DIMACS Technical Report 2001-40
October 2001

Characterization of superbipartite graphs

by

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[1]Supported by DIMACS Summer Research Salary 2001.

DIMACS is a partnership of Rutgers University, Princeton University, AT&T Labs-Research, Bell Labs, NEC Research Institute and Telcordia Technologies (formerly Bellcore).

DIMACS was founded as an NSF Science and Technology Center, and also receives support from the New Jersey Commission on Science and Technology.

ABSTRACT

Let G and H be graphs. A *substitution* of H in G instead of a vertex $v \in V(G)$ is the graph $G(v \to H)$, which consists of disjoint union of H and $G - v$ with the additional edge-set $\{xy : x \in V(H), y \in N_G(v)\}$.
For a hereditary class of graphs $\mathcal{P}$, the *substitutional closure* of $\mathcal{P}$ is defined as the class $\mathcal{P}^*$ consisting of all graphs which can be obtained from graphs in P by repeated substitutions.
Let $\mathcal{B}$ be the class of all graphs G such that $G - N[v]$ is a bipartite graph for every vertex v of G. Here $N[v]$ denotes the closed neighborhood of v. A graph is called a *superbipartite graph* if it is in the substitutional closure of $\mathcal{B}$.
We characterize superbipartite graphs in terms of forbidden induced subgraphs. Note that the weighted stability number in $\mathcal{B}^*$ can be found in polynomial time.

1 Introduction

The *neighborhood* of a vertex $x \in V(G)$ is the set $N_G(x) = N(x)$ of all vertices in G that adjacent to x; $N[x] = \{x\} \cup N(x)$.

Definition 1. *Let G and H be graphs. A* substitution of H in G instead of a vertex *$v \in V(G)$ is the graph $G(v \to H)$ consisting of disjoint union of H and $G - v$ with the additional edge-set $\{xy : x \in V(H), y \in N_G(v)\}$.*

Definition 2. *For a class $\mathcal{P}$ of graphs, its* substitutional closure $\mathcal{P}^*$ *consists of all graphs that can be obtained form $\mathcal{P}$ by repeated substitutions, i.e., $\mathcal{P}^*$ is generated by the following rules:*

(S1) *$\mathcal{P} \subseteq \mathcal{P}^*$;*

(S2) *if $G, H \in \mathcal{P}^*$ and $v \in V(G)$, then $G(v \to H) \in \mathcal{P}^*$.*

Let $\mathrm{ISub}(G)$ be the set of all induced subgraphs of a graph G (considered up to isomorphism). A class of graphs $\mathcal{P}$ is called *hereditary* if $\mathrm{ISub}(G) \subseteq \mathcal{P}$ for every $G \in \mathcal{P}$.

For a set of graphs Z, the class of *Z-free* graphs consists of all graphs G such that $\mathrm{ISub}(G) \cap Z = \emptyset$.

Proposition 1. *If $\mathcal{P}$ is a hereditary class then $\mathcal{P}^*$ is also a hereditary class.*

Open Problem 1. *For a hereditary class $\mathcal{P}$ given by a set Z of forbidden induced subgraphs, find a forbidden induced subgraph characterization of the substitutional closure $\mathcal{P}^*$.*

Bertolazzi, De Simone, and Galluccio [1], [2] note that this problem is especially interesting in the case where $\mathcal{P}$ is a good class for the vertex packing problem, i.e., the weighted stability number can be found in polynomial time for all graphs in $\mathcal{P}$.

Definition 3. *Let $\mathcal{B}$ be the class of all graphs G such that $G - N[v]$ is a bipartite graph for every vertex v of G. A graph is called a* superbipartite graph *if it is in the substitutional closure of $\mathcal{B}$.*

We characterize superbipartite graphs in terms of forbidden induced subgraphs. For this, we need to describe Reducing Copath Method proposed by I. Zverovich [4] for solving Problem 1

Definition 4. *A set $W \subseteq V(G)$ is called* homogeneous *in a graph G if*

(H1) $2 \leq |W| \leq |V(G)| - 1$, *and*

(H2) *$N(x) \backslash W = N(y) \backslash W$ for each $x, y \in W$.*

According to (H2), a homogeneous set W defines a partition

$$W \cup W^+ \cup W^- = V(G)$$

such that

- every vertex of W is adjacent to every vertex of W^+ (notation $W \sim W^+$), and
- every vertex of W is non-adjacent to every vertex of W^- (notation $W \not\sim W^-$).

By (H1), $W^+ \cup W^- \neq \emptyset$ for every homogeneous set W.

Definition 5. *A graph without homogeneous sets is called* prime*.*
A graph H is called a (primal) extension *of a graph G if*

(E1) *G is an induced subgraph of H, and*

(E2) *H is a prime graph.*

Definition 6. *An extension H of G is* minimal *if there are no extensions of G in the set* $\mathrm{ISub}(H)\backslash\{H\}$*.*
We denote by $\mathrm{Ext}(G)$ *the set of all minimal extensions of a graph G.*

Definition 7. *For a set of graphs Z we put*

$$\mathrm{Ext}(Z) = \bigcup_{G \in Z} \mathrm{Ext}(G),$$

and we define Z^o as the set of all minimal graphs in $\mathrm{Ext}(Z)$ *with respect to the partial order 'to be an induced subgraph'.*

Theorem 1. *If Z is the set of all minimal forbidden induced subgraphs for a hereditary class $\mathcal{P}$ then Z^o is the set of all minimal forbidden induced subgraphs for $\mathcal{P}^*$.*

The notation $x \sim y$ (respectively, $x \not\sim y$) means that x and y are adjacent (respectively, non-adjacent). For disjoint sets X and Y, the notation $X \sim Y$ (respectively, $X \not\sim Y$) means that every vertex of X is adjacent to (respectively, non-adjacent) to every vertex of Y. In case of $X = \{x\}$ we also will write $x \sim Y$ and $x \not\sim Y$ instead of $\{x\} \sim Y$ and $\{x\} \not\sim Y$, respectively.

Definition 8. *Let G be an induced subgraph of a graph H, and let W be a homogeneous set of G.*
We define a reducing W-copath (with respect to G) *in H as a sequence*

$$R = (u_1, u_2, \ldots, u_t), t \geq 1,$$

of pairwise distinct vertices of $V(H)\backslash V(G)$ satisfying the following conditions:

(R1) *there exist vertices* $w_1, w_2 \in W$ *such that*

(R1a) $u_1 \sim w_1$, *and*

(R1b) $u_1 \not\sim w_2$;

(R2) *for each* $i = 2, 3, \ldots, t$ *either*

(R2a) $u_i \sim u_{i-1}$ *and* $u_i \not\sim W \cup \{u_1, u_2, \ldots, u_{i-2}\}$, *or*

(R2b) $u_i \not\sim u_{i-1}$ *and* $u_i \sim W \cup \{u_1, u_2, \ldots, u_{i-2}\}$
[when $i = 2$, $\{u_1, u_2, \ldots, u_{i-2}\} = \emptyset$];

(R3) *for every* $i = 1, 2, \ldots, t-1$

(R3a) $u_i \sim W^+$, *and*

(R3b) $u_i \not\sim W^-$;

(R4) *either*

(R4a) $u_t \not\sim x$ *for a vertex* $x \in W^+$, *or*

(R4b) $u_t \sim y$ *for a vertex* $y \in W^-$.

Theorem 2 (Zverovich [4]). *Let* H *be an extension of its induced subgraph* G, *and let* W *be a homogeneous set of* G. *Then there exists a reducing* W*-copath (with respect to* G*) in* H.

2 Characterization of superbipartite graphs

As usual, C_n (respectively, K_n) is the cycle (respectively, the complete graph) of order n.

Theorem 3. *A graph is superbipartite if and only if it has no induced subgraphs* $G_1, G_2, \ldots, G_{12}$ *and* X_n, Y_n, Z_n $(n = 1, 2, \ldots)$ *(Figure 1).*

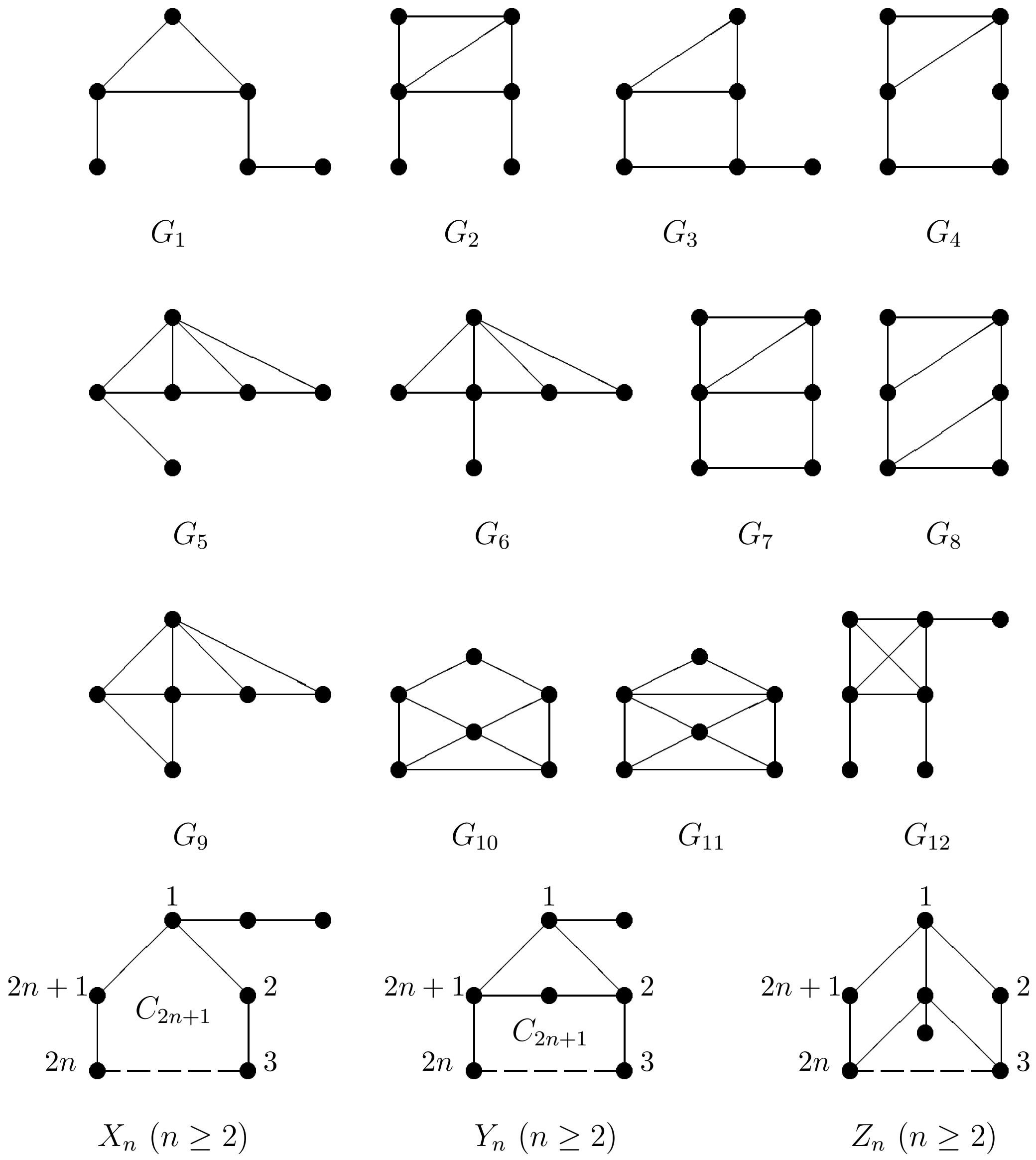

Figure 1.

Proof. We recall that $\mathcal{B}$ is the class of all graphs G such that $G - N[v]$ is a bipartite graph for every vertex v of G. It is obvious that $\mathcal{Z} = \{K_1 \cup C_{2n+1} : n \geq 1\}$ is the set of all minimal forbidden induced subgraphs for $\mathcal{B}$.

Let W be a homogeneous set of a graph G. We denote by $\mathcal{H}_t(G, W)$ the set of all graphs that can be obtained from G by adding a reducing W-copath of length t.

Claim 1. *Let W be a homogeneous set of a graph G. If $W^+ = \emptyset$ and $W^- = \{w\}$ then $\mathcal{H}(G,W)$ is reducible to $\mathcal{H}_1(G,W) \cup \mathcal{H}^b(G,W)$, where $\mathcal{H}^b(G,W)$ consists of all reducing W-copaths $R = (u_1, u_2, \dots, u_t)$ such that $t \geq 2$ and each vertex u_i $(i = 2, 3, \dots, t)$ satisfies (R2b).*

Proof. Let H be a graph that obtained from G by adding a reducing W-copath $R = (u_1, u_2, \dots, u_t)$. If the statement is not true, then $t \geq 2$ and there is a vertex u_i satisfying (R2a). We will assume that i is the minimal number in $\{2, 3, \dots, t\}$ such that u_i satisfies (R2a).

By (R2a), $u_i \not\sim W$ and $u_i \sim u_{i-1}$. The set $W \cup \{u_i\}$ induces a subgraph G' isomorphic to G (with u_i instead of w). Moreover, W is a homogeneous set in G'. It follows from $u_i \sim u_{i-1}$ and (R4) that $(u_1, u_2, \dots, u_{i-1})$ is a reducing W-copath with respect to G'. □

First, we construct the set $\mathrm{Ext}(G_3)$, where $G_3 \cong K_1 \cup C_3$. Let G_3 consists of a triangle (v_1, v_2, v_3) and an isolated vertex w. The set $W = \{v_1, v_2, v_3\}$ is homogeneous in G_3.

We consider a reducing W-copath $R = (u_1, u_2, \dots, u_t)$. By Claim 1, we may assume that either $t = 1$ or $t \geq 2$ and all the vertices $u_1, u_2, \dots, u_t$ satisfy (R2b).

If $t = 1$ the we obtain graphs A_1 and A_2 shown in Figure 2.

Suppose that $t \geq 2$. By (R2b) and (R1), u_{t-1} is adjacent to at least one vertex of W, say $u_{t-1} \sim v_1$. If $u_{t-1} \not\sim \{v_2, v_3\}$ then the set $V(G_3) \cup \{u_{t-1}, u_t\}$ induces B (Figure 2). Otherwise we may assume that u_{t-1} is adjacent to v_2. Then the set $\{u_{t-1}, u_t, v_1, v_2, w\}$ induces A_2 (Figure 2).

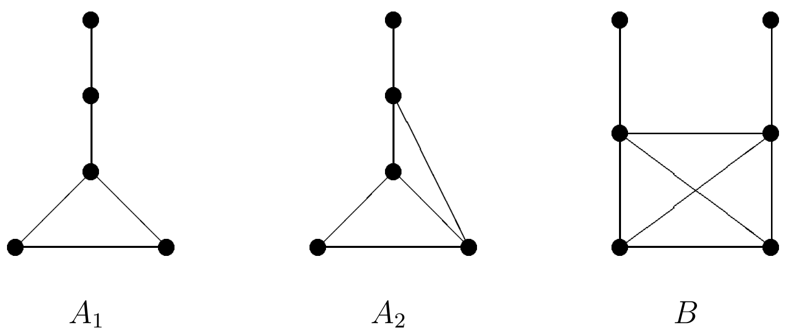

Figure 2.

Each of A_1, A_2, B has a unique homogeneous set W' of cardinality two. As it was shown by Zverovich and Zverovich [5], in that case it is sufficient to construct the set of all reducing W'-copath of length 1. This simple and straightforward procedure gives the following results:

- A_1 produces $G_1, G_3, G_4, G_5, G_7, G_8$ and G_9;
- A_2 produces $G_2, G_5, G_6, G_7, G_9, G_{10}$ and G_{11};
- B produces G_{12} or graphs that contain at least one of A_1, A_2 as an induced subgraphs.

Now we consider the graph $G_n \cong K_1 \cup C_{2n+1}$, where $n \geq 2$. Specifically, G_n consists of an induced cycle $C = (v_1, v_2, \ldots, v_n)$ and an isolated vertex w.

It is sufficient to construct those graphs in $\mathrm{Ext}(G_n)$ that do not contain any G_m ($m = 3, 4, \ldots, n-1$) as an induced subgraph.

The unique homogeneous set of G_n is $W = V(C)$. We have $W^+ = \emptyset$ and $W^- = \{w\}$. By Claim 1, we need to consider the set $\mathcal{H}_1(G_n, W) \cup \mathcal{H}^b(G_n, W)$ only.

Let $R = (u_1, u_2, \ldots, u_t) \in \mathcal{H}_1(G_n, W) \cup \mathcal{H}^b(G_n, W)$

Claim 2. *The vertex u_1 can not be adjacent to three successive vertices of C.*

Proof. Suppose that $u_1 \sim \{v_1, v_2, v_3\}$. We consider triangles $T_1 = (u_1, v_1, v_2)$ and $T_2 = (u_1, v_2, v_3)$. The set $T_1 \cup \{v_j\}$ does not induce $K_1 \cup C_3$. Therefore $u_1 \sim \{v_4, v_5, \ldots, v_{2n}\}$. Similarly, $u_1 \sim \{v_5, v_6, \ldots, v_{2n+1}, v_1\}$. Thus, $u_1 \sim V(C)$, a contradiction to (R1b). □

Case 1 $t = 1$.

By (R1a), u_1 is adjacent to some vertex of C. The condition (R4) implies that $u_1 \sim w$. If $|N(u_1) \cap V(C)| = 1$ then $V(G_n) \cup \{u_1\}$ induces X_n and the proof is finished.

Let $|N(u_1) \cap V(C)| \geq 2$. It is easy to see that u_1 and some subpath of C constitute an induced odd cycle. Without loss of generality we may assume that $D = (u_1, v_1, v_2, \ldots, v_{2k})$ ($1 \leq k \leq n$) is the shortest such cycle.

Subcase 1.1 $k = n$.

We have $u_1 \sim \{v_1, v_{2n}\}$ and $u_1 \not\sim \{v_2, v_3, \ldots, v_{2n-1}\}$. By Claim 2, $u_1 \not\sim v_{2n+1}$. Thus, $V(G_n) \cup \{u_1\}$ induces Y_n, and we are done.

Subcase 1.2 $k \leq n - 1$.

Let $m = |V(D)|$. We have $m < |V(C)|$. By the assumption, the set $V(D) \cup \{v_j\}$ does not induce G_m. It follows that $u_1 \sim U$, where $U = \{v_{2k+2}, v_{2k+3}, \ldots, v_{2n}\}$. By Claim 2, $|U| \leq 2$, or $2n - 2k - 1 \leq 2$, or $k \geq n - 1$. Thus, $k = n - 1$.

We have $u_1 \sim \{v_1, v_{2n-2}, v_{2n}\}$ and $u_1 \not\sim \{v_2, v_3, \ldots, v_{2n-3}\}$. By Claim 2, $u_1 \not\sim \{v_{2n-1}, v_{2n+1}\}$. Thus, $V(G_n) \cup \{u_1\}$ induces Z_n, and we are done.

Case 2 $R \in \mathcal{H}^b$.

We have $t \geq 2$. Suppose that that $t \geq 3$. By (R2b), $u_2 \sim V(C)$. By (R3b), $u_2 \not\sim w$. Thus, the set $\{u_2, v_1, v_2, w\}$ induces $G_3 = K_1 \cup C_3$, a contradiction.

Now $t = 2$. By (R3), $u_1 \not\sim w$ and $u_2 \sim w$. If u_1 and a subpath of C constitute an induced odd cycle C' of length $m < n$, then $V(C') \cup \{w\}$ induces G_m, a contradiction. Otherwise we have $|N(u_1) \cap V(C)| \leq 2$. Since C has $2n + 1 \geq 5$ edges, there is an edge e of C (say, $e = v_1 v_2$) such that $u_1 \not\sim \{v_1, v_2\}$.

According to (R2b), $u_2 \not\sim u_1$ and $u_2 \sim V(C)$. The set $\{u_1, u_2, v_1, v_2\}$ induces $G_3 = K_1 \cup C_3$, a contradiction. $\square$

References

[1] P. Bertolazzi, C. De Simone, and A. Galluccio, A nice class for the vertex packing problem, Discrete Appl. Math. **76** (1-3) (1997) 3–19

[2] C. De Simone, On the vertex packing problem, Graphs and Combin. **9** (1) (1993) 19–30

[3] S. Olariu, On the closure of triangle-free graphs under substitution, Inform. Process. Lett. **34** (2) (1990) 97–101

[4] I. E. Zverovich, Extension of hereditary classes with substitutions, Discrete Appl. Math. (to appear); available as RUTCOR Research Report 14-2001 (Rutgers University, New Brunswick, 2001) http://rutcor.rutgers.edu/∼rrr/

[5] I. E. Zverovich and I. I. Zverovich, The reducing copath method for simple homogeneous sets, RUTCOR Research Report 19-2001 (Rutgers University, New Brunswick, 2001) http://rutcor.rutgers.edu/∼rrr/