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On 3-Simplicial Vertices in Planar Graphs[1]

by

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ABSTRACT

A vertex v in a graph $G = (V, E)$ is *k-simplicial* if the neighborhood $N(v)$ of v can be vertex-covered by k or fewer complete graphs. The main result of the paper states that a planar graph of order at least four has at least four 3-simplicial vertices of degree at most five. This result is a strengthening of the classical corollary of Euler's Formula that a planar graph of order at least four contains at least four vertices of degree at most five.

1 Introduction

A simple consequence of the classical Euler Formula for planar graphs is that any planar graph of order at least four has at least four vertices of degree at most five. Grünbaum and Motzkin [5] showed that this result is best possible.

In this paper we strengthen the result on the number of vertices of degree at most five in the following sense. A vertex is 3-*simplicial* if its neighborhood can be vertex-covered by at most three cliques. Clearly any vertex of degree at most three is 3-simplicial. But a vertex of degree four needs at least one edge in its neighborhood to make it 3-simplicial. And a vertex of degree five needs a triangle or two independent edges in its neighborhood to make it 3-simplicial. Our main result is that each planar graph of order at least four has at least four 3-simplicial vertices of degree at most five. We also exhibit an infinite class of planar graphs that contain exactly four 3-simplicial vertices. So our result is in a sense best possible. But in our example the four 3-simplicial vertices are all of degree two. Hence it is natural to ask, (if the order is large enough) whether excluding vertices of degree two forces *more* than four 3-simplicial vertices. The icosahedron is a particularly intriguing example because all twelve of its vertices have degree five and all twelve are 3-simplicial.

In Section 3 we prove our main result. In Section 4 we prove the analog for outerplanar graphs: an outerplanar graph of order at least four has at least four 2-simplicial vertices of degree at most three unless it is the 3-sun or $K_{1,3}$. This result is best possible.

2 Preliminaries

Let $G = (V, E)$ be a graph. The *neighborhood* $N(v)$ of a vertex v of G consists of all vertices adjacent to v. A vertex v in G is k-*simplicial* if the neighborhood $N(v)$ of v can be vertex-covered by k or fewer complete graphs. That is, there are (not necessarily distinct) cliques $C_1, C_2, \ldots, C_k$ in G such that each *vertex* in $N(v)$ lies in at least one C_i. The *simpliciality* $\sigma(v)$ of a vertex v is the smallest k such that v is k-simplicial. Clearly, we have $\sigma(v) \leq d(v)$, where $d(v)$ is the degree of v. By definition v is k-simplicial for all $k \geq \sigma(v)$. A *proper clique cover* of $N(v)$ is a covering of $N(v)$ by mutually disjoint non-empty cliques. So, if the simpliciality of v is m, then any proper clique cover of $N(v)$ consists of m or more cliques. Of course the number of cliques in a proper clique cover of $N(v)$ is at most $d(v)$.

Notice that our notion of a 1-simplicial vertex coincides with the now classical notion of "simplicial vertex" (a vertex of which the neighborhood is a single clique). This concept plays an central role in the algorithmic theory of chordal graphs [3]. Recall that a graph G is *chordal* if every cycle of length four or more in G has a chord – i.e., an edge joining two vertices of the cycle that are not consecutive on the cycle. The notion of a k-simplicial vertex was introduced by Jamison and Mulder [4], who showed that 3-simplicial vertices always exist in a certain class of graphs representable by sufficiently overlapping subtrees of a binary tree. This is analogous to the representation of chordal graphs by intersecting subtrees of a tree [2].

3 Planar Graphs

We now embark on the proof of the main result. Recall that the *order* of a graph is the number of its vertices and the *size* is the number of its edges.

Theorem 1 *Every planar graph $G = (V, E)$ of order at least four has at least four vertices that are both 3-simplicial and of degree at most 5.*

Proof. Assume the contrary, and let $G = (V, E)$ be a counterexample of minimum order, and amongst those of minimum order one of maximum size. Consider a fixed plane drawing of G. Let

$v = \lvert V\rvert$	be the number of vertices
v_i	be the number of vertices of degree i
$e = \lvert E\rvert$	be the number of edges
f	be the number of faces
f_i	be the number of faces with exactly i vertices.

For brevity, a vertex z in a plane graph that is both 3-simplicial and has $d(z) \leq 5$ will be called a *good vertex* of that graph. Any other vertex will be called *bad*. Note that G certainly has order at least five, since otherwise every vertex is good. We begin by establishing two claims of G that are needed for the effective application of Euler's formula in the main body of the proof.

Claim 1. *G is 2-connected.*

If G is disconnected, then we can join two components by an edge. This results in a planar graph which still has less then four good vertices but has more edges than G. Since this contradicts the maximality of the size of G, it follows that G is connected.

Suppose G has a cutvertex z. Let $H_1, H_2, \ldots, H_m$ denote the components of $G - z$, and let B_i denote the subgraph induced by $H_i \cup \{z\}$, for $i = 1, 2, \ldots m$. Note that $m \geq 2$. Now say B_i is a *small branch* (respectively, *big branch*) if B_i has order at most three (respectively, four or more). By the choice of G as a counterexample of minimal order, each big branch has at least four good vertices. Of course, in each small branch, all vertices are good. When the branches are glued back together at z, the only neighborhood that changes is that of z. Hence z is the only vertex whose simpliciality and degree can differ in G from what they are in the branches. Therefore each big branch will contribute at least three good vertices to G itself, and each small branch will contribute at least one. Thus since G is a counterexample, it has at most three branches and no big branch. It follows that G must consist of either 1) two branches of order two, or 2) three branches of order two, or 3) one branch of order three and one of order two. Since G is of order at least five, these are all impossible. Hence G contains four good vertices after all, which settles the proof of Claim 1.

Note that it follows from Claim 1 that G has no isolates or leaves, so that we have $v_0 = v_1 = 0$.

Since G is 2-connected, every edge lies on a cycle, which in the drawing is a Jordan curve separating the plane into two components. Each of these contains a face with the edge in its boundary. This implies that a face with i vertices on its boundary has exactly i edges on its boundary. So f_i also counts the number of faces with i edges on its boundary.

Claim 2. *$f_i = 0$ for all $i \geq 6$.*

The idea behind the proof is to try to find an edge to add to G to obtain a counterexample of larger size, contrary to the maximality of the size of G. Notice that if we join two non-adjacent vertices x and y of G by an edge, two things may happen. First, the neighborhoods of x and of y are enlarged by one vertex each. This cannot reduce the simpliciality of either x or of y. So if x is bad in G, then it remains bad after adding the edge xy. The other thing that may happen is that we may create a larger clique in the neighborhood of some vertex z and thereby decrease the simpliciality z. This is only possible if z is a common neighbor of x and y, and hence $d(x, y) = 2$. If z is good, then we do not care about decreasing its simpliciality. If z is bad and we make z good by adding xy to its neighborhood, then we say that we can *convert* z. Clearly a convertible vertex must have degree 4 or 5 and its simpliciality must be 4 and must go down to 3 by adding the new edge.

Now suppose that F is a face with 6 or more vertices. Without loss of generality we may assume that F is the outerface. Let $C = v_1 \to v_2 \to \ldots \to v_n \to v_1$ be the cycle bounding F, where C is drawn in the plane such that the labeling of C is clockwise. Note that *all other* edges of G lie in the interior of C. If any two vertices on C have distance at least three in G, then we can join them by an edge in the outerface without converting any bad vertex of G. As this would contradict the maximality of the size of G, we must have

$$d(v_i, v_j) \leq 2, \quad \text{for } 1 \leq i, j \leq n. \tag{1}$$

A *chord* of C is an edge of G in the interior region of C joining two non-consecutive vertices of C. A *short chord* of C is a chord joining two vertices of C whose indices differ by exactly 2 (mod n). A *long chord* is a chord of C that is not a short chord. We distinguish three cases.

Case 1. C has a long chord.
Without loss of generality we may assume that v_1v_k is a long chord, for some k with $4 \leq k \leq n-2$. Then this chord divides the interior of C into two regions. Hence v_i and v_j are not adjacent for any $1 < i < k$ and $k < j \leq n$. So, by (1), they are at distance two, that is, they must have a common neighbor. The only possible common neighbors of v_2 and v_{k+1} are v_1 and v_k. Without loss of generality, we may assume that v_1 is a common neighbor of v_2 and v_{k+1} (otherwise we relabel the vertices of C clockwise starting at v_k, that is v_k becomes v_1). So v_1v_{k+1} is a chord as well.

Now v_n and v_i can only have v_1 as common neighbor, for $3 \leq i \leq k-1$, so that v_1 is adjacent to $v_3, \ldots, v_{k-1}$. Hence v_{k-1} and v_j can only have v_1 as common neighbor, for $k+2 \leq j \leq n-1$, so that v_1 is adjacent to $v_{k+2}, \ldots, v_{n-1}$. So v_1 is adjacent to all other vertices of C. In particular, we have $d(v_1) \geq n-1$.

Assume that $n \geq 7$, so that $d(v_1) \geq 6$. Then v_1 is a bad vertex that cannot be converted by adding edges in its neighborhood. Now v_3 and v_{n-1} have only v_1 as common neighbor. So, if we join them in the outerface by a new edge, then no bad vertex is converted. Hence we have a counterexample with one more edge than G. This contradicts the maximality of the size of G.

So we have $n = 6$. Now v_2 and v_5 have only v_1 as common neighbor. So by maximality of the size of G, joining v_2 and v_5 by an edge must convert v_1 from bad to good. Thus $d(v_1) \leq 5$, and hence v_2, v_3, v_4, v_5, v_6 are precisely the neighbors of v, and $d(v_1) = 5$. But then $\{v_2, v_3\}$, $\{v_4\}$, and $\{v_5, v_6\}$ are three cliques covering $N(v_1)$, so v_1 is actually good in G. This contradiction settles Case 1.

Case 2. C has a short chord.
By Case 1, C has no long chords. Without loss of generality we may assume that v_1v_3 is a short chord of C in the interior of C. Now v_2 cannot be adjacent to v_j, for $4 \leq j \leq n$. So we have $d(v_2, v_j) = 2$, for $4 \leq j \leq n$. Notice that the only possible common neighbors of v_2 and v_{n-1} are v_1 and v_3.

If $n \geq 7$, then v_3v_{n-1} would be a long chord, so v_1 must be the common neighbor and v_1v_{n-1} is a short chord. In order for v_2 and v_5 to be at distance two, we need the short chord v_3v_5. But then v_4 and v_n cannot have distance two, contradicting (1). So we necessarily we have $n = 6$.

Note that the chord v_1v_3 prevents v_2 and v_5 from being adjacent. By the maximality of the size of G, we are not allowed to join v_2 and v_5 by an edge in the outerface. This means that v_2 and v_5 have a common neighbor that is bad in G but good in $G + v_2v_5$. There are only two possible common neighbors of v_2 and v_5: namely, v_1 in case v_1v_5 is a chord and v_3 in case v_3v_5 is a chord (or both if both chords occur). By symmetry, we may assume that v_1 is the convertible common neighbor of v_2 and v_5, so in particular, v_1v_5 is a chord.

Now v_1 being good in $G + v_2v_5$ means that $d(v_1) \leq 5$. But v_2, v_3, v_5, v_6 are four of the possibly five neighbors of v_1, and these are covered by the two cliques $\{v_2, v_3\}$ and $\{v_5, v_6\}$. If there is a fifth neighbor, it can be taken as a clique on its own. Thus we have a covering of $N(v_1)$ by at most three cliques, contradicting the assumption that v_1 is a convertible bad vertex. This settles Case 2.

Case 3. C has no chords.
Note that in this case non-consecutive vertices on C are at distance two. Moreover, for v_1 and v_4 to be at distance two, there must exist a common neighbor z of v_1 and v_4 in the interior of C. The path $v_1 \to z \to v_4$ divides the interior of C into two regions. Consider v_j with $4 < j \leq n-1$. Since G has no chords, v_j is not adjacent to v_1, and v_2 is not adjacent to v_4. So the only way that v_2 and v_j have a common neighbor is that they both are adjacent to z. By a similar argument v_3 and v_n are adjacent to z as well. Hence z and C induce an n-wheel, so that $d(z) \geq n \geq 6$, which implies that z is bad and cannot be converted. Therefore, since z is the only common neighbor of v_1 and v_4, we can join v_1 and v_4 by an edge in the outerface without converting any bad vertex. This contradicts the maximality of $|E|$, by which Case 3 is settled. This concludes the proof of Claim 2.

As in the proof of the result on the existence of four vertices of degree at most five, our computations are based on Euler's Formula $v - e + f = 2$ and the basic edge-counting formula:

$$\sum_{u \in V(G)} d(u) = 2e = \sum_{i \geq 1} i v_i. \tag{2}$$

Note that in (2) we may start the summation with $i = 2$, since G does not contain isolated vertices. As observed above, f_i counts the number of faces with exactly i edges on its boundary, so we also have

$$\sum_{i \geq 3} i f_i = 2e. \tag{3}$$

Then from (3) and Claim 2, we have

$$3f_3 + 4f_4 + 5f_5 = 2e \tag{4}$$

and

$$f_3 + f_4 + f_5 = f. \tag{5}$$

Using $v = \sum v_i$ we deduce from (2) that

$$\begin{aligned} 2e &= \sum i v_i = 2v_2 + 3v_3 + 4v_4 + 5v_5 + 6v_6 + 7v_7 + \ldots \\ &\geq 6v - v_5 - 2v_4 - 3v_3 - 4v_2. \end{aligned} \tag{6}$$

By Euler's formula and (5) we have

$$\begin{aligned} 2e &= 2v + 2f - 4 \\ &= 2v + 2(f_3 + f_4 + f_5) - 4. \end{aligned} \tag{7}$$

By (7) and (4) we have

$$\begin{aligned} 2v &= 2e - 2f_3 - 2f_4 - 2f_5 + 4 \\ &= f_3 + 2f_4 + 3f_5 + 4. \end{aligned} \tag{8}$$

Now by (4), (6), and (8) we get

$$\begin{aligned} 3f_3 + 4f_4 + 5f_5 &\geq 6v - v_5 - 2v_4 - 3v_3 - 4v_2 \\ &= 3f_3 + 6f_4 + 9f_5 + 12 - v_5 - 2v_4 - 3v_3 - 4v_2. \end{aligned}$$

Hence

$$v_5 + 2v_4 \geq 2f_4 + 4f_5 + 12 - 3v_3 - 4v_2. \tag{9}$$

Now let v_i^* denote the number of 3-simplicial vertices of degree i. From (9) we have

$$(v_5 - v_5^*) + 2(v_4 - v_4^*) \geq 2f_4 + 4f_5 - v_5^* - 2v_4^* + 12 - 3v_3 - 4v_2. \tag{10}$$

Recall that we are considering a plane drawing of $G = (V, E)$. Let N be the number of pairs (u, F) where u is a vertex of degree four or five that is not 3-simplicial and F is a face with 4 or 5 vertices and u is on F. Suppose $d(u) = 4$ and let x_1, x_2, x_3, x_4 be the neighbors of u in cyclic order in the plane embedding. If there is an edge between two x_i's, then they form a triangle with u and u is 3-simplicial. Hence if u is not 3-simplicial, there are no edges among the x_i's. But $x_i u$ and $u x_{i+1}$ (mod 4) are two consecutive edges of a face containing u. By Claim 1, these faces must all be different. Since none of these are triangles, u lies on at least four faces each with four or more sides. Thus the non-3-simplicial vertices of degree 4 contribute $4(v_4 - v_4^*)$ to N.

Let u be a vertex of degree 5 with x_1, x_2, x_3, x_4, x_5 as its neighbors in cyclic order in the plane. Suppose that at most two consecutive pairs x_i, x_{i+1} (mod 5) are *not* adjacent. Then u is 3-simplicial. So at least three consecutive pairs of neighbors of u are non-adjacent. Any of these pairs are on a 4-face or 5-face. By Claim 1, these faces are all distinct. So u contributes at least 3 to N. Hence we have

$$N \geq 3(v_5 - v_5^*) + 4(v_4 - v_4^*).$$

On the other hand, we have

$$4f_4 + 5f_5 \geq N.$$

Combining these two inequalities we get

$$4f_4 + 5f_5 \geq N \geq 3(v_5 - v_5^*) + 4(v_4 - v_4^*). \tag{11}$$

From (10) and (11) we have

$$4f_4 + 5f_5 \geq 4f_4 + 8f_5 - 3v_5^* - 4v_4^* + 24 - 6v_3 - 8v_2 + v_5. \tag{12}$$

This implies that

$$3v_5^* + 4v_4^* + 6v_3 + 8v_2 \geq 3f_5 + 24 + v_5, \tag{13}$$

so that

$$3v_5^* + 4v_4^* + 6v_3 + 8v_2 \geq 24, \tag{14}$$

so that

$$8(v_5^* + v_4^* + v_3 + v_2) \geq 24, \tag{15}$$

whence

$$v_5^* + v_4^* + v_3 + v_2 \geq 3. \tag{16}$$

Note that

$$\begin{aligned} v_5^* &= \text{number of 3-simplicial vertices of degree 5}, \\ v_4^* &= \text{number of 3-simplicial vertices of degree 4}, \\ v_3 &= \text{number of vertices of degree 3}, \\ v_2 &= \text{number of vertices of degree 2}. \end{aligned}$$

So, (16) asserts that the number of good vertices in G is at least three.

Finally, since G was supposed to be a counterexample, it follows that G has exactly three good vertices. Note that strict inequality in (14) implies strict inequality in (16). So equality in (16) implies equality in (14). By (15), this is only possible when $v_3 = 0$, $v_4^* = 0$, $v_5^* = 0$, and $v_2 = 3$. To have equality in (14) we need to have equality throughout our computations above. In particular, we have $f_5 = 0$ and $v_5 = 0$ by equality in (13), and then, by equality in (11), we have

$$4f_4 = N = 4v_4. \tag{17}$$

Now (17) implies that all vertices of degree four are 4-simplicial, and that each 4-face has four 4-simplicial vertices (all of degree four) on its boundary. Take any of the three vertices of degree two. Then it cannot be on a 4-face. By Claim 1 it must be on two faces. Hence these two faces must both be triangles. But this would imply the existence of a multiple edge in the simple graph G, which is impossible. This final impossibility concludes the proof of the theorem. □

The result in Theorem 1 is sharp as is shown by the following example.

Theorem 2 *For each composite, positive integer m, there is a plane graph G of order $4m+2$ with all faces bounded by 4-cycles, exactly four vertices of degree 2, and the other $4m-2$ vertices of degree 4.*

Proof. Let $m = (a+1)(b+1)$ be a composite number with $a \geq 1$ and $b \geq 1$. Now form a $(2a+1) \times (2b+1)$ rectangular grid which can be viewed as a plane drawing of the Cartesian product of the two paths P_{2a+1} and P_{2b+1}. Let $v_1, v_2, \ldots, v_{2a+1}$ be the vertices along the top row of this grid. As illustrated in the figure, place a vertices $w_1, w_2, \ldots, w_a$ in a vertical row above v_{a+1}, with w_1 at the top and w_a closest to v_{a+1}. Introduce another vertex u between v_{a+1} and w_a. Join the vertices $v_{a+1}, u, w_a, w_{a-1}, \ldots, w_1$ into a path. Now join v_i and v_{2a+2-i} to w_i for $1 \leq i \leq a$, to form "tents" over the top row of the grid.

Repeat this for each side of the grid. Now adjoin a vertex at infinity which is joined to the top tent vertex over each side. Figure 1 illustrates this construction for $a = 2$ and $b = 3$. Note that the vertex at infinity is not drawn. The four u-vertices are the vertices of degree 2. All other vertices are of degree 4, and all faces are 4-gons. It remains to count the number of vertices. There are

$(2a+1)(2b+1)$	vertices in the grid
$2a$	w-vertices extending out from the top and bottom
$2b$	w-vertices extending out from the sides
4	u-vertices of degree 2
1	vertex at infinity.

Thus the number of vertices is $4ab+2a+2b+1+2a+2b+4+1 = 4(ab+a+b+1)+2 = 2m+2$ as claimed. □

The next theorem is motivated by observing that the graph constructed in Theorem 2 (and shown in Figure 1 for m = 6) has no triangles and $2v_2 = 8$.

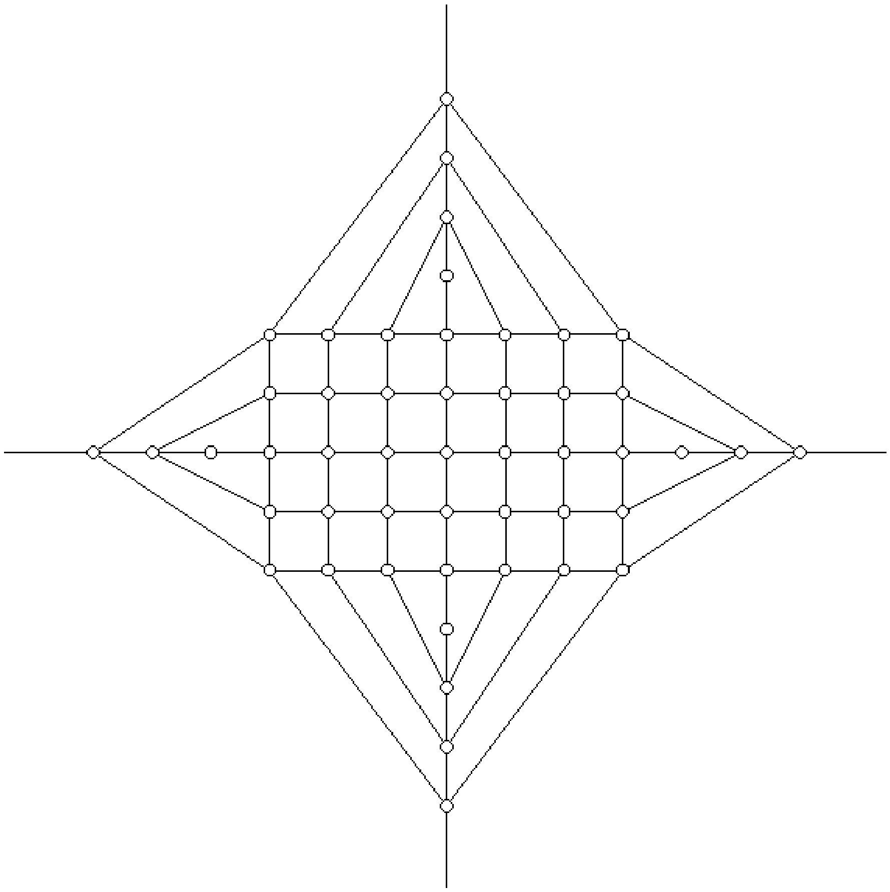

Figure 1: A planar graph with exactly four 3-simplicial vertices. (One of the vertices is at infinity.)

Theorem 3 *Let G be a triangle-free plane graph with $v_1 = 0$ and $f_i = 0$ for $i \geq 6$. Then $2v_2 + v_3 \geq 8 + f_5$, and this bound is achieved.*

Proof. From (4) and (2) of Theorem 1, we have

$$2e = 4f_4 + 5f_5 \tag{18}$$

$$2e = 2v_2 + 3v_3 + 4v_4 + 5v_5 + \ldots \tag{19}$$

By Euler's formula we have $v - e + f = 2$. Hence

$$(4v - 2e) - 2e + 4f = 8 \tag{20}$$

Substituting (18) and (19) respectively in (20) and noting that $\sum v_i = v$ and $f = f_4 + f_5$, we get

$$4(v_2 + v_3 + \ldots) - (2v_2 + 3v_3 + 4v_4 + \ldots) - 4(f_4 + 5f_5) + (4f_4 + f_5) = 8.$$

This gives $2v_2 + v_3 - (v_5 + 2v_6 + 3v_7 + \ldots) - f_5 = 8$. So, $2v_2 + v_3 \geq 8 + f_5$. Note that in Figure 1, G has $v_2 = 4$, $v_3 = 0$ and $f_5 = 0$ and so this graph G attains the lower bound. $\square$

The dodecahedron also achieves the the lower bound in Theorem 3, with $v_2 = 0, v_3 = 20$, and $f_5 = 12$.

4 Outerplanar Graphs

In this section we study the outerplanar case. It turns out that now we can find at least four 2-simplicial vertices of degree at most three, whenever the order is at least four, unless the graph is one of two exceptional graphs.

The *3-sun*, or the *triangle of triangles* is the graph on six vertices consisting of a central triangle and three extra vertices each adjacent to a different pair of vertices of the central triangle. In the terminology of the proof of Theorem 1 the 3-sun consists of a 6-cycle and three short chords pairwise sharing an end. The 3-sun is a chordal graph and a maximal outerplanar graph as well. The 3-sun has three vertices of degree two and three vertices of degree four. The vertices of degree four have simpliciality 3. The vertices of degree two have simpliciality 1, so that they are 2-simplicial as well. They are mutually non-adjacent. The star $K_{1,3}$ also has three mutually non-adjacent 2-simplicial vertices.

Theorem 4 *Let G be an outerplanar graph of order at least four. Then G contains at least four 2-simplicial vertices of degree at most three unless G is the 3-sun or $K_{1,3}$.*

Proof. Assume the contrary, and let G be a counterexample of minimum order, and amongst the counterexamples of minimum order one of maximum size. Now we call a vertex *good* if it is 2-simplicial and has degree at most three. Let G be embedded in the plane with an outerplanar embedding.

Claim. G is 2-connected.

If G is disconnected, then we can join two components by an edge. Thus we obtain an outerplanar graph which still has less then four good vertices but has more edges than G. Since this contradicts the maximality of the size of G, it follows that G is connected.

Suppose G has a cutvertex z. Let $H_1, H_2, \ldots, H_m$ denote the components of $G - z$, and let B_i denote the subgraph induced by $H_i \cup \{z\}$, for $i = 1, 2, \ldots m$. Note that $m \geq 2$. Again say B_i is a *small branch* (respectively, *big branch*) if B_i has order at most three (respectively, four or more). By the choice of G as a counterexample of minimal order, each big branch has at least four good vertices — unless it happens to be a 3-sun or $K_{1,3}$. As before all vertices are good in each small branch. When the branches are glued back together at z, the only neighborhood that changes is that of z. Hence z is the only vertex whose simpliciality and degree can differ in G from what they are in the branches. Therefore each big branch (except a 3-sun or $K_{1,3}$) will contribute at least three good vertices to G itself, and each small branch will contribute at least one. Thus since G is a counterexample, it has at most three branches and no big branch (except possibly a 3-sun or $K_{1,3}$). It follows that G must consist of either 1) two branches of order two, or 2) three branches of order two, 3) one branch of order three and one of order two, or 4) a 3-sun or $K_{1,3}$ branch and one branch of order two. The $K_{1,2}$ which arises in case 1) has only three vertices, and the $K_{1,3}$, which arises from case 2), is excluded by the theorem. Since the five graphs that arise from cases 3) and 4) all have four good vertices, this settles the proof of the Claim.

Let C be the cycle that is the boundary of the outerface. Then C is of length at least four. If $G = C$, then all vertices of G are of degree two, whence all are good. This is impossible, so C must have chords. Take any chord xy. Then $\{x, y\}$ is a cutset in G that cuts G into two components H_1 and H_2. Let G_i be the subgraph of G induced by xy and H_i, for $i = 1, 2$. If both G_1 and G_2 are of order at least four, then they both contain at least two good vertices distinct from x and y (note that in the 3-sun the three good vertices are mutually non-adjacent.) So G is not a counterexample after all. Hence at least one of G_1 and G_2 is of order three, say it consist of x, y and a third vertex z. Then z is necessarily a vertex of degree two in G with x and y as its neighbors. We call such a vertex z of degree two on the outerface with its neighbors x and y joined by a short chord a *cap* on the chord xy.

Thus we have shown that every chord of C implies the existence of a cap on that chord. The only way that a chord can have two caps is that G is a 4-cycle with exactly one chord. But this is not a counterexample. So each chord gives rise to exactly one cap. Clearly caps are good vertices. So G has at most three caps, whence has at most three chords. Let D be the cycle in G obtained after removing all the caps of G. If D is of length at least four, then there is an edge uv of D on the outerface. If u is of degree three then one of its neighbors is a cap, say w. Then both u and w are good vertices. If u is a vertex of degree two, then let w be the neighbor of u distinct from v. Then again u and w are two good vertices. Similarly, we find two good vertices amongst v and its neighbors distinct from u. Hence G has four good vertices. To avoid this D must be a triangle. If D has no caps, then G has only the three vertices of D. If D has one cap, then all four vertices of G are good. If D has two caps, then G is a 5-cycle with two chords and again has four good vertices. Finally, if every edge of D is capped, then D is the 3-sun, which was excluded in theorem as an exceptional case. This concludes the proof of the Theorem. □

Consider $K_2 \square P_n$, the cartesian product of an edge with the path on n vertices. This is a triangle-free outerplanar graph with exactly four vertices of degree two, $2n - 4$ vertices of degree three, and all (inner) faces 4-cycles. This yields an infinite family of graphs where Theorem 4 is best possible.

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