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On a Conjecture of Loebl

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ABSTRACT

Martin Loebl conjectured that if in a graph G on n vertices at least half the vertices have degree at least $n/2$, then G contains, as subgraphs, all trees with at most $n/2$ edges. We prove the conjecture for sufficient large n.

1. Introduction

In the paper on discrepancy of trees [1] written by Erdős, Füredi, Loebl and Sós, Loebl gave the following conjecture:

Conjecture 1 (Loebl). *If G is a graph on n vertices, and at least $n/2$ vertices have degrees at least $n/2$, then G contains, as subgraphs, all trees with at most $n/2$ edges.*

This conjecture has a strong similarity to $k = n/2$ case of the celebrated Erdős and Sós Conjecture: Every graph on n vertices and more than $(k-1)n/2$ edges contains, as subgraphs, all trees with k edges.

Ajtai, Komlós and Szemerédi proved the following approximate version, which was sufficient for the application in [1].

Theorem 1 (Ajtai, Komlós and Szemerédi). *For every $\varepsilon > 0$ there is a threshold $n_0 = n_0(\varepsilon)$ such that the following statement holds for all $n \geq n_0$: If G is a graph on n vertices, and at least $(1+\varepsilon)n/2$ vertices have degrees at least $(1+\varepsilon)n/2$, then G contains, as subgraphs, all trees with at most $n/2$ edges.*

In this paper, we will prove Conjecture 1 for sufficient large n. Some techniques used in this paper could be applied even for small n. We hope these lead to a final settlement of the conjecture.

Theorem 2. *There is a threshold n_0 such that the following statement holds for all $n \geq n_0$: If G is a graph on $2n$ vertices, and at least n vertices have degrees at least n, then G contains, as subgraphs, all trees with at most n edges.*

Theorem 4 gives an immediate estimate on *Ramsey number* of trees. In general, $R(H)$, the Ramsey number of a graph H, denotes the minimum integer n such that a monochromatic copy of H can be found in any 2-coloring of the edges of K_n.

Corollary 3. *$R(T_n) \leq 2n-2$, where T_n denotes a tree on n vertices.*

For convenience, we will prove an equivalent theorem:

Theorem 4 (Main Theorem). *There is a threshold n_0 such that the following statement holds for all $n \geq n_0$: If G is a graph on $2n$ vertices, and at least n vertices have degrees at least $n-1$, then G contains, as subgraphs, all trees with at most $n-1$ edges.*

It is easy to see that Theorem 4 is best possible: Suppose a graph G of $2n$ vertices is made up of two isolated isomorphic subgraphs G_1, G_2, in which $V(G_i) = A_i \cup B_i, i = 1, 2$, $|A_i| = n/2 - 1$ and $|B_i| = n/2 + 1$. There are all possible edges within A_i and between A_i and B_i. Thus G has $n-2$ vertices of degree $n-1$ but does not contain a path of n vertices.

2. Notations and main tools

$V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G, and we write $v(G) = |V(G)|$ (order of G) and $e(G) = |E(G)|$ (size of G). G_n denotes n-graphs, that is graphs of order n. $N(v, X)$ is the set of neighbors of $v \in V$ in the set $X \subset V$. We use $N(v)$ to denote $N(v, V)$. Hence $|N(v, X)| = \deg(v, X) = \deg_G(v, X)$ is the degree of v in set X, and $\deg(v) = \deg(v, V)$. $\delta(G)$ stands for the minimum, and $\Delta(G)$ for the maximum degree in G. When A and B are disjoint subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of G with one endpoint in A and the other in B. A bipartite graph G with color classes A and B and edge set E will be denoted by $G = (A, B, E)$, with $E \subset A \times B$. The *density* between disjoint sets X and Y is defined as:

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}.$$

The average degree of vertices in X to Y is defined as follows:

$$deg(X, Y) = \frac{1}{|X|} e(X, Y) = d(X, Y)\,|Y| \tag{1}$$

$\delta(X, Y)$ is the minimum of $\deg(v, Y)$ among all the vertices $v \in X$. Note the asymmetry of $\deg(X, Y)$ and $\delta(X, Y)$.

Let T be a rooted tree. For a vertex $x \in T$, $p(x)$ stands for the parent of x, $c(x)$ for the set of its children. $T(x)$ is the subtree containing x and all the vertices below x, $t(x) = |T(x)|$. In the tree T, T_{even}, T_{odd} denote the vertices of T whose distance from the root are even, odd, respectively. The *ratio* of the tree T is defined as $|T_{odd}|/|T_{even}|$.

Given parameters k and β, a tree with root r is called a *β-tree* if it has size at most k and all the components of the forest $T \setminus \{r\}$ (called the component subtree) have size less than βk.

A *bi-polar tree* is a tree of order k which consists of two β-trees with their roots connected. We also regard a single β-tree as a bi-polar tree.

In the proof of the Main Theorem the *Regularity Lemma* [3] plays a pivotal role. We will need the following definition to state the Regularity Lemma:

Definition 1 (Regularity Condition). *Let $\varepsilon > 0$. A pair (A, B) of disjoint vertex-sets in G is ε-***regular** *if for every $X \subset A$ and $Y \subset B$, satisfying*

$$|X| > \varepsilon|A|, \ |Y| > \varepsilon|B|,$$

we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

This definition implies that regular pairs are highly uniform bipartite graphs; the density of any reasonably large induced subgraph is almost as same as the density of the whole regular pair. We will use the following form of the Regularity Lemma:

Lemma 5 (Degree Form). *For every $\varepsilon > 0$ there is an $M(\varepsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set V into $\ell + 1$ clusters $V_0, V_1, \ldots, V_\ell$, and there is a subgraph G' of G with the following properties:*

- $\ell \leq M(\varepsilon)$,
- $|V_0| \leq \varepsilon|V|$,
- *all clusters V_i, $i \geq 1$, are of the same size $N \leq \lceil \varepsilon|V| \rceil$,*
- $\deg_{G'}(v) > \deg_G(v) - (d + \varepsilon)|V|$ *for all $v \in V$,*
- $G'|_{V_i} = \emptyset$ *(V_i is an independent set in G'), for all $i \geq 1$,*
- *all pairs (V_i, V_j), $1 \leq i < j \leq \ell$, are ε-regular in G', each with density either 0 or greater than d.*

We will need a variation of Tutte's one-factor Theorem.

Lemma 6. *Let H be a graph on $2h$ vertices, and let L be the set of vertices of degree greater than $(1 - \varepsilon)h$. If $|L| \geq (1 - \varepsilon)h$ then either there is a matching M in H covering all but $2\varepsilon h$ vertices, or there exist a partial matching M and a vertex set S such that*

- $\forall v \notin S$, *M covers all but at most one vertex in $N(v)$.*
- *Let $M^{(2)}(v)$ denote the edges in M whose both ends are neighbors of v. The vertices in $M^{(2)}(v)$ are not contained in S.*
- *$H \setminus S$ contains at least two adjacent vertices $A, B \in L$*

Proof: in Appendix.

We make an easy observation on the relationship between the number of leaves and that of vertices of degree larger than 2 in a tree.

Lemma 7. *Suppose $M(T) = \{x \in T : \deg(x) > 2\}$. Then $\sum_{x \in M(T)}(\deg(x) - 2) < l$, where l is the number of leaves of T. In particular, a n-tree with l leaves has fewer than l vertices of degrees larger than 2.*

Proof: in Appendix.

3. Sketch of proof

Suppose a $2n$-vertex graph G satisfies

$$|L| \geq n, \quad \text{where } L = \{v \in G : \deg(v) \geq n\} \tag{2}$$

The vertices of L will be referred to as *large* vertices, while the other vertices are the *small* vertices. Since deleting the edges between small vertices will not change the condition of the theorem, we will assume that no edges exist between small vertices in G.

Throughout the paper, we will assume that n is sufficiently large and will use the following main parameters:

$$\beta \ll \varepsilon \ll d \ll \gamma \ll \delta \ll \rho \ll 1 \quad \text{and} \quad \beta \leq \frac{1}{\ell} \tag{3}$$

For simplicity we do not compute the actual dependencies. Some new parameters will be introduced in certain theorems. Since they are used locally and their values are subject to change, we do not list them here.

Unless in extremal cases (will be explained later), we will apply Lemma 5 to G, with ε and d as in (3), and get a partition of $V(G)$ into clusters $V_0, V_1, \ldots, V_\ell$. We will use G'' to represent $G' \setminus V_0$. The large vertices now have degree

$$\deg_{G''}(v) \geq n - (d + 2\varepsilon)2n > n - 3dn$$

in G'' and number of large vertices in G'' is at least $|L| - |V_0| \geq n - 2\varepsilon n$. We define a *reduced graph* H as follows: The vertices of H are the clusters of V_i, $1 \leq i \leq \ell$, and we have an edge between two clusters if their density in G' exceeds d.

We will show (in Section 5.3) that the embedding of an arbitrary tree can be reduced to that of bi-polar trees. Thus we focus on the embedding of bi-polar trees in G''. We will find two adjacent clusters A, B in which most vertices have degrees greater than $(1 - 3d)n$ and also the neighborhoods (in H) of A and B are covered by a (partial) matching in H. To embed a bi-polar tree T with a and $b(\leq a)$ vertices in its two β-trees, we will apply the greedy algorithm. Unfortunately, the maximal degree of our G'' might be smaller than the order of T. After a careful case by case analysis, we formulate the situation that we can not find an embedding of T in G'' to two extremal cases: G'' can be split into two almost equal parts such that between them, there exist either almost no edges or almost all possible edges. In Section 4, we will show that, under the extremal case, any tree with at most n edges can be embedded into original graph G directly. Next in Section 5, we will prove the non-extremal case, i.e. the case assumed that extremal cases do not hold.

4. The Extremal Cases

There are two extremal cases. The first one (EC1)is rather trivial: the edges between large vertices and small vertices are almost full. The second case (EC2) is that vertices of G can be divided into two almost disconnected equal parts. In this case, we first prove a special case (EC2') where embedding of trees takes place merely in one part and secondly show that EC2 implies EC2'.

Using the bi-partite nature of trees, we first make a few observations on tree embedding. Assume that T is a tree with t edges. T_{even}, T_{odd} are defined as before, we let $t_e = |T_{even}|, t_o = |T_{odd}|$.

The following proposition is a simple conclusion of the greedy Algorithm.

Proposition 8. *G contains T as a subgraph if either of following holds:*

1. *$G = (L, R, E)$ is a bi-partite graph satisfying $\delta(L, R) \geq t_o, \delta(R, L) \geq t_e$,*
2. *The vertices of T can be partitioned into two sets T_1 and T_2 while T_2 is independent. On the other hand, $V(G) = L \cup R$ with $\delta(L, R) \geq |T_2|, \delta(L, L) \geq |T_1|, \delta(R, L) \geq |T_1|$.*

□

Proposition 9. *A graph G contains a bipartite subgraph $G = (L, R)$ in which $\delta(G) \geq \min(t_o, t_e)$, and $\delta(L, G) \geq t$. Then G contains T as a subgraph.*

Proof. Suppose $t_o \geq t_e$. Then there are at least $t_o - t_e$ odd leaves. We put (at most t_e) non-leaf odd vertices in R, and all the vertices of T_{even} in L using greedy algorithm. Since the parents of un-embedded leaves are in L and $\delta_G(L) \geq t$, we can add them in G greedily. □

4.1. Extremal Case 1 (EC1)

Lemma 10. *If a $2n$-vertex graph G satisfies (2) and $d(L, L) \leq c_1 = (\frac{1}{4})^5$, then $T \subset G$.*

Proof. Let $c_2 = \sqrt{c_1}$. We can find $L_0 \subset L$, such that $|L_0| \geq n - c_2 n$ and $\delta(L_0, L) \leq c_2 n$. Thus $\delta(L_0, R) \geq n - c_2 n$, where $R = V \setminus L$. Next we find $R_0 \subset R$, such that $|R_0| \geq n - c_3 n$ and $\delta(R_0, L_0) \geq |L_0|(1 - c_3)$, where $c_3 = \sqrt{2c_2}$. Together, we have $\delta(L_0, R_0) \geq n - 2c_3 n$, and $\delta(R_0, L_0) \geq n - 2c_3 n$. $n - 2c_3 n \geq \frac{n}{2} \geq t_e$, thus we can apply Proposition 9. □

4.2. Extremal Case 2 (EC2)

The extremal case 2 is that for $\alpha \ll 1$, $2n$-graph G satisfies (2) and

- $V(G) = V_1 \cup V_2, |V_1| = |V_2| = n,$
- $d(V_1, V_2) \leq \alpha$

We first prove that embedding holds under one special case: Extremal Case 2' (EC2') then show that EC2 implies EC2'.

4.2.1. Extremal Case 2' (EC2')

Lemma 11. *For $\alpha \ll \varepsilon_1 \ll 1$, if a $2n$-vertex graph G satisfies (2) and*

1. $V(G) = V_1 \cup V_2, |V_1| = |V_2| = n$, $|V_1 \cap L| \geq \frac{n}{2}$,
2. $\deg(v, V_2) < \varepsilon_1 n, \forall v \in V_1 \cap L$,

G contains any n-tree T as a subgraph.

Proof. Let $B = V_1 \cap L, A = V_1 \setminus B$. Since we can always treat any vertex of B as the one in A, we can assume $|B| = \frac{n}{2}$. Later we will note that a larger value of $|B|$ might simplify the proof.

Since $\delta(B, A) \geq |A| - \varepsilon_1 n$, as the way we derived R_0 in the EC1, we can find $A_1 \subset A$ such that $\delta(A_1, B) \geq (1 - \varepsilon_2)|B|$ and $|A_1| \geq |A| - \varepsilon_2 n$ for some $\varepsilon_2 = \sqrt{2\varepsilon_1}$.

We can write our situation as following:

1. $A, B \subset V(G), |A| = |B| = \frac{n}{2}, A \cap B = \emptyset$,
2. $\deg(v) \geq n, \forall v \in B$
3. $\delta(B, A \cup B) \geq (1 - \varepsilon_1)n$
4. $\exists A_1 \subset A, |A_1| \geq \frac{n}{2} - \varepsilon_2 n, \delta(A_1, B) \geq (1 - \varepsilon_2)\frac{n}{2}$

Note, $\delta(B, A_1) \geq |A_1| - \varepsilon_1 n \geq |A| - 2\varepsilon_2 n$ Thus Proposition 9 show that any tree of order at most $n - 4\varepsilon_2 n$ or any n-tree satisfying $\min(|T_{even}|, T_{odd}|) \leq \frac{n}{2} - 2\varepsilon_2 n$ can be embedded in G.

We will distinguish two cases according to the number of leaves of to-be-embedded n-tree T.

Case 1. T has at least ρn leaves ($\varepsilon_2 \ll \rho \ll 1$).
Without loss of generality, we assume that $|T_{odd}| \geq |T_{even}|$. We define a gap funtion $d(T) = |T_{odd}| - |T_{even}|$. During the proof, the vertices of T will be partitioned into two sets O_T and E_T, where at least one of them are independent. Let $d(O_T, E_T) = |O_T| - |E_T|$. Initially we set $O_T = T_{odd}$, $E_T = T_{even}$. Thus $d(O_T, E_T) = d(T)$.

We can assume that $d(T) < 4\varepsilon_2 n$, otherwise $|T_{even}| \leq \frac{n}{2} - 2\varepsilon_2 n$.

We denote sets of leaves in O_T and E_T by L_o, L_e, respectively. Suppose $l_o = |L_o|, l_e = |L_e|$. Let us further distinguish three cases according to the vales of l_o, l_e. (Note, $l_o + l_e \geq \rho n$.)

We choose ε_3 such that $\varepsilon_2 \ll \varepsilon_3 \ll \rho$.
(1.a) $l_e, l_o > \varepsilon_3 n$.
We first move $d(T)/2$ leaves from O_T to E_T.
Define $L_e^1 = \{v \in L_e, \deg(p(v)) = 2\}$, the leaves in E_T whose parents have only one child.

When $|L_e^1| > \frac{\varepsilon_3}{2}n$, we use Proposition 8 to embed all the vertices of T except for L_e^1 and $\varepsilon_3 n$ leaves in O_T to $A_1 \cup B$. Suppose B_R is the remaining set in B, and A_P is the set of mappings of $p(L_e^1)$. So $|B_R| = |A_P| = |L_e^1|$. Since $\delta_G(A_P, B_R) \geq |B_R| - \varepsilon_2 n, \delta_G(B_R, A_P) \geq |A_P| - \varepsilon n$, the matching lemma guarantees the embedding of L_e^1. At last, we embed the $\varepsilon_3 n$ odd-leaves greedily.

When $|L_e^1| \leq \frac{\varepsilon_3}{2}n$, we let $L_e^2 = L_e \setminus L_e^1$. We bring $p(L_e^2)$ to E_T and remove L_e^2 from E_T. Then we will embed new E_T and all but $\varepsilon_3 n$ leaves of O_T to B and A_1, respectively. Finally add the leaves (whose parents are mapped in B) greedily.

Note, Since B is almost complete, an embedding of T exists even if O_T contains edges.

(1.b) $l_o < \varepsilon_3 n$.

Since we can always increase the number of leaves by moving the parents of leaves to E_T from O_T, we are back to Case 1.a.

(1.c) $l_e < \varepsilon_3 n$.

We assume that $d(T) \geq 1$, otherwise we can exchange E_T and O_T (that is, $E_T = T_{odd}, O_T = T_{even}$) to back to Case b. Similarly, if we have that $|B| \geq |T_{odd}| = \frac{1+d(T)}{2}n$, we can also exchange E_T and O_T. So $|B| \geq \frac{n}{2} + \varepsilon_3 n$ is enough to close the case.

It is easy to see that for any positive integer $k \leq n$, there is an induced subtree of order between $k/2$ and k. We call it a $[k/2, k]$ subtree.

Indeed, we can always find $x_0 \in T$ such that 1. $t(x_0) \geq \frac{k}{2}$ 2. $\forall u \in c(v_0), t(y) < \frac{k}{2}$ If $t(x_0) \leq k$, $t(x_0)$ is what we are looking for. Otherwise, from $T(x_0)$, we cut subtree $T(y)$, $y \in c(x_0)$ one by one until the remaining subtree (rooted at x_0) has degree not more than k. We know the size of this tree is at least $k/2$ because the last removed y satisfies $t(y) < \frac{k}{2}$.

Similarly we can always find a subtree of T whose number of leaves is between $k/2$ and k, if $k \leq \rho n$.

Suppose T' is a subtree containing $[\frac{\rho}{4}n, \frac{\rho}{2}n]$ leaves, with root r'. Let T_R represent the remaining tree $T \setminus T'$. $d_{T'} = |T'_{odd}| - |T'_{even}|$.

Assume $d_{T'} \geq \frac{d(T)}{2}$ first. If the root $r' \in O_T$, we "flip" $T' \setminus \{r'\}$, i.e. move T'_{odd} from O_T to E_T, and move T'_{even} except for r from E_T to o_T. Hence O_T, E_T have been changed. The new $d(O_T, E_T)$ equals to $d(T) - 2(d_{T'} + 1) < 0$. Note E_T is still independent, and O_T may contain internal edges. Moreover, both O_T and E_T contain at least $\frac{\rho}{4}n$ leaves. We are now in Case 1.a. If $r' \in E_T$, we flip the entire subtree T'. It again only creates edges in O_T and causes $d(O_T, E_T) = d(T) - 2d_{T'} \leq 0$. We are also in Case 1.a.

When $d_{T'} < \frac{d(T)}{2}$, if $r \in O_T$, we flip T. It results in internal edges in E_T and $d(O_T, E_T) = d(T) - 2d_{T'} > 0$. If $r \in E_T$, we need the assumption that n and $d(T)$ are both even. Then we get the same result that $d(O_T, E_T) = d(T) - 2(d_{T'} + 1) \geq 0$ Actually, although an odd value of n might cause $d(O_T, E_T) = -1$, we have $|B| \geq |A| + 1$. So it does not stop us from backing to Case 1.a. □

Case 2. Embedding trees with fewer than ρn leaves

Recall that T satisfies $t_e \leq t_o < t_e + 4\varepsilon_2 n$.

¿From lemma 7, we know that fewer than ρn vertices are of degree larger than 2, or, more than $n - 2\rho n$ vertices have degree equal to 2.

When the number of leaves is small, Proposition 8 can be extended:

Proposition 12. *Suppose $\mu_1 \ll \mu_2 \ll 1$ and n_1 is large enough. T is n_1-tree with at most $\mu_2 n_1$ leaves and G is a bipartite n_1-graph with classes B and A, satisfying*

- $|B| \geq |A|$,
- $\delta(B, A) \geq (1-\mu_1)|T_{odd}|, \delta(A, B) \geq (1-\mu_1)|T_{even}|$

Then G contains T as a (spanning) subgraph.

Proof of Proposition. Assume $|A| = cn_1$, where c is a constant between 0 and $\frac{1}{2}$. We will first reduce T to a smaller graph to which we apply Proposition 8 then we use the matching lemma to add remaining vertices of T.

Since

$$\sum_{x \in M(T) \cap T_{even}} \deg(x) \leq 2|M(T)| + \mu_2 n_1 < 3d_2 n_1,$$

$$|\{x \in T_{odd} : N(x) \cap M(T) = \emptyset\}| \geq cn_1 - 3\mu_2 n_1$$

Thus,

$$|\{x \in T_{odd}, \deg(x) = 2, N(x) \cap M(T) = \emptyset\}| \geq cn_1 - 5\mu_2 n_1.$$

Therefore, we can find at least $\frac{c}{2}n_1 - 10\mu_2 n_1$ vertex-disjoint 2-paths (3-vertex path) whose end-vertices are in T_{even} and middle vertices in T_{odd}.

The way we derive a tree T' is following: We pick $\mu_2 n_1$ such paths. We omit their middle vertices and connect two ends of each path by a "fake" edge. Thus T' has $\mu_2 n_1$ fewer odd vertices than T. We do the same thing (finding 2-paths with ends in T_{odd} and middle vertices in T_{even}, omitting their middle vertices by fake edges) to reduce the number of even vertices.

Now we embed new tree T' in G using Proposition 8, because $\delta(B, A) \geq |T'_{odd}|, \delta(A, B) \geq |T'_{even}|$. Note, those fake edges are not preserved in G. Their existence is merely to keep T' be connected.

Suppose $\{u_i, v_i\}_{i=1}^{\mu_2 n_1} \subset B$ are the images of end-vertices of paths and $\{w_i\}_{i=1}^{\mu_2 n_1} \subset A$ are the unoccupied vertices. We treat each u_i, v_i as one "combined" vertex uv_i. The neighbors of uv_i are vertices who are adjacent to both u_i and v_i. Thus, $\deg(uv_i, \{w_i\}) \geq (\mu_2 - 2\mu_1)n_1$. On the other hand, $\deg(w_i, \{uv_i\}) \geq (\mu_2 - 2\mu_1)n_1$. By the matching lemma, there exists a matching between $\{uv_i\}$ and $\{w_i\}$ in G. Equivalently, the missing vertices in T_{odd} are added to T'.

Similarly, we can add the missing vertices in T_{even}. Thus, G contains T as a subgraph. □

Recall that our situation is:

- $V_1 = B \cup A, |V_1| = n, |B| = n/2$
- $\delta(B, V_1) \geq (1-\varepsilon_1)n$
- $A = A_1 \cup C, \deg(A_1, B) \geq (1-\varepsilon_2)|B|, |C| \leq \varepsilon_2 n$

It is slightly different from the conditions of the above proposition. As before, we can leave the embedding of odd leaves to the end while we put T_{even} in B. The problem is that vertices in $A \setminus A_1$ do not satisfy the degree condition. If we can find $|A \setminus A_1|$ 2-paths in V_1 containing these vertices as mid-vertices then the following revised proposition will guarantee the embedding.

Proposition 13. *G is as before, except that $\delta(A, B) \geq (1-\mu_1)|T_{even}|$ is replaced by*

- $A = A_0 \cup A_1, |A_0| \leq \mu_1 n_1, \delta(A_1, B) \geq (1-\mu_1)|T_{even}|$
- *There are $|A_0|$ disjoint 2-paths, each of them contains vertices of as mid-points.*

then G contains the spanning tree T with fewer than $\mu_2 n_1$ leaves.

Proof of Proposition. As Proposition 12, we will embed modified tree T' in G and then add the missing vertices by the matching lemma. The difference is that we will include those 2-paths in the mapping of T'.

Let $k = |A_0|$. Denote 2-paths containing A_0 by $P_1, P_2, \dots, P_k$. Let $B' = B \setminus \{P_i\}_1^k$.

Recall that T' contains many 2-paths whose mid-vertices are in T_{odd}. Assume the mid-vertices of first k 2-paths are $x_1, x_2, \dots, x_i$. Suppose the three nearest ancestors of x_i are $p(x_i), s_i, t_i$ in the order of increment distance from x_i. It was the greedy algorithm that we use to embed T' in Proposition 12. Before $p(x_i)$ is embedded, only a few ($\leq 10\mu_2 n_1$) T_{even} vertices (denote this set Q) were embedded in B. Thus we can let them embedded to B'. We will map $p(x_i), x_i$ and $c(x_i)$ to the path P_i for $i = 1, \dots, k$. If $p(x_i)$ and t_i are mapped to u_i and v_i, respectively, s_i will be (immediately) embedded to a common neighbor of u_i and v_i. $\square$

Now the only question left is how to find disjoint 2-paths that contain $A \setminus A_1$.

Let $C = A \setminus A_1$. We first change C slightly: remove $v \in C$ to V_2 in which $\deg(v, B) < \rho n$, and move $v \in V_2$ to C in which

$$\deg(v, B) \geq \rho n \tag{4}$$

whenever $|V_1| < n$. Thus size of V_1 might become $\varepsilon_2 n$ less than n.

In V_1, by the greedy algorithm, we find $|C|$ vertex- disjoint 2-paths containing vertices of C as the middle points. This is possible because of (4) and the fact $|C|$ is much smaller than ρn.

Finally, if $|V_1| = n$, the above proposition immediately closes this section. Now assume that $n - |V_1| = i > 0$. Then $\deg(v, V_2) \geq i$ for any $v \in B_1$, we can find at least i vertex disjoint 2-paths whose mid-points are in V_2, end points from B_1. In fact, suppose the maximal size of 2-paths families is m. We have $i\frac{n}{2} \leq e(B_1, V_2) \leq m\rho n + |V_2| - m$ implies $m > i$. Therefore, i V_2-vertices can be treated as C vertices when we apply Proposition 13. $\square$

4.2.2. EC2 $\Rightarrow$ EC2'

We will show that, the extremal case 2 can be converted to Extremal Case 2' unless we can find an embedding of T in G directly.

The following parameters are used:

$$\alpha \ll \alpha_2 \ll \alpha_3 \ll \alpha_1$$

We can assume that $|V_1 \cap L| < \frac{n}{2} + \alpha_1 n$. Otherwise we remove at most $\alpha_2 n$ vertices v from V_1 to V_2 in which $\deg(v, V_2) \geq \alpha_2 n$. The resulting V_1 still contains at least $\frac{1+\alpha_1}{2} n$ large vertices. When we arbitrarily move vertices from V_2 to V_1 to make $|V_1| = n$, we reach the Extremal Case 1.

We need the following lemma:

Lemma 14. *Under the condition of EC2, if some large vertex v_0 has more than $\alpha_1 n$ neighbors in each of V_1 and V_2, T can be embedded in G.*

Proof. Say, $V_0 \in V_2$. We assume that $\deg(v_0, V_1) \geq \frac{n}{2}$.

As before we remove a small number of vertices v from V_1 to V_2 in which $\deg(v, V_2) \geq \alpha_2 n$. Then V_1 contains a large vertex set B, which have fewer than $\alpha_2 n$ neighbors in V_2 and $|B| \geq \frac{n}{2} - 2\alpha_2 n$. As explained in Extremal Case 2', any tree of order $n - \alpha_3 n$ can be found in V_1 if $\alpha_3 \gg \alpha_2$.

In the tree T, we can always find an induced subtree T' of size $[\frac{\alpha_1}{4}n, , \frac{\alpha_1}{2}n]$ vertices. Without loss of generality, we assume T' is rooted at the root r of T. We denote the rest of the tree, actually a forest, by F. Since any leaf that is adjacent to r can be embedded at last, we will assume that F contains no isolated vertices. Thus the number of sub-trees in F (or $\deg(r, F)$) is at most $\frac{n}{2} - \frac{\alpha_1}{8}n$. We will put r in v_0 and embed T' in V_2, F in V_1.

First consider V_2. By assumption it has to contain at least $\frac{n}{2} - \alpha_1 n$ large vertices. As the way we define B and A_1 in V_1, we will find two sets B_2 and A_2 of sizes almost $n/2$ such that edges between them are complete. Since $\deg_{T'}(r) < \deg(v_0, A_2 \cup B_2)$, we will map the neighbors of r at T' to $A_2 \cup B_2$ and embed the rest of T' by greedy algorithm.

Similarly in V_1, we can put the roots of F in $N(v_0, B \cup A_1)$. Some of roots will go to B, while the others go to A_1. Then, according to the distance from their roots, the vertices of F are divided to F_1 (those with even distance from the roots embedded in B and those with odd distance from the roots embedded in A_1) and F_2. If $\max(|F_1|, |F_2|) \leq \frac{n}{2} - \alpha_3 n$, the greedy algorithm will complete the embedding.

Otherwise, say, $|F_1| > \frac{n}{2} - \alpha_3 n$. Since $|F| = |F_1| + |F_2| \leq n - \frac{\alpha_1}{4}n$, we have $|F_1| - |F_2| > (\frac{\alpha_1}{4} - 2\alpha_3)n$. We will change T_{even} and T_{odd} to E_T and O_T such that embedding the whole tree is possible. We move the neighbors of r that would be embedded to A_1 from T_{odd} to T_{even} and flip their sub-trees accordingly. If the resulting structure has a gap $||O_T| - |E_T||$ greater than $\alpha_3 n$, Proposition 9 will embed T only using V_1. Otherwise, when $||O_T| - |E_T|| \leq \alpha_3 n$, we can exchange F_1 and F_2. Then the resulting gap will be at least $\frac{\alpha_1}{4}n$, again Proposition 9 is enough to close the case. □

We let $L_1 = \{v \in L : \deg(v, V_1) > \alpha_1 n\}$ and $L_2 = \{v \in L : \deg(v, V_2) > \alpha_1 n\}$. The above proposition shows that $L_1 \cap L_2 = \emptyset$ and $\delta(L_1, V_1) \geq (1 - \alpha_1)n, \delta(L_2, V_2) \geq (1 - \alpha_1)n$. Since $L_1 \cup L_2 = L$, one of $|L_1|, |L_2|$ is at least $n/2$. We arbitrarily move non-L vertices to make $|V_1| = |V_2| = n$. We are in the Extremal Case 2'.

5. The Non-extremal Case

When none of extremal cases hold for G. We apply Lemma 5 to G, and obtain the *purer* graph G'' and the reduced graph H. We will show there always exists an embedding of T in G''.

5.1. Preparation of embedding

It is easy to see the following proposition holds. Note, we treat G'' as G, so all the following numbers are meant in G'', (thus they underestimate what happens in G.) Note, for a cluster $X \in H$, its average degree in G'', "$\deg(X)$" is defined as in (1), while "$\deg_H(X)$" denote the degree of X in H.

Proposition 15. Let $\{X, Y\}$ be a pair of clusters with density d.

- All but at most εN ($N = |X|$) vertices $u \in X$ have degrees

  $$\deg(u, Y) > (d - \varepsilon)|Y| \quad \text{(that is,} \quad d(u, Y) > d - \varepsilon)$$

  The same holds if we replace Y with a subset $Y' \subset Y$, $|Y'| > \varepsilon|Y|$.

- All but at most εN vertices $u \in X$ have degrees $\deg(u) > \deg(X) - 4\varepsilon n$.

- In general, for every cluster Y a subset $\mathcal{S}_Y$ such that either $\mathcal{S}_Y = \emptyset$ or $|\mathcal{S}_Y| > \varepsilon N$. Then, writing $\mathcal{S} = \cup \mathcal{S}_Y$, all but at most εN vertices $u \in X$ have the following property:

  $$\deg(u, \mathcal{S}) > \deg(X, \mathcal{S}) - 4\varepsilon n \qquad \square$$

We define *large* clusters as ones whose degree is at least $(1 - 3d)n$. The above property implies that any cluster containing more than ε vertices is large. Denote large clusters $\mathcal{L}$. Since the number of large vertices in G'' is at least $n - \varepsilon n$, the size of $\mathcal{L}$ is at least:

$$(n - \varepsilon n - \#\{\text{large vertices in small clusters}\})\,/N \geq \frac{n - 2\varepsilon n - \varepsilon 2N\ell}{N} \geq (1 - 4\varepsilon)\ell$$

As what we assumed in the original graph G, we assume that no edges between two small clusters in H. It means that we treat the large vertices in the small vertices as small vertices. It does not change any above conclusion about large clusters.

We will use the following results from [2] on the embedding of β-trees, where $\gamma = \dfrac{\varepsilon + \beta\ell/2}{d - \varepsilon}$.

Proposition 16. *Let (X, Y) be an edge in the reduced graph H. Any β-forest of size less than $(1 - \gamma)N$ can be embedded into (X, Y) with roots in X (same for Y).* $\square$

Remark, if all the trees in the forest have the same ratio $R < 1$, the size of embedded forest could be $(1 - \gamma)(1 + R)N$.

Lemma 17. *Suppose A is a cluster and M is a matching in $H \setminus A$. We also write M for the set of clusters covered by M, $\deg(A, M) = \sum_{y \in M} \deg(A, y)$ for the average degree of a vertex in A into M, and m for total number of vertices of G included in M. Then every β-tree of size at most $(1 - \gamma)\deg(A, M) - 2\varepsilon m$ can be embedded into $A \cup M$. Moreover, all but at most εN vertices of A can be chosen the root.*

□

The following lemma asserts that we can assume that, the component subtrees of a β-tree have equal ratios.

Lemma 18. *If the ratio of β-tree is R, after grouping certain component subtrees together (to sub-forests), we can let every component subtrees or sub-forests have a ratio between $(1-\beta_1)/R, (1+\beta_1)/R$, where $\beta_1 = \beta^{\frac{1}{2}}$.*

Proof of Lemma 18 in Appendix A.

The bi-polar tree T have a vertices in one β-tree T_a, $b(b \leq a)$ vertices in another β-tree T_b. The ratios of T_a and T_b are R_a, R_b, respectively. And we assumed that all the subtrees under the roots r_a, r_b have the same ratios R_a, R_b, respectively. The following proposition gives the range of R_a or R_b:

Proposition 19. *If two values of $R_a, R_b, \frac{1}{R_a}, \frac{1}{R_b}$ are less than c_0, then bi-polar tree T can be embedded in G. Here, $c_0 = \frac{c_1}{2}$, c_1 was defined in Extremal Case 1.*

Proof. Assume that $R_a, \frac{1}{R_b} < c_0$. Other cases are similar.

Since $R_i = |T_{i,even}|/|T_{i,odd}|,\ i \in \{a, b\}$, we have $t_{a,even} < c_0 a, t_{b,odd} < c_0 b$. Thus we know that T_a and T_b must have $(1-c_0)a, (1-c_0)b$ leaves, respectively, or T has at least $(1-c_0)n$ leaves.

Since $d(L, L) > c_1$, a basic result of graph theory concludes that there exist a subgraph in L whose minimum degree is at least $\frac{c_1}{2}|L| > c_0 n$. Therefore we can embed all non-leaves of T in this subgraph and add the leaves greedily. □

¿From now on, we assume that either $c_0 \leq R_a \leq \frac{1}{c_0}$ or $c_0 \leq R_b \leq \frac{1}{c_0}$.

5.2. Embedding bi–polar trees

Assume that T is a bi-polar tree. Two β-trees T_a and T_b have roots r_a and r_b, respectively.

When there are no two large adjacent clusters, we are in Extremal case 1. We have

$$|\{X \in H, \deg_H(X) \geq (1-3d)\ell\}| \geq |\{X \in H, \deg(X) \geq (1-3d)\ell\}| \geq (1-4\varepsilon)\ell,$$

so we can apply Lemma 6 to H. Summarize two possible conclusions of Lemma 6, we find a cluster set $O(\subset H \setminus S)$ containing two adjacent large clusters A and B, such that all but $6d\ell$ neighbors of any cluster in O are covered by a partial matching M. We also write M for the set of clusters covered by the matching. After deleting certain edges of H, we assume that $\forall C \in O, N(C) \subset M$ and $\deg(C, M) = (1-10d)n$

We will show non-extremal case falls into following cases, each of them leads to an embedding of T:

- Sizes of both β-trees is sufficiently large, there exist

1. reasonably many edges $e = (X, Y) \in M$, in which $d(A, \{X, Y\})$ is not equal to $d(B, \{X, Y\})$,
2. reasonably many clusters $C \in N(A)$ with $d(A, C) < 1$,
3. reasonably many small neighbors of A have enough neighbors outside $N(A)$.

- Size of one β-tree is small,

1. for reasonably many clusters $C \in N(A)$, $d(A, C) < 1$.
2. Reasonably many small neighbors of A have enough neighbors outside $N(A)$
3. Reasonably many large neighbors of A have enough neighbors outside $N(A)$

Let $M(C) = \{e(X, Y) \in M : \{X, Y\} \cap N(C) \neq \emptyset\}$ for $C \in O$. Moreover, $M^{(1)}(C) = \{e(X, Y) \in M :$ only one of X, Y in $N(C)\}$, $M^{(2)}(C) = M(C) \setminus M^{(1)}(C)$. Lemma 6 asserts $M^{(2)}(C) \subset O$. For an edge $e = (X, Y) \in M$, we write $d_A(e)$ for the weight $d(A, X) + d(A, Y)$ and $d_B(e)$ for the weight $d(B, X) + d(B, Y)$.

We will map r_a, r_b to two adjacent large vertices (one from A, another from B), whose degree to M is almost $\deg(A, M), \deg(B, M)$. Observe that leaves directly under r_a or r_b can always be embedded at last by the greedy algorithm if other vertices of T have been embedded. Thus we assume that T does not contain such leaves.

We will use the following statement heavily, which is a corollary of Lemma 17:

Corollary 20. *If the matching M can be split into two (disjoint) sub-matching M_a and M_b such that:*

$$(1-\gamma)\deg(B, M_b) \geq b + 2\varepsilon n \qquad (1-\gamma)\deg(A, M_a) \geq a + 2\varepsilon n \tag{5}$$

Then G contains the bi-polar tree T as a subgraph.

Proof. Apply Lemma 17 twice: embedding T_b in $\{B\} \cup M_b$, T_a in $\{A\} \cup M^a$. □

Let $\gamma_1 = \sqrt{5\gamma}$. We first assume that $b > \gamma_1 n$. The case that $b \leq \gamma_1 n$ will be discussed later.

5.2.1. $b > \gamma_1 n$

Proposition 21. *If $\sum_{e \in M} |d_A(e) - d_B(e)| > 2\gamma_1 \ell$, (5) holds.*

Proof. Since $\deg(A, M) = \deg(B, M)$, $\exists M^0 \subset M$ such that $\sum_{e \in M^0} d_A(e) - d_B(e) > \gamma_1 \ell$.

Let $M_1 = \{e \in M : d_A(e) \geq d_B(e)\}$, $M_2 = M - M_1$.
Set $c = (\deg(A, M_1) - \deg(B, M_1))/n$. Thus $c \geq \gamma_1$ and $\deg(B, M_2) - \deg(A, M_2) = cn$.

Let $a^{(i)} = \deg(A, M_i)$, $b^{(i)} = \deg(B, M_i)$, $i = 1, 2$. Assume that $a^{(1)} \leq b^{(2)}$ (the other case is similar). We have $b^{(2)} - b^{(1)} \geq c$. Combining it with the fact $b^{(2)} + b^{(1)} = (1 - 10d)n$, we get $b^{(2)} \geq \frac{1+c-10d}{2} n$. (Note, that $a^{(1)} > b^{(2)}$ would imply that, $a^{(1)} > \frac{1+c-10d}{2} n$.)

Arrange the edges in M_2 in the descending order of $\frac{d_B(e)-d_A(e)}{d_B(e)}$. We start from the empty set and add edges of M_2 in order once at a time. We stop as soon as $\sum d_B(e)L \geq (1+2\gamma)b$, (such moment exists because $b^{(2)} > (1+\frac{\gamma_1}{2})\frac{n}{2} > (1+2\gamma)$). Denote this sub-matching M_b. Let $M_a = M - M_b$. The way we obtained M_b guarantees:

$$\frac{\deg(B, M_b) - \deg(A, M_b)}{\deg(B, M_b)} \geq \frac{cn}{b^{(2)}}$$

$$\deg(B, M_b) - \deg(A, M_b) > cn\frac{b}{b^{(2)}} > cb \geq 5\gamma n$$

Since $d_B(e) \leq 2\ \forall e$, we also have $\deg(B, M_b) < (1+2\gamma)b + 2L$. Thus, $\deg(A, M_a) = \deg(A, M) - \deg(A, M_b) \geq deg(A, M) - (1+2\gamma)b - 2L + 5\gamma n > a - 3\gamma n + 5\gamma n > a + 2\gamma n$. □

Because the large clusters in $M_{A,2}$ play same roles as B, we have the following corollary:

Corollary 22. *If some $C \in M_{A,2} \cap \mathcal{L}$ satisfies $\sum_{e\in M} |d_A(e) - d_C(e)| > 2\gamma_1\ell$, G contains T as a subgraph.* □

We thus know that except for $\gamma_2\ell$ edges, all edges in M satisfy $|d_A(e) - d_B(e)| < \gamma_2$, where $\gamma_2 = \sqrt{2\gamma_1}$. We assume that M contains these edges only. Assume $\gamma_2 \ll \eta \ll \rho$.

Proposition 23. *Suppose edge $e(X, Y) \in M$ satisfies $d(B, Y) \geq \eta, d(B, X) \leq 1-\eta$, and the β-tree T_b has ratio $c_0 < R < 1/c_0$ and size exceeding $\gamma_1 n$. More than $(1-\gamma)(d_B(e) + \frac{c_0}{2}\eta)N$ vertices of T_b can be embedded in $B \cup e$.*

Proof. Pick a typical vertex $v \in B$ such that following holds: assume $X_1 = N(v, X), Y_1 = N(v, Y)$, $|Y_1| \geq (\eta - \varepsilon)N$ and $|X \setminus X_1| \geq (\eta - \varepsilon)N$. Pick sets $X_0 \subset X \setminus X_1$, and $Y_0 \subset Y_1$ such that $|X_0| = |Y_0| = (\eta - \varepsilon)N$. Because of Lemma 18, we assume that all the component subtrees of T_b share the same ratio $1/R$. Because $c_0 < R < 1/c_0$, we can embed $(1-\gamma)(1+c_0)(\eta-\varepsilon)N$ vertices of β-tree in $X_0 \cup Y_0$ according to the remark after Proposition 16. In the rest of X and Y, we can at least embed $(1-\gamma)(d_B(e)-\eta+\varepsilon)N - 4\varepsilon N$ vertices by Lemma 17. Thus in total, at least $(1-\gamma)(d_B(e) + \frac{c_0}{2}\eta)N$ vertices of the tree could be embedded in $B \cup e$. □

REMARK, if the "β-tree" are actually a forest consisting of constant number of β-trees, above algorithm still applies, the only change is that we choose a new typical vertex as the root of each new tree. (in fact, multi-roots even provide additional room during embedding because of the regularity.)

Corollary 24. *Let $M_3 = \{e(X, Y) \in M : 3\eta \leq d_A(e) \leq 2 - 3\eta\}$. If $|M_3| > \gamma_1\ell$, G contains T as a subgraph.*

Indeed, from Proposition 19, we have that, say, $c_0 < R_b < \frac{1}{c_0}$. Since $|d_A(e) - d_B(e)| \leq \gamma_2$ for all $e \in M$, edges in M_3 satisfy $2\eta < d_B(e) < 1 - 2\eta$, (thus satisfy the condition of Proposition 23).

We first embed the β-tree T_b: use as many edges in M_3 as possible and apply Proposition 23 when embedding. Let M_b denote the sub-matching that T_b-embedding takes place.

$\frac{c_0}{2}\eta\min(|B|, \gamma_1 n)$ more vertices than $(1-\gamma)\deg(B,Q)$ will be put in $M_b \cap M_3$ because of Proposition 23. Since $\eta \gg \gamma$, we have $\frac{c_0}{2}\eta\gamma_1 > 5\gamma$. Thus we can embed T_a by the greedy algorithm in the remaining edges (denoted by M_a). $\square$

We define a cluster set U_1 for those in the edges $e \in M$ satisfying $d_A(e) > 2 - 3\eta$. It is clear that U_1 has a one-factor and $U_1 \subset N(A)$. Corollary 24 says,

$$\deg(A, U_1) \geq deg(A) - 4\eta n \tag{6}$$

thus $(1-4\eta)\ell \leq |U_1| < \ell$. Let $U_2 = H \setminus U_1$ and $M_{out} = M \setminus U_1$. By Lemma 6, the neighbors in U_2 of any cluster in U_1 is covered by M_{out}.

Let P and Q denote the set of large and small clusters in U_1, respectively. By Corollary 22 and (6), $\forall C \in P, \deg(C, U_2) < 5\eta n$. Since any edge of H contains at least one large clusters, the number of large clusters in U_1 is at least $|U_1|/2$, or $(1-2\eta)\ell/2$. If this number exceeds $(1+\rho)\ell/2$, we immediately reach the Extremal Case 2': Let V_1 denote vertices of G included in the clusters of U_1. The number of large G-vertices in V_1 is at least $(1+\frac{\varrho}{2})n$. All but εn of large vertices have at most $6\eta n$ neighbors outside V_1. Remove these εn vertices from V_1 and add or remove more vertices to make V_1 have order n. We now reach the situation of Extremal Case 2'. We thus assume $|Q| > (1-2\rho)\ell/2$.

Proposition 25. *If $d(Q, U_2) > \rho_1 \gg \rho$, G contains T as a subgraph.*

Proof. We know, more than $\frac{\rho_1}{2}|Q|$ small clusters have degree at least $\frac{\rho_1}{2}|U_2| > \frac{\rho_1}{2}\ell$ in M_{out}. Pick $\frac{\rho_1}{6}\ell$ of such clusters and denote them by Q_1.

Our embedding depends on the structure of β-tree T_a. Recall that T_a contains no leaves directly under the root r_a. Let a_1 be the number of children of r_a. The order of tree $a(\geq \frac{n}{2})$ is at least $2a_1$.

(1) $a \geq \frac{5}{2}a_1$.

Pick a cluster $X \in Q_1$. Suppose its partner, the (large) cluster in the same edge of M as X is Y. We know that the neighbors of X in U_2 is in the matching M_{out}. For each component subtree of T_a, we put its root into X then embed the rest in M_{out} by Lemma 17. We repeat this procedure until X is filled. Because $a \geq \frac{5}{2}a_1$, the number of embedded is at least $(1-\gamma)\frac{5}{2}N$, which exceeds an embedding in $\{X, Y\}$ by at least $(1-\gamma)\frac{1}{2}N$. We repeat this to other clusters in Q_1 until total gain (over the embedding in $M \cup U_1$) is $(1-\gamma)\frac{1}{2}N\frac{\rho_1}{6}\ell > \rho n$. Then we use greedy algorithm to embed the rest of T_a and T_b in $M \cup U_1$.

Remark: the above algorithm has nothing to do with that the fact Q are *small* clusters.

(2) $a < \frac{5}{2}a_1$.

We know, T_a contains at least $\frac{1}{2}a_1 > \frac{1}{10}n$ 2-paths whose commend ends is r_a. Denote the vertices in these paths (excluding r_a) by D. We will let D be the last vertices to be embedded. Using the degree condition of clusters of Q_1, we choose distinct neighbors for all the elements of Q_1 and use these new (disjoint) edges to replace corresponding edges of M. In this new matching, by the greedy algorithm, we can embed all but $\frac{\rho_1}{6}n$ vertices of T, thus left vertices are from 2-paths. Since partners of the clusters in Q_1 were not used and they are large clusters, we can add the remaining 2-paths greedily.

Corollary 22 and Proposition 25 together show that $d_H(U_1, H \setminus U_1) < \rho_1$, which leads to Extremal Case 2: The corresponding G vertices in the clusters of $N(A)$ make a set V_1 of order approximately n, such that $d(V_1, G \setminus V_1)$ is small.

5.2.2. $b \leq \gamma_1 n$

We can assume that $c_0 \leq R_a \leq \frac{1}{c_0}$. Otherwise similar as in Proposition 19, an embedding can be found directly. Since we can always embed T_b in the edges of M where $d_B(e) \geq d_A(e)$ first, our goal is to embed the tree T_a (of size $(1-\gamma_1)n$) in the remaining matching (still denoted by $M(A)$). Corollary 24 still applies. We define U_1, U_2 as before. (6) still holds.

As before, we distinguish cases based on the values of a and a_1. When $a \geq \frac{5}{2}a_1$, the Remark in Proposition 25 says if $d(U_1, U_2) > \rho_1$, there exists an embedding of T. When $d(U_1, U_2) > \rho_1$, we are in Extremal Case 2.

Therefore assume that $a < \frac{5}{2}a_1$. That means, T_a contains at least $\frac{1}{2}a_1 > \frac{1}{10}n$ paths of length 2 starting from the root r_a. Denote the large, small clusters in U_1, by P and Q, respectively. It is easy to see almost all the edges of U_1 are made of a large and a small clusters. Otherwise, if there exists enough edges made up of two large clusters, we will save the clusters in these edges to the end and embed the middle points of 2-paths into the (large) vertices in these clusters and add leaves finally. In this way sufficiently more vertices can be embedded.

As showed in Proposition 25, we can also exclude the case $d(Q, U_2) \geq \rho_1$. The last case to consider is that $d(P, U_2) > \rho_2$ for $\rho_2 \gg \rho_1$. *Note*, Corollary 22 does not hold because $b \leq \gamma_1 n$.)

We then find P_1, a subset of P containing $\frac{\rho_2}{4}\ell$ large clusters, each of them has at least $\frac{\rho_2}{2}\ell$ neighbors in U_2. Consider $X \in P_1$. Since clusters in P_1 can be treated as A, Corollary 24 implies most of the neighbors of X in U_2 appear as pairs in M (occupying both ends of edges). Pick one large cluster from each of such pairs and denote this set $P_2(X)$.

Let $P_2 = \cup_{X \in P_1} P_2(X)$. In the bipartite graph (P_1, P_2), the degree of $X \in P_1$ is at least $\frac{\rho_2}{6}\ell$. Thus, we can find $P_3 \subset P_2$ such that (1) $\forall B \in P_3, \deg(B, P_1) \geq \frac{\rho_2^2}{32}\ell$, (2) $|P_3| \geq \frac{\rho_2^2}{8}\ell$.

Since clusters in P_3 play the same role as A, by Corollary 24, they should be adjacent to most of small clusters that share the edges of M with P_1. But since $\rho_2 \gg \rho_1$, it contradicts with the assumption $d(Q, U_2) < \rho_1$.

5.3. Embedding arbitrary trees

5.3.1. The bi-polar tree generated from a arbitrary tree

Let $0 < \beta < 1$ be fixed. To every tree T we will assign a bi-polar tree as follows.

Consider a rooted tree T on m vertices. A vertex x is called a *β-vertex* if

a) $t(x) \geq \beta m$

b) $t(y) < \beta m$ for all $y \in T(x) \setminus \{x\}$.

It is easy to see that the number of β-vertices is constant (not more than $1/\beta$).

We define the following (bottom up) transformation. A β-vertex is called a B-root if it is at an odd distance from the root, otherwise it is an A-root. We declare the parent of each B-root (A-root) an A-parent (B-parent) if its closet β-vertex ancestor is a A-root (B-root). Prune all subtrees growing out from A-roots and B-roots, but leave a leaf in place of the original subtree. Apply the same transformation repeatedly until the original tree shrinks to a single tree of size at most βk. At this point, we also define the original root to be an A-root. The forest defined by the A-roots will be called the A-forest, and the one defined by the B-roots the B forest. Merging the roots together within each one of these two forests, we get two β-trees, and connecting their roots with an edge, we get the bi-polar tree assigned to T.

5.3.2. Embedding the original tree

After we know how to embed the corresponding bi-polar tree, we can embed the original tree. We will do it from top down. We will only use vertices that are typical with respect to A and B in that if u is in cluster x then $d(u, A) > d(x, A) - \varepsilon$, and the same holds with B replacing A. Recall before embedding the matching M was split into two parts M_a (for a-trees) and M_b (for b-trees).

First we embed the original root and the corresponding tree of size at most βk into M_a. Then we take the first leaf which corresponds to an A-root or A-parent, and reroute it (move it back) to A. We can do that, for its parent is typical with respect to A, and thus it has a large degree into A (which will be filled only by the constant number of A-roots and A-parents). We put this A-root/parent to a vertex of A which has a degree within $2\varepsilon n$ of the average of degrees in A into the leftover of the matching M_a (according to the last itemized property of Proposition 15, all but at most εN vertices of A are like that). Then, we embed into the leftover of M_a the tree of that new root (if any). We reroute all “seen” A-roots and A-parents and embed all seen a-trees.

Next we embed a B-root, a children of a A-parent, to a neighbor of vertex to which A-parent was embedded. Then we embed into M_b the tree of the new root. If any B-root/parent exists in this tree, we will reroute them as what we did to a-trees.

We repeat this procedure till the last A- or B-root. □

A. Appendix

A.1. Proof of Lemma 6

We use the Gallai–Edmonds decomposition. Let S denote the usual cutset (such that the following is true for the components outside S: every even component has a one–factor, deleting any vertex from an odd component will result in a one–factor (they are factor–critical), and there is a one–factor of size $|S|$ connecting S with $|S|$ odd components). We will show that either there is a matching M in H covering almost all the vertices, or there is unique component outside S that contains two adjacent vertices of L. This clearly suffices.

1. *No component has vertices of L.*
Then we would have $(1-\varepsilon)h \leq |L| \leq |S|$. The matching covering S misses at most $2\varepsilon h$ vertices.

2. *At least two components contain vertices of L.*
Then each of them must be of size greater than $(1-\varepsilon)h - |S| + 1$, since they have vertices from L and their neighbors outside the components are all in S. Thus, the matching covering S and these two components consists of at least $2|S| - 2 + 2((1-\varepsilon)h - |S| + 1) = 2h - 2\varepsilon h$ vertices.

Therefore, except for S, there is an unique component O having vertices of L. In fact, almost half of vertices of O are in L. Because, set $a = |O \cap L|$, $b = |O \setminus L|$, we have $(1-\varepsilon)h \leq |L| \leq a + |S|$ and naturally $|O| + 2|S| - 1 \leq 2h$. Hence $a \geq \frac{|O|}{2} - \varepsilon h$.

Furthermore, if no two vertices in $O \cap L$ were connected then we would have $b + |S| \geq (1-\varepsilon)h$. It would cause $2|S| + |O| \geq (1-\varepsilon)h + (1-\varepsilon)h = 2h - 2\varepsilon h$, or the matching covering S and O misses at most $2\varepsilon h + 1$ vertices. □

A.2. Proof of Lemma 7

We will use induction on n and l, the numbers of vertices and leaves of T, respectively. For convenience, we define function $f(T) = \sum_{x \in M(T)} (\deg(x) - 2)$.

When $l = 2$, the tree is a path. Thus, all vertices are of degree less than 2. Hence $f(T) = 0 < l$.

Now assume that $n > l > 2$. We assume that our observation holds for either the order of tree is smaller than n or the number of leaves is smaller than l. Arbitrarily pick a leaf x_0 with parent y_0. Let $T' = T \setminus \{x_0\}$. If $\deg(y_0) > 2$, T' is a $n-1$ tree with $l-1$ leaves. By hypothesis, $f(T') < l - 1$. When $\deg(y_0) = 3$, then $\deg_{T'}(y_0) = 2$ thus $y_0 \notin M(T')$. So $f(T') = f(T) - 1$. When $\deg(y_0) > 3$, we also get $f(T') = f(T) - 1$. Hence $f(T) < l$. If $\deg(y_0) = 2$, T' is a $n-1$ tree with l leaves and the same f-value as original tree T. The hypothesis says, $f(T') < l$. But since $f(T) = f(T')$, we again get $f(T) < l$. □

A.3. Proof of Lemma 18

For convenience, we use $(0,1)$-*ratio* $r(T) = |T_{odd}|/|T|$, instead of ratio $R(T) = |T_{odd}|/|T_{even}|$, when T is a tree or forest. We re-state Lemma 18 as follows:
Given a β-tree of size n and $r(F) = r$. We can wrap its component trees into certain sub-forests $F_1, F_2, \ldots, F_s$, such that $|F_i| \leq \beta_1 n$ and the $(0,1)$-ratio of F_i is between $r(1-d)$ and $r(1+d)$, for $i = 1, \ldots, s$, where $\beta_1 = \beta^{\frac{1}{3}}$, $d = \beta^{\frac{2}{3}}$.

Proof. We use an ordered list L to store all the component trees in the order of ascending $(0,1)$-ratios. We will move trees to a forest one by one according to following rules. F^t represent this forest at the time t, or there are t trees in the forest. Initially $F^0 = \emptyset$. Let $\varepsilon = \beta^{\frac{1}{3}}$.

1. Stop when none of trees in current L has $(0,1)$-ratio outside the range $[(1-\varepsilon)r, (1+\varepsilon)r]$.

2. Otherwise, when $r(F^t) > r$, we let $F^{t+1} = F^t \cup L_{start}$. And update L as $L \setminus L_{start}$. When $r(F^t) \leq r$ we let $F^{t+1} = F^t \cup L_{end}$ and $L = L \setminus L_{end}$.

The above algorithm makes following propositions: Let $x = \max\left(\frac{1}{r\varepsilon}\beta n, \frac{1}{\varepsilon}\beta n\right)$.

Proposition 26. $(1-\varepsilon)r \leq r(F^t) \leq (1+\varepsilon)r$ *whenever* $|F^t| > x + \beta n$.

Indeed, assume that $r(F^{t_1}) < (1-\varepsilon)r$ and $|F^{t_1}| > x + \beta n$. We will show that there is a contradiction. Similarly, $r(F^t) > (1+\varepsilon)r$ can be showed impossible when $|F^t| > x + \beta n$.

Let $t_0 = \max_{t \leq t_1}\{t : r(F^{t-1}) \geq r\}$ (such t_0 exists because of the algorithm of constructing F^t). Then $\forall t_0 \leq t \leq t_1, r(F^t) < r$. The algorithm also makes $r(F^{t_0}) \leq r(F^{t_0+1}) \leq \ldots \leq r(F_1^t)$. Before the moment t_0, $(0,1)$-ratio of the forest was at least r; after t_0, trees added to F^t are those with $(0,1)$-ratio at least r. We thus have

$$r(F^{t_0}) \geq \frac{xr}{\beta n + x} \geq (1-\varepsilon)r$$

Contradiction. □

The way to find desired forests $F_1, F_2, \ldots, F_s$ is to choose time $t_1, t_2, \ldots, t_{s-1}$ to divide F such that $F_i = F^{t_i} \setminus Ft_{i-1}$ $(t_0 = 0)$. We choose t_1 as the minimum t such that $|F^t| > x + \beta n$ and choose $t_i, i \geq 2$ such that $|F^{t_i}| - |Ft_{i-1}| \in [y - \beta n, y + \beta n]$, where $y = \beta^{\frac{1}{3}}n$. It is easy to see that $r(F_i) \in [(1-d)r, (1+d)r]$. Because, for example, $r(F^{i-1}) \geq (1-\varepsilon)r$ and $r(F^i) \leq (1+\varepsilon)r$ implies that $r(F_i) \leq (1+\varepsilon)r + \frac{|F^{i-1}|}{\beta^{\frac{1}{3}}n}2\varepsilon r \leq (1+d)r$. □

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