $\mathcal{B}$ over the field GF_2^n.

The method of using basis elements as if they were vectors of GF_2^n, introduced in the paper [5], happened to be very useful in studying NEPS. Using this method, the diameter of NEPS of connected, bipartite graphs is determined in [1]. It is also used to disprove the conjecture of Cvetković [2].

Two graphs are said to be *almost cospectral* if their eigenvalues, different from 0, coincide together with their multiplicities. Cvetković [2] posed the following conjecture.

Conjecture 1 *The components of NEPS of connected, bipartite graphs are almost cospectral.*

Let $S \subset \{1, 2, \ldots, n\}$ and let

$$\mathrm{Ann}\,(\mathcal{B}, S) = \{\beta \in \mathcal{B} \colon (\forall i \in S)\, \beta_i = 0\}.$$

For $u \in GF_2^n$ define $u^{-S} \in GF_2^{n-|S|}$, with coordinates in the set $\{1, 2, \ldots, n\} \setminus S$, by $u_i^{-S} = u_i$ for $i \notin S$. Now, denote

$$\mathrm{ann}\,(\mathcal{B}, S) = \{\beta^{-S} \colon \beta \in \mathrm{Ann}\,(\mathcal{B}, S)\}.$$

Observe that it holds $r(\mathrm{Ann}\,(\mathcal{B}, S)) = r(\mathrm{ann}\,(\mathcal{B}, S))$. The following theorem is proven in [6].

Theorem 3 ([6]) *Let* $G = \mathrm{NEPS}\,(B_1, B_2, \ldots, B_n; \mathcal{B})$*, where* $B_1, B_2, \ldots, B_n$ *are arbitrary bipartite graphs. If for every* $S \subset \{1, 2, \ldots, n\}$*, for which it holds that* $\mathrm{Ann}\,(\mathcal{B}, S) \neq \emptyset$*, we have*

$$|S| + r(\mathrm{ann}\,(\mathcal{B}, S)) \leq r(\mathcal{B}), \tag{1}$$

then the multiplicity of every nonzero eigenvalue of G *is divisible by* $2^{n-r(\mathcal{B})}$.

If the components of G are almost cospectral, then the multiplicity of every nonzero eigenvalue is divisible by the number of components, which is $2^{n-r(\mathcal{B})}$. It is proved in [6] that if basis $\mathcal{B}$ does not satisfy condition (1) then there exist connected, bipartite graphs $B_1, B_2, \ldots, B_n$ such that the components of $\mathrm{NEPS}\,(B_1, B_2, \ldots, B_n; \mathcal{B})$ are not almost cospectral, which disproves Cvetković's conjecture. Thus the condition (1) is necessary for the components of G to be almost cospectral. Also, new conjecture is posed which claims that the condition (1) is also sufficient for the components of NEPS to be almost cospectral.

After refusal of his original conjecture, Cvetković formulated new conjecture which appeared in [6].

Conjecture 2 *Let* $G = \mathrm{NEPS}\,(B_1, B_2, \ldots, B_n; \mathcal{B})$*, where* $B_1, B_2, \ldots, B_n$ *connected, bipartite graphs. If* Λ *is an eigenvalue of* G*, then* Λ *is the eigenvalue of each component of* G*.*

In Section 2 we prove this conjecture. Then in Section 3 we give an upper bound on the multiplicity of an eigenvalue of NEPS as an eigenvalue of a component of NEPS. Finally, in Section 4 we obtain new sufficient condition for the components of NEPS to be almost cospectral and give a characterization of bases which satisfy it. This condition is weaker than condition (1) and the characterization of bases for which the components of NEPS are almost cospectral lies somewhere between the two, which indicates that it will remain the topic of future research.

2 Each eigenvalue of NEPS is an eigenvalue of each factor

Theorem 4 *Let* $G = \mathrm{NEPS}\,(G_1, \ldots, G_n; \mathcal{B})$*, where* $G_1, \ldots, G_n$ *are connected, bipartite graphs. If* Λ *is an eigenvalue of* G*, then* Λ *is an eigenvalue of each component of* G*.*

Proof Let Λ be an arbitrary eigenvalue of G. Denote by T the set of n-tuples $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where λ_i is an eigenvalue of G_i, $i = 1, 2, \ldots, n$, such that

$$\Lambda = \sum_{\beta \in \mathcal{B}} \lambda_1^{\beta_1} \lambda_2^{\beta_2} \cdots \lambda_n^{\beta_n}.$$

Let $(\mu_1, \mu_2, \ldots, \mu_n)$ be an n-tuple of T which has the smallest number of elements equal to 0. W.l.o.g. suppose that $\mu_i \neq 0$ for $i = 1, 2, \ldots, k$, and

$$\mu_{k+1} = \mu_{k+2} = \ldots = \mu_n = 0.$$

Lemma 1 *For each* $i = k+1, k+2, \ldots, n$ *there exists at least one* $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathcal{B}$ *such that* $\beta_i = 1$ *and* $\beta_j = 0$ *for* $j \in \{k+1, k+2, \ldots, n\} \setminus \{i\}$*.*

Proof of Lemma Suppose on the contrary that for each $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathcal{B}$ with $\beta_i = 1$ there exists $j_\beta \in \{k+1, k+2, \ldots, n\} \setminus \{i\}$ such that $\beta_{j_\beta} = 1$. Consider new n-tuple $(\mu_1, \ldots, \mu_k, 0, \ldots, \lambda, \ldots, 0)$, where λ is an arbitrary nonzero eigenvalue of G_i and it stands instead of 0 in i-th coordinate. We have that

$$\sum_{\beta \in \mathcal{B}} \mu_1^{\beta_1} \cdots \mu_k^{\beta_k} 0^{\beta_{k+1}} \cdots \lambda^{\beta_i} \cdots 0^{\beta_n} =$$

$$\sum_{\beta \in \mathcal{B}, \beta_i = 0} \mu_1^{\beta_1} \cdots \mu_k^{\beta_k} 0^{\beta_{k+1}} \cdots \lambda^0 \cdots 0^{\beta_n} + \sum_{\beta \in \mathcal{B}, \beta_i = 1} \mu_1^{\beta_1} \cdots \mu_k^{\beta_k} 0^{\beta_{k+1}} \cdots \lambda^1 \cdots 0^{\beta_n}.$$

Since $\lambda^0 = 0^0 = 1$, it holds that

$$\sum_{\beta \in \mathcal{B}, \beta_i = 0} \mu_1^{\beta_1} \cdots \mu_k^{\beta_k} 0^{\beta_{k+1}} \cdots \lambda^0 \cdots 0^{\beta_n} = \sum_{\beta \in \mathcal{B}, \beta_i = 0} \mu_1^{\beta_1} \cdots \mu_k^{\beta_k} 0^{\beta_{k+1}} \cdots 0^0 \cdots 0^{\beta_n}.$$

Since $\beta_{j_\beta} = 1$ it holds that $0^{\beta_{k+1}} \cdots \lambda^1 \cdots 0^{\beta_n} = 0$, and it follows that

$$\sum_{\beta \in \mathcal{B}, \beta_i = 1} \mu_1^{\beta_1} \cdots \mu_k^{\beta_k} 0^{\beta_{k+1}} \cdots \lambda^1 \cdots 0^{\beta_n} = 0.$$

On the other hand, it is obvious that

$$\sum_{\beta \in \mathcal{B}, \beta_i = 1} \mu_1^{\beta_1} \cdots \mu_k^{\beta_k} 0^{\beta_{k+1}} \cdots 0^1 \cdots 0^{\beta_n} = 0,$$

and we conclude that

$$\sum_{\beta \in \mathcal{B}} \mu_1^{\beta_1} \cdots \mu_k^{\beta_k} 0^{\beta_{k+1}} \cdots \lambda^{\beta_i} \cdots 0^{\beta_n} = \sum_{\beta \in \mathcal{B}} \mu_1^{\beta_1} \cdots \mu_k^{\beta_k} 0^{\beta_{k+1}} \cdots 0^{\beta_i} \cdots 0^{\beta_n} = \Lambda.$$

From this it follows that $(\mu_1, \dots, \mu_k, 0, \dots, \lambda, \dots, 0) \in T$. But this is a contradiction, since it has a smaller number of zero elements than $(\mu_1, \dots, \mu_k, 0, \dots, 0, \dots, 0)$. ■

For $i = k+1, k+2, \dots, n$ let γ^i be an element of $\mathcal{B}$ such that its i-th coordinate is equal to 1 and its j-th coordinate is equal to 0, for $j \in \{k+1, k+2, \dots, n\} \setminus \{i\}$.

Now, let C be an arbitrary component of G and let u be an arbitrary vertex of C. As we already saw, it holds that

$$V(C) = \{v \in V(G) \colon \overrightarrow{bd}(u, v) \in \mathcal{L}(\mathcal{B})\}.$$

Further, for $i = 1, 2, \dots, n$ let x_i be an arbitrary eigenvector of μ_i in G_i. Then

$$X = x_1 \otimes x_2 \otimes \dots \otimes x_n$$

is the eigenvector of Λ in G.

For $i \le k$ it holds that $\mu_i \ne 0$ and since G_i is bipartite graph in each of its parts there exists at least one nonzero coordinate of x_i. Denote by a_i^0 the vertex of G_i that is in the same part as u_i and for which $x_{i,a_i^0} \ne 0$. Denote by a_i^1 the vertex of G_i that is in different part of G_i than u_i and for which $x_{i,a_i^1} \ne 0$.

For $i > k$ it holds that $\mu_i = 0$ and we can only claim that there exists at least one nonzero coordinate of x_i. Denote by a_i the vertex of G_i for which $x_{i,a_i} \ne 0$.

Let $S = \{1, 2, \dots, k\}$. Form the vector δ' with indices in $\{k+1, k+2, \dots, n\}$ such that

$$\delta'_i = \begin{cases} 0, & \text{if } u_i \text{ and } a_i \text{ are in the same part of } G_i \\ 1, & \text{if } u_i \text{ and } a_i \text{ are in distinct parts of } G_i \end{cases}$$

For $\beta \in \mathcal{B}$ denote by β^{-S} the restriction of β to indices not in S. Then we have that

$$\delta' = \sum_{\{i \colon \delta'_i = 1\}} (\gamma^i)^{-S}.$$

Further, let

$$\delta = \sum_{\{i\,:\,\delta'_i=1\}} \gamma^i \in \mathcal{L}(\mathcal{B}).$$

Consider now the vertex v of G for which $v_i = a_i^{\delta_i}$ for $i \leq k$, and $v_i = a_i$ for $i > k$. Vertex v belongs to C, because $\overrightarrow{bd}(u,v) = \delta \in \mathcal{L}(\mathcal{B})$. The component of eigenvector X corresponding to v is

$$X_v = x_{1,a_1^{\delta_1}} \cdots x_{k,a_k^{\delta_k}} \cdot x_{k+1,a_{k+1}} \cdots x_{n,a_n} \neq 0.$$

This shows that the restriction of X on C is nonzero eigenvector of Λ, and subsequently, that Λ is the eigenvalue of C. ■

3 Upper bound on the multiplicities of eigenvalues in components of NEPS

In this section we are interested to bound the multiplicity of each nonzero eigenvalue of NEPS in each component of NEPS. We will first define choices of signs of eigenvalues and consider their equivalence, according to [6].

Let $G = \mathrm{NEPS}(G_1,\ldots,G_n;\mathcal{B})$, where $G_1,\ldots,G_n$ are connected bipartite graphs, and let $\lambda_1,\ldots,\lambda_n$ be the eigenvalues of graphs $G_1,\ldots,G_n$, resp., and let Λ be the corresponding eigenvalue of G represented by

$$\Lambda = \sum_{\beta\in\mathcal{B}} \lambda_1^{\beta_i} \cdots \lambda_n^{\beta_n}. \tag{2}$$

Suppose that

$$S = \{i \in \{1,\ldots,n\} \colon \lambda_i = 0\} \neq \{1,\ldots,n\}$$

and let

$$\mathrm{Ann}(\mathcal{B},S) = \{\beta \in \mathcal{B} \colon (\forall i \in S)\, \beta_i = 0\}.$$

If $S = \emptyset$ then $\mathrm{Ann}(\mathcal{B},\emptyset) = \mathcal{B}$. For $u \in GF_2^n$ define $u^{-S} \in GF_2^{n-|S|}$, with coordinates in the set $\{1,\ldots,n\} - S$, by $u_i^{-S} = u_i$ for $i \notin S$. Now, denote

$$\mathrm{ann}(\mathcal{B},S) = \{\beta^{-S} \colon \beta \in \mathrm{Ann}(\mathcal{B},S)\}.$$

Observe that it holds $r(\mathrm{Ann}(\mathcal{B},S)) = r(\mathrm{ann}(\mathcal{B},S))$.

It is easy to see that if $\beta \notin \mathrm{Ann}(\mathcal{B},S)$, then $\prod_{i=1}^n \lambda_i^{\beta_i} = 0$, so we can write the equation (2) in the form

$$\Lambda = \sum_{\beta\in\mathrm{Ann}(\mathcal{B},S)} \prod_{i\notin S} \lambda_i^{\beta_i} = \sum_{\beta^{-S}\in\mathrm{ann}(\mathcal{B},S)} \prod_{i\notin S} \lambda_i^{\beta_i^{-S}}. \tag{3}$$

Now, if $\mathrm{Ann}(\mathcal{B},S) = \emptyset$ then $\Lambda = 0$. Since we are interested in nonzero eigenvalues, we may suppose that $\mathrm{Ann}(\mathcal{B},S) \neq \emptyset$ (and also $\mathrm{ann}(\mathcal{B},S) \neq \emptyset$).

The graph G_i is bipartite, so for every eigenvalue $\lambda_i > 0$ of G_i it holds that $-\lambda_i$ is also the eigenvalue of G_i with the same multiplicity (see, e.g., Theorem 1.3.14 in [4]), so in (2) we can put either λ_i or $-\lambda_i$. Therefore, we can always choose only nonnegative eigenvalues $\lambda_1, \dots, \lambda_n$ of graphs $G_1, \dots, G_n$, resp., and later choose the signs of eigenvalues λ_i, $i \notin S$. To every choice of signs of eigenvalues λ_i, $i \notin S$, we can assign unique vector $s \in GF_2^{n-|S|}$, with coordinates in the set $\{1, \dots, n\} - S$, defined by

$$s_i = \begin{cases} 0, & \text{if } \lambda_i > 0 \text{ is chosen} \\ 1, & \text{if } -\lambda_i < 0 \text{ is chosen} \end{cases}$$

Observe that, after we choose the signs corresponding to the vector s, we have $(-1)^{s_i}\lambda_i$ instead of λ_i in (3), and therefore

$$\begin{aligned} \Lambda = \sum_{\beta^{-S} \in \operatorname{ann}(\mathcal{B},S)} \prod_{i \notin S} ((-1)^{s_i}\lambda_i)^{\beta_i^{-S}} &= \sum_{\beta^{-S} \in \operatorname{ann}(\mathcal{B},S)} (-1)^{\sum_{i \notin S} s_i \cdot \beta_i^{-S}} \prod_{i \notin S} \lambda_i^{\beta_i^{-S}} \\ &= \sum_{\beta^{-S} \in \operatorname{ann}(\mathcal{B},S)} (-1)^{s \cdot \beta^{-S}} \prod_{i \notin S} \lambda_i^{\beta_i^{-S}}, \end{aligned} \tag{4}$$

where $s \cdot \beta^{-S}$ is the usual inner product of vectors in $GF_2^{n-|S|}$. It is obvious that the choices of signs s_1 and s_2 yield the same eigenvalue Λ if

$$(\forall \beta^{-S} \in \operatorname{ann}(\mathcal{B}, S)) \quad s_1 \cdot \beta^{-S} = s_2 \cdot \beta^{-S}.$$

But then $s_1 - s_2 \in \ker \operatorname{ann}(\mathcal{B}, S)$, and such choices of signs are called *equivalent*. The linear subspace $\ker \operatorname{ann}(\mathcal{B}, S)$ has the dimension $n - |S| - r(\operatorname{ann}(\mathcal{B}, S))$, so the set $s_1 + \ker \operatorname{ann}(\mathcal{B}, S)$ has cardinality $2^{n-|S|-r(\operatorname{ann}(\mathcal{B},S))}$.

Therefore, we have proved the following

Proposition 1 *Let m_{λ_i} be the multiplicity of the eigenvalue λ_i of G_i, $i = 1, 2, \dots, n$. A choice of signs s of these eigenvalues and all the choices of signs equivalent to it (i.e. belonging to $s + \ker \operatorname{ann}(\mathcal{B}, S)$) take part in the multiplicity of the eigenvalue Λ of G with the summand*

$$2^{n-|S|-r(\operatorname{ann}(\mathcal{B},S))} \prod_{i=1}^{n} m_{\lambda_i}. \quad \blacksquare \tag{5}$$

Next we want to bound the multiplicity of Λ in an arbitrary component of G. Let u be an arbitrary vertex of G and let C_u denote the component of G containing u. In [5] it is proved that the set of vertices of C_u is given by

$$V(C_u) = \{v \in V(G) \colon \overrightarrow{bd}(u, v) \in P\}$$

where

$$P = \{\mathcal{B}^{\mathrm{T}} \cdot a \colon a \in GF_2^{|\mathcal{B}|}\} = \mathcal{L}(\mathcal{B}).$$

In order to bound the multiplicity of Λ in C_u we will consider eigenvectors of G corresponding to Λ restricted to the component C_u. Denote the parts of bipartite graph G_i by A_i and B_i in such a way that $u_i \in A_i$. For $i \notin S$, let y_i^0 be the eigenvector of G_i corresponding to λ_i and let y_i^1 be the eigenvector of G_i corresponding to $-\lambda_i$. We know that these vectors may be chosen such that

$$y^1_{i,j} = \begin{cases} y^0_{i,j} & \text{if } j \in A_i \\ -y^0_{i,j} & \text{if } j \in B_i \end{cases} \tag{6}$$

For $i \in S$, let y_i be the eigenvector of G_i corresponding to $\lambda_i = 0$.

W.l.o.g. suppose that $S = \{k+1, k+2, \dots, n\}$. Given the choice of signs s, the eigenvector $x^{(s)}$ of G corresponding to Λ is given by

$$x^{(s)} = y_1^{s_1} \otimes \cdots \otimes y_k^{s_k} \otimes y_{k+1} \otimes \cdots \otimes y_n.$$

We want to find out when two different choices of signs s and t, giving the same eigenvalue Λ ($t \in s + \ker \operatorname{ann}(\mathcal{B}, S)$), do also give the same eigenvector $x^{(s)} = x^{(t)}$?

For a vertex v of C_u, the component $x_v^{(s)}$ of $x^{(s)}$ corresponding to the vertex v and the choice of signs s is equal to

$$x_v^{(s)} = y^{s_1}_{1,v_1} \cdots y^{s_k}_{k,v_k} \cdot y_{k+1,v_{k+1}} \cdots y_{n,v_n}.$$

From (6) it follows that $y^1_{i,v_i} = (-1)^{(\overrightarrow{bd}(u_i,v_i))_i} \cdot y^0_{i,v_i}$ and we can write $x_v^{(s)}$ as

$$\begin{aligned} x_v^{(s)} &= \prod_{s_i=1} (-1)^{(\overrightarrow{bd}(u_i,v_i))_i} \cdot \prod_{i=1}^{k} y^0_{i,v_i} \cdot \prod_{j=k+1}^{n} y_{j,v_j} \\ &= \prod_{i=1}^{k} (-1)^{s_i \cdot (\overrightarrow{bd}(u_i,v_i))_i} \cdot \prod_{i=1}^{k} y^0_{i,v_i} \cdot \prod_{j=k+1}^{n} y_{j,v_j} \\ &= (-1)^{\sum_{i=1}^{k} s_i \cdot (\overrightarrow{bd}(u_i,v_i))_i} \cdot \prod_{i=1}^{k} y^0_{i,v_i} \cdot \prod_{j=k+1}^{n} y_{j,v_j} \\ &= (-1)^{s \cdot \overrightarrow{bd}^{-S}(u,v)} \cdot \prod_{i=1}^{k} y^0_{i,v_i} \cdot \prod_{j=k+1}^{n} y_{j,v_j}. \end{aligned}$$

Notice how we transformed $\prod_{s_i=1}(-1)^{(\overrightarrow{bd}(u_i,v_i))_i}$ into $\prod_{i=1}^{k}(-1)^{s_i \cdot (\overrightarrow{bd}(u_i,v_i))_i}$.

The component $x_v^{(t)}$ of $x^{(t)}$ corresponding to the vertex v and the choice of signs t is equal to

$$x_v^{(t)} = (-1)^{t \cdot \overrightarrow{bd}^{-S}(u,v)} \cdot \prod_{i=1}^{k} y^0_{i,v_i} \cdot \prod_{j=k+1}^{n} y_{j,v_j},$$

and we see that eigenvectors $x^{(s)}$ and $x^{(t)}$ coincide for all vertices of C_u if and only if

$$\begin{aligned} & s \cdot \overrightarrow{bd}^{-S}(u,v) = t \cdot \overrightarrow{bd}^{-S}(u,v) \text{ for all } v \in V(C_u) \\ \Leftrightarrow\ & s \cdot \alpha^{-S} = t \cdot \alpha^{-S} \text{ for all } \alpha \in P = \mathcal{L}(\mathcal{B}) \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow\ s\cdot\alpha^{-S}=t\cdot\alpha^{-S} \text{ for all } \alpha\in\mathcal{B}\\
&\Leftrightarrow\ s\cdot\beta=t\cdot\beta \text{ for all } \beta\in\mathcal{B}^{-S}\\
&\Leftrightarrow\ t\in s+\ker\mathcal{B}^{-S}.
\end{aligned}$$

For $\beta\in P^{-S}=\mathcal{L}(\mathcal{B}^{-S})$, let

$$C_{u,\beta}=\{v\in C_u\colon bd^{-S}(u,v)=\beta\}.$$

The components of $x^{(t)}$ corresponding to the vertices from $C_{u,\beta}$ share the same sign $(-1)^{t\cdot\beta}$ and the signs can be represented in the matrix, denoted by $A(\mathcal{B},S,s)$, as follows:

$$\begin{array}{cc}
 & s+\ker\operatorname{ann}(\mathcal{B},S)/\ker\mathcal{B}^{-S} \\
\mathcal{L}(\mathcal{B}^{-S}) & \overbrace{\begin{array}{cccc} s_1+\ker\mathcal{B}^{-S} & s_2+\ker\mathcal{B}^{-S} & s_3+\ker\mathcal{B}^{-S} & \dots \end{array}} \\
\begin{array}{c}\beta_1\\ \beta_2\\ \beta_3\\ \vdots\end{array} &
\left[\begin{array}{cccc}
(-1)^{s_1\cdot\beta_1} & (-1)^{s_2\cdot\beta_1} & (-1)^{s_3\cdot\beta_1} & \dots\\
(-1)^{s_1\cdot\beta_2} & (-1)^{s_2\cdot\beta_2} & (-1)^{s_3\cdot\beta_2} & \dots\\
(-1)^{s_1\cdot\beta_3} & (-1)^{s_2\cdot\beta_3} & (-1)^{s_3\cdot\beta_3} & \dots\\
\vdots & \vdots & \vdots & \ddots
\end{array}\right]
\end{array}$$

The component $x_v^{(t)}$ is then obtained by multiplying the sign $(-1)^{t\cdot\overrightarrow{bd}^{-S}(u,v)}$ with $\prod_{i=1}^{k}y_{i,v_i}^{0}\cdot\prod_{j=k+1}^{n}y_{j,v_j}$.

The multiplicity of Λ in C_u, obtained from one set $s+\ker\operatorname{ann}(\mathcal{B},S)$ of equivalent choices of signs and one choice of eigenvectors of λ_i, is thus equal to the number of linearly independent columns in the above matrix $A(\mathcal{B},S,s)$, i.e. to the rank r^* of $A(\mathcal{B},S,s)$ in R. Notice that the matrix $A(\mathcal{B},S,s)$ does not depend on the component C_u, so that we get the same matrix and the same rank whatever the component of G we start with.

However, this does not imply that the components are almost cospectral. The point is that the eigenvectors of Λ are obtained by multiplying the sign from $A(\mathcal{B},S,s)$ with $\prod_{i=1}^{k}y_{i,v_i}^{0}\cdot\prod_{j=k+1}^{n}y_{j,v_j}$. In general, if we are given another choice of eigenvectors $z_1^0,\dots,z_k^0$ of nonzero eigenvalues $\lambda_1,\dots,\lambda_k$, resp., and eigenvectors $z_{k+1},\dots,z_n$ of zero eigenvalues $\lambda_{k+1},\dots,\lambda_n$, it may happen that for each vertex v of some component holds $\prod_{i=1}^{k}y_{i,v_i}^{0}\cdot\prod_{j=k+1}^{n}y_{j,v_j}=\prod_{i=1}^{k}z_{i,v_i}^{0}\cdot\prod_{j=k+1}^{n}z_{j,v_j}$, so that we get the same eigenvectors of Λ, starting with different choice of eigenvectors of λ_i.

Anyway, we can still bound the multiplicity of Λ in C_u. The columns of $A(\mathcal{B},S,s)$ correspond to the cosets of $s+\ker\operatorname{ann}(\mathcal{B},S)/\ker\mathcal{B}^{-S}$, and the number of columns is thus equal to $2^{r(\mathcal{B}^{-S})-r(\operatorname{ann}(\mathcal{B},S))}$. From this we get that $r^*\le 2^{r(\mathcal{B}^{-S})-r(\operatorname{ann}(\mathcal{B},S))}$ and so we proved

Theorem 5 *Let $G=\mathrm{NEPS}(G_1,\dots,G_n;\mathcal{B})$, where $G_1,\dots,G_n$ are connected bipartite graphs. For each $i=1,2,\dots,n$, let λ_i be a nonnegative eigenvalue of G_i with multiplicity m_{λ_i} such that*

$$S=\{i\in\{1,\dots,n\}\colon\lambda_i=0\}\neq\{1,\dots,n\}$$

and

$$\operatorname{ann}(\mathcal{B}, S) \neq \emptyset.$$

Further, let $s \in GF_2^{n-|S|}$ be a choice of signs of nonzero eigenvalues, and let Λ be an eigenvalue of G given by

$$\Lambda = \sum_{\beta^{-S} \in \operatorname{ann}(\mathcal{B},S)} (-1)^{s \cdot \beta^{-S}} \prod_{i \notin S} \lambda_i^{\beta_i^{-S}}.$$

If C is an arbitrary component of G, then the choice of eigenvalues λ_i and the choice of signs s take part in the multiplicity of Λ as an eigenvalue of C with value at most $2^{r(\mathcal{B}^{-S}) - r(\operatorname{ann}(\mathcal{B},S))} \cdot \prod_{i=1}^{n} m_{\lambda_i}$.

4 Sufficient condition for the almost cospectrality of components of NEPS

The main result of this section is the new sufficient condition for the almost cospectrality of components of NEPS.

Theorem 6 *Let $G = \operatorname{NEPS}(G_1, \ldots, G_n; \mathcal{B})$, where $G_1, \ldots, G_n$ are connected bipartite graphs. If for each $S \subset \{1, 2, \ldots, n\}$, such that $\operatorname{ann}(\mathcal{B}, S) \neq \emptyset$, holds*

$$r(\mathcal{B}^{-S}) = r(\mathcal{B}) - |S|, \tag{7}$$

then the components of G are almost cospectral.

Proof For each $i = 1, 2, \ldots, n$, let λ_i be a nonnegative eigenvalue of G_i with multiplicity m_{λ_i} such that

$$S = \{i \in \{1, \ldots, n\} : \lambda_i = 0\} \neq \{1, \ldots, n\}$$

and

$$\operatorname{ann}(\mathcal{B}, S) \neq \emptyset.$$

Further, let $s \in GF_2^{n-|S|}$ be a choice of signs of nonzero eigenvalues and let Λ be an eigenvalue of G given by

$$\Lambda = \sum_{\beta^{-S} \in \operatorname{ann}(\mathcal{B},S)} (-1)^{s \cdot \beta^{-S}} \prod_{i \notin S} \lambda_i^{\beta_i^{-S}}.$$

We have seen from Proposition 1 that the multiplicity of Λ obtained from one choice of eigenvalues $\lambda_1, \ldots, \lambda_n$ and one set of equivalent choices of signs $s + \ker \operatorname{ann}(\mathcal{B}, S)$ is equal to

$$2^{n-|S|-r(\operatorname{ann}(\mathcal{B},S))} \prod_{i=1}^{n} m_{\lambda_i}. \tag{8}$$

On the other hand, the number of components is equal to $2^{n-r(\mathcal{B})}$, and from Theorem 5 the multiplicity of Λ in each component is bounded from above by $2^{r(\mathcal{B}^{-S}) - r(\operatorname{ann}(\mathcal{B},S))} \cdot \prod_{i=1}^{n} m_{\lambda_i}$.

Therefore, the total multiplicity of Λ in G obtained from one choice of eigenvalues $\lambda_1, \ldots, \lambda_n$ and one set of equivalent choices of signs $s + \ker \operatorname{ann}(\mathcal{B}, S)$ is bounded from above by

$$2^{n-r(\mathcal{B})} \cdot 2^{r(\mathcal{B}^{-S})-r(\operatorname{ann}(\mathcal{B},S))} \cdot \prod_{i=1}^{n} m_{\lambda_i} = 2^{n-r(\mathcal{B})} \cdot 2^{r(\mathcal{B})-|S|-r(\operatorname{ann}(\mathcal{B},S))} \cdot \prod_{i=1}^{n} m_{\lambda_i}$$

$$= 2^{n-|S|-r(\operatorname{ann}(\mathcal{B},S))} \cdot \prod_{i=1}^{n} m_{\lambda_i}. \tag{9}$$

This equality between the actual value (8) and the upper bound (9) shows that in every component of G all $2^{r(\mathcal{B}^{-S})-r(\operatorname{ann}(\mathcal{B},S))} \cdot \prod_{i=1}^{n} m_{\lambda_i}$ eigenvectors of Λ, corresponding to all possible choices of eigenvectors of eigenvalues $\lambda_1, \ldots, \lambda_n$, must be linearly independent. Therefore, the multiplicity of Λ in every component of G, obtained from one choice of eigenvalues $\lambda_1, \ldots, \lambda_n$ and one set of equivalent choices of signs $s + \ker \operatorname{ann}(\mathcal{B}, S)$, is equal to $2^{r(\mathcal{B}^{-S})-r(\operatorname{ann}(\mathcal{B},S))} \prod_{i=1}^{n} m_{\lambda_i}$, and we conclude that the components of G are almost cospectral. ■

The sufficient condition $r(\mathcal{B}^{-S}) = r(\mathcal{B}) - |S|$ from Theorem 6 is stronger than necessary condition $r(\operatorname{ann}(\mathcal{B}, S)) \le r(\mathcal{B}) - |S|$ from [6], because $\operatorname{ann}(\mathcal{B}, S)$ is submatrix of $\mathcal{B}^{-S}$ and therefore $r(\operatorname{ann}(\mathcal{B}, S)) \le r(\mathcal{B}^{-S})$.

Next we characterize bases $\mathcal{B}$ which satisfy condition $r(\mathcal{B}^{-S}) = r(\mathcal{B}) - |S|$ for each $S \subset \{1, 2, \ldots, n\}$ for which $\operatorname{ann}(\mathcal{B}, S) \neq \emptyset$.

Theorem 7 *The $0-1$ matrix $\mathcal{B}$ with n columns satisfies*

$$r(\mathcal{B}^{-S}) = r(\mathcal{B}) - |S| \tag{10}$$

for each $S \subset \{1, 2, \ldots, n\}$ for which $\operatorname{ann}(\mathcal{B}, S) \neq \emptyset$ if and only if it consists of $n - r(\mathcal{B})$ columns equal to all-1 vector and $r(\mathcal{B})$ linearly independent columns $c_1, c_2, \ldots, c_{r(\mathcal{B})}$ forming the submatrix $\mathcal{B}'$ such that for each $S \subset \{c_1, c_2, \ldots, c_{r(\mathcal{B})}\}$ for which $\operatorname{ann}(\mathcal{B}', S) \neq \emptyset$ it holds that the linear span of columns from $\{c_1, c_2, \ldots, c_{r(\mathcal{B})}\} \setminus S$ contains the all-1 vector.

Proof Suppose that matrix $\mathcal{B}$ satisfies (10). Let $r(\mathcal{B}) = k$ and let $c_1, c_2, \ldots, c_k$ be linearly independent columns. If there exists 0 in column j, such that $j \neq c_i$ for $i = 1, 2, \ldots, k$, then for $S = \{j\}$ it must hold $r(\mathcal{B}^{-S}) = k - 1$, which is impossible, since $\mathcal{B}^{-S}$ contains linearly independent columns $c_1, c_2, \ldots, c_k$. Therefore, $n - k$ columns different than $c_1, c_2, \ldots, c_k$ are equal to all-1 vector.

Let $S \subset \{c_1, c_2, \ldots, c_k\}$ such that $\operatorname{ann}(\mathcal{B}', S) \neq \emptyset$. The matrix $\mathcal{B}^{-S}$ contains $k - |S|$ linearly independent columns from $\{c_1, c_2, \ldots, c_k\} \setminus S$ and the remaining $n-k$ columns equal to all-1 vector. From (10) follows that $r(\mathcal{B}^{-S}) = k - |S|$ and we conclude that the linear span of columns from $\{c_1, c_2, \ldots, c_k\} \setminus S$ contains the all-1 vector.

On the other hand, suppose that $\mathcal{B}$ consists of $n - r(\mathcal{B})$ columns equal to all-1 vector and $r(\mathcal{B})$ linearly independent columns $c_1, c_2, \ldots, c_{r(\mathcal{B})}$ forming the submatrix $\mathcal{B}'$ such that for each $S \subset \{c_1, c_2, \ldots, c_{r(\mathcal{B})}\}$ for which $\operatorname{ann}(\mathcal{B}', S) \neq \emptyset$ it holds that the linear span of columns from $\{c_1, c_2, \ldots, c_{r(\mathcal{B})}\} \setminus S$ contains the all-1 vector.

In this case, $\text{ann}(\mathcal{B}, S) \neq \emptyset$ only if $S \subset \{c_1, c_2, \ldots, c_{r(\mathcal{B})}\}$. But then the linear span of columns from $\{c_1, c_2, \ldots, c_{r(\mathcal{B})}\} \setminus S$ contains the all-1 vector, and we immediately have that $r(\mathcal{B}^{-S}) = r(\mathcal{B}) - |S|$. ■

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