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On Performance Prediction of Cellular Telephone Networks

by

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ABSTRACT

This paper considers large cellular mobile networks in which the arrival rates of calls for different cells are different. The lower and upper bound of how much the given number of channels would satisfy the arriving calls are presented. Numerical examples are given.

1 Introduction

A cellular telecommunication network covers a certain geographic area, which is divided into regions or cells. In each cell there is a base station which uses radio frequencies to connect mobile users and the cable network. It might seem natural to choose a circle to represent the coverage area of a base station, but adjacent circles cannot be overlaid upon a map without leaving gaps or creating overlapping regions. So in literature we assume cells are hexagons. By using the hexagon geometry, the fewest number of cells can cover a geographic region, and the hexagon closely approximates a circular radiation pattern which would occur for an omni-directional base station antenna and free space propagation (Rappaport 1999).

In the last decade, the rapid development of new wireless services like digital cellular phone networks resulted in a run out of the most important resource, frequencies in the radio spectrum. Like with all scarcely available resources, the cost of frequency-use provides the need for economic use of the available frequencies. Reuse of frequencies within a cellular telecommunication network can offer considerable economies. However, reuse of frequencies also leads to loss of quality of communication links. The use of (almost) the same frequencies for multiple wireless communications can cause an interference between the signals that is unacceptable. The frequency assignment problem balances the economies of reuse of frequencies and the loss of quality in the network.

The frequency band $[f_{min}, f_{max}]$ available to some provider of wireless communication is usually partitioned in a set of channels, all with the same bandwidth δ of frequencies. For this reason the channels are usually numbered from 1 to a given maximum N, where $N = (f_{max} - f_{min})/\delta$. On each channel available one can communicate information from a transmitter to a receiver. For bidirectional traffic one needs two such channels, one for each direction. In the models considered in the literature the second channel is almost always ignored. The reasons for ignoring this aspect of the frequency assignment problem depends on the application. Instead of one band $[f_{min}, f_{max}]$, in most applications two bands $[f^1_{min}, f^1_{max}]$ and $[f^2_{min}, f^2_{max}]$ of N channels are available: one with the channels $\{1, ..., N\}$ and one with the channels $\{s+1, ..., s+N\}$, where $s >> N$. Thus, the backward connection uses a channel which is shifted s channels up. The choices of s values prevents any interference of backward channels with forward channels. As a consequence, each assignment for the forward channels can directly be transformed to an assignment for the backward channels with similar performance.

Before the actual frequency assignment strategy is provided to the network, it is important to examine how well the current available channels can serve the mobile users in the network.

In this paper we only consider FDMA (Frequency Division Multiple Access) strategy. The results from this paper can easily be extended to TDMA (Time Division Multiple Access) strategy. We assume that there are a total of fixed number of channels, say N, available to the network, and a channel can be used simultaneously in any two cells provided that these two cells are not immediate neighbors.

This paper is organized as follows: Section 2 attempts to characterize the probability distribution of the number of calls in each cell by using the tool of point processes and secondary processes generated by Poisson point processes. Section 3 presents the necessary and sufficient conditions for a feasible assignment in cellular telecommunication networks. Cluster-based network and general network are exploited in this section. Section 4 gives the lower and upper bounds of how well the supply of channels can serve the call demand by using the probability bounding technique. Numerical examples are presented in Section 5.

2 Probability Distribution of Number of Calls in the Cells

In this section we characterize the joint probability distribution of the number of calls in the cells in such a way that we take into account that the users perform random motions.

Let $\tau = 0$ be the initial time and suppose that at this time the users are located at various points of the geographic area we are looking at. We designate it by T and it may mean a portion of the plane, or, if it is appropriate from the modeling point of view, the entire plane. The type of the set, however, can be quite general in our discussion. It can be, e.g., a surface, a Euclidean space or an even more general space. We assume that in T a σ-algebra, $\mathcal{S}_T$ is given with the property that if $t \in T$, then $\{t\} \in \mathcal{S}_T$. We impose some additional condition on T and $\mathcal{S}_T$. In order to formulate it we need some definitions.

Definition 1. The finite system of sets $\mathcal{B} = \{A_1, ..., A_r\}$ is a decomposition of the set $A \in \mathcal{S}_T$, if $A_i \in \mathcal{S}_T$, $i = 1, ..., r$, $\cup_{i=1}^r A_i = A$ and $A_i \in \mathcal{S}_T$, $A_i \cap A_j = \phi$, if $i \neq j$.

Definition 2. Let $\mathcal{B}_1 = \{A_1^{(1)}, ..., A_{r_1}^{(1)}\}$, $\mathcal{B}_2 = \{A_1^{(2)}, ..., A_{r_2}^{(2)}\}$ be two decompositions of the set A. We say that $\mathcal{B}_1$ precedes $\mathcal{B}_2$, $\mathcal{B}_1 \prec \mathcal{B}_2$, if every $A_i^{(1)}$ can be decomposed by the use of some of the sets $A_j^{(2)}$, $j = 1, ..., r_2$.

Assumption. For every set $A \in \mathcal{S}_T$ there exists a sequence of subdivisions $\mathcal{B}_n$ such that $\mathcal{B}_n \prec \mathcal{B}_{n+1}$, $n = 1, 2, ...$, and if $t_1 \in A$, $t_2 \in A$, $t_1 \neq t_2$, then for large n these two points are separated by the sets in $\mathcal{B}_n$, i.e., $t_1 \in A_i^{(n)}$, $t_2 \in A_j^{(n)}$ for some $i \neq j$.

We assume that at time $\tau = 0$ the users are randomly distributed in T, i.e., they form a random point distribution. A random point distribution is a random selection of a finite number of points from T. We also call it a finite random point distribution. A σ-finite random point distribution is a random selection of a finite or countably infinite number of points such that there exists a decomposition of the set T into a countable number of pairwise disjoint sets A_1, A_2,... , such that $A_i \in \mathcal{S}_T$, $i = 1, 2, ...$ and each A_i contains a finite number of random points with probability 1, $i = 1, 2, ...$

Even though the number of users is finite, sometimes, it is appropriate to assume, from the mathematical point of view, that there are infinitely many random points. Let $X(A)$, $A \in \mathcal{S}$ designate the number of random points that fall in the set A.

Definition 3. A finite random point distribution in T is said to be of a Poisson type, if the following conditions hold:

(i) if $A_1, ..., A_n$ are pairwise disjoint subsets of T and all are elements of $\mathcal{S}_T$, then the random variables $X(A_1)$,...,$X(A_n)$ are independent;

(ii) there exists a finite measure $\lambda(A)$, $A \in \mathcal{S}_T$ such that for any A in $\mathcal{S}_T$ we have the equality

$$P(X(A) = k) \;=\; \frac{\lambda^k(A)}{k!} e^{-\lambda(A)},\; k = 0, 1, ... \tag{1}$$

Assumption (ii) means that $X(A)$ has Poisson distribution with expectation $\lambda(A)$.

Definition 4. A measure $\lambda(A)$, $A \in \mathcal{S}_T$ is called σ-finite if T can be decomposed into A_1, A_2,... such that $\lambda(A_i) < \infty$, $i = 1, 2, ...$

Definition 5. A random point distribution is of a σ-finite Poisson type, if condition (i) in Definition 3 holds and there exists a σ-finite measure $\lambda(A)$, $A \in \mathcal{S}_T$ such that for any $A \in \mathcal{S}_T$ with $\lambda(A) < \infty$, equation (1) holds.

A σ-finite random point distribution, that satisfies (i) in Definition 3, has the property that for any $A \in \mathcal{S}_T$, the random variable $X(A)$ is finite valued or infinite valued with probability 1. It is impossible that $0 < P(X(A) < \infty) < 1$. This fact follows from the $0 - 1$ law of Kolmogorov.

If T is a Euclidean space and $\lambda(A) = \lambda|A|$, where $|A|$ is the area of the set A or a Lebesgue measure of A, then the random point distribution is called homogeneous. If $T = R^2$, then $|A|$ is the area of A.

Mobile users move around in time and we look at this phenomenon as a secondary or marked Poisson process. If the random points chosen from T at time $\tau = 0$ are $t_1, t_2, ...$ and their random displacement vectors are $y_1(\tau), y_2(\tau), ..$, then the whole phenomenon is described by the pairs

$$(t_1, y_1(\tau)), (t_2, y_2(\tau)), ... \tag{2}$$

The positions of the new points are

$$t_1 + y_1(\tau), t_2 + y_2(\tau), ... \tag{3}$$

More generally, if we designate by $y_1, y_2, ...$ the entire trajectories as time functions of the random displacements as they happen in time, then the entire phenomenon (initial positions and motions) is described by the pairs

$$(t_1, y_1), (t_2, y_2), ... \tag{4}$$

Let Y designate the space from where the displacements are randomly chosen. Then the pairs (4) are chosen from the product space $Z = T \times Y$. We assume that a σ-algebra $\mathcal{S}_Y$ of some subsets of Y, together with a family of measures $\mu(C, t)$, $C \in \mathcal{S}_Y$, $t \in T$ is given, where $\mu(C, t)$ is the probability that if the motion starts at the point $t \in T$ at time $\tau = 0$, then the randomly chosen displacement y is an element of C.

Let $\mathcal{S}_Z$ be the Cartesian product of the σ-algebras $\mathcal{S}_T$ and $\mathcal{S}_Y$. This means that $\mathcal{S}_Z$ is the smallest σ-algebra in Z that contains the sets

$$\{\{t, y\} | t \in A \in \mathcal{S}_T, y \in C \in \mathcal{S}_Y\}. \tag{5}$$

Definition 6. A random point distribution, finite or σ-finite, is said to be non-atomic (or atomless), if

$$P_0(\{t\}) \;=\; P(X(\{t\}) = 0) \;=\; 1 \tag{6}$$

The following theorem provides us with a background for the assumption that users are distributed spatially according to a random point distribution of Poisson type.

Theorem 2.1 *(Prékopa 1958). If a random point distribution (finite or σ-finite) satisfies condition (i) in Definition (3), and is non-atomic, then it is of a (finite or σ-finite) random point distribution of Poisson type.*

The next "product space theorem" is fundamental from the point of view of applications of random point distributions. In this theorem the set Y is arbitrary, i.e., not necessarily the set of possible displacements of the random points.

Theorem 2.2 *(Prékopa 1958). Given a random point distribution (finite or σ-finite) in T. Suppose that to each random point t_i, $i = 1, 2, ...$, the corresponding elements y_i from Y, $i = 1, 2, ...$, are independent. Then the random point distribution (4) in the product space $T \times Y$ is of a Poisson type (finite or σ-finite). The expectation of $V(D)$, the number of random points in $D = A \times C$, $A \in \mathcal{S}_T$, $C \in \mathcal{S}_Y$ is given by*

$$E(V(D)) \;=\; \nu(D) \;=\; \int_C \mu(A - t, t)\lambda(dt) \tag{7}$$

and if D is an arbitrary element of $\mathcal{S}_Z$, then

$$E(V(D)) \;=\; \nu^*(D) \tag{8}$$

where ν^* is the unique extension of the measure ν from the product sets to all sets in $\mathcal{S}_Z$.

For the mobile users, distributed in $T = R^2$ or a part of it, Theorem 2.2 implies that if at time $\tau = 0$ the users are distributed according to a random point distribution of Poisson type, then the same holds at any time $\tau > 0$. We will state this fact in a more general form in Theorem 2.3. Now we apply Theorem 2.2 for the number of calls that hold out at time τ.

We assume that the mobile users in each cell i form a homogeneous random point distribution of Poisson type. In order to make the discussion more realistic, we also assume that each mobile user can be in two alternative states: in a *call* mode and in a *think* mode according to whether his/her cellular phone is in use or not. We assume that the modes are driven by a Markov Chain with two states. So the time spent in the *think* and the *call* modes have exponential distributions with parameters k_0 and k_1, respectively. The user is in a *call* or *think* mode with probability $\theta = \frac{k_0}{k_0+k_1}$ and $1-\theta$, respectively. Usually it can be assumed that θ is a constant (Baccelli et al. (1996)). Hence, we can assume that the number of calls that arrive at each cell i occurs according to a Poisson distribution with parameter $\theta\lambda_i$.

Let $\mathcal{S}_T$ and $\mathcal{S}_Y$ designate the Borel subsets of the sets T and Y, respectively. Then $\mathcal{S}_Z = \mathcal{S}_T \times \mathcal{S}_Y$ is the collection of all Borel subsets of Z.

If $X(A)$ is the number of calls that arrive in the set $A \subset [0, t_0]$, where $A \in \mathcal{S}_T$, and the holding times, that correspond to different calls, are independent, then it gives rise, by Theorem 2.2, to a σ-finite random point distribution in the product space Z.

We can also derive, by the use of the product space theorem, that the number of calls H_s that hold out at time s, has a Poisson distribution. In fact, $H_s = V(D)$, where

$$D = \{(t,y)|t+y \geq s, 0 \leq t \leq s, y \geq 0\}. \tag{9}$$

It remains to compute $E(H_s)$. By Theorem 2.2 we obtain that

$$E(H_s) = \theta\lambda_i \int_0^s (1 - F(s-\tau,\tau))d\tau. \tag{10}$$

If $F(y,t) = F(y)$, i.e., $F(y,t)$ is independent of t, then (10) reduces to

$$E(H_s) = \theta\lambda_i \int_0^s (1 - F(s-\tau))d\tau. \tag{11}$$

In the special case when the holding time is exponential, $F(y) = 1 - e^{-cy}$, for $y \geq 0$, we obtain from (11):

$$E(H_s) = \frac{\lambda_i\theta}{c}(1 - e^{-cs}). \tag{12}$$

We can also look at the case $T = (-\infty, \infty)$ and assume that the random point distribution representing the call process is homogeneous in T. In this case the set D changes to

$$D = \{(t,y)|t+y \geq s, t \leq s, y \geq 0\}, \tag{13}$$

the lower limit of the integrals (10)-(11) is $-\infty$, and for the case of $F(y,\tau) = F(y) = 1-e^{-cy}$ $(y \geq 0)$, (11) yields

$$E(H_s) = \frac{\lambda_i\theta}{c}. \tag{14}$$

Note that (14) is obtained from (12) by letting $s \to \infty$.

We can derive another important fact by the use of Theorem 2.2. This is expressed in

Theorem 2.3 *If at time $\tau = 0$, the users form a random point distribution of Poisson type (finite or σ-finite) in the set T, further they move independently of each other according to arbitrary random processes (that may depend on the user and also on the initial position), then at any time $\tau > 0$, the users form a random point distribution of Poisson type in T.*

Let $X_\tau(A)$ designate the number of users at time τ in the set $A \in \mathcal{S}_T$. Then we have

$$E(X_\tau(A)) \;=\; \int_C \mu(A-t,t)\lambda(dt), \tag{15}$$

where

$$C \;=\; \{(t,y)|y(\tau) \in A-t\}.$$

Proof . The assertions follow from Theorem 2.2 in a straightforward manner.

In the next theorem we assume that $T = R^n$, the n-dimensional Euclidean space. Even though contemporary applications need only the case $n = 2$, we state it in a general form.

Theorem 2.4 *Suppose that in $T = R^n$ a homogeneous random point distribution of Poisson type is given at time $\tau = 0$. Suppose further, that each random point performs a random motion in time and the displacement stochastic processes are independent, governed by the same probability distribution. Under these conditions the points form a homogeneous random point distribution at any time $\tau > 0$.*

Proof . We only have to show that the value in (7) is equal to $\lambda(A)$, where $|A|$ is the n-dimensional Lebesgue measure of the set A.

Since the displacements of the different points are governed by the same probability distribution, it follows that $\mu(A-t,t) = \mu(A-t)$. On the other hand the measure $\lambda(\cdot)$ is the Lebesgue measure multiplied by a positive constant λ. We may assume that Y is the set of possible displacements that the random points undergo until time τ. In fact, we are only interested in their positions at time τ. In this case $Y = R^n$. Let y designate the general element of $Y = R^n$. Then the right hand side of (7) can be written in the following way

$$\begin{aligned} \lambda \int_{R^n} \mu(A-t)dt \;&=\; \lambda \int_{R^n} \left(\int_{y \in A-t} d\nu \right) dt \\ =\; \lambda \int_{R^n} \left(\int_{t \in A-y} dt \right) d\nu \;&=\; \lambda \int_{R^n} |A| d\nu \;=\; \lambda |A|. \end{aligned}$$

This proves the theorem.

3 Necessary and Sufficient Conditions

Consider a cellular network which has a total of N duplex channels available for use. Assume the number of cells in the cellular network is C. First we investigate cellular telecommunication networks which employ cluster concept in the frequency reuse strategies. A set of S cells which collectively use the complete set of available frequencies is called a cluster. The factor S is called the cluster size and is typically equal to $4, 7$ or 12. For the sake of simplicity, we use cluster size 7 as an example in this paper.

As in Section 2, assume the number of calls in each cell i, denoted by X_i has a distribution of Poisson type with parameter $\frac{\lambda_i \theta}{c_i}$, independently of all other cells, where c_i is the parameter

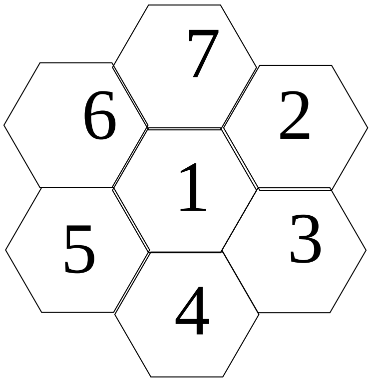

Figure 1: A Cellular Network with 7 Cells

of the call holding time. We assume that for each cell c_i is the same, i.e., $c_i = c$, $i = 1, 2, ..., C$. Let $\mathbf{X} = (X_1, ..., X_C)$ be the vector which designates the number of calls in the cellular network. Let $\mathbf{n} = (n_1, ..., n_C)$ be one realization of $\mathbf{X}$.

It is enough to examine the partial network with 7 cells if the cluster size for the overall cellular network is 7. That is, it consists of a central cell surrounded by six other cells (see Figure 1). The following theorem can easily be established.

Theorem 3.1 *For a cellular network with* 7 *cells, the six inequalities of the form* $n_i + n_j + n_k \leq N$ *for every* i, j *and* k *cells which share a vertex, provide us with the necessary and sufficient condition for* $\mathbf{n}$ *to be feasible.*

The condition $n_i + n_j + n_k \leq N$ where i, j and k are three cells which share a vertex in the above theorem is called a triple condition.

The necessity is easy to show. If $\mathbf{n}$ does not satisfy one of the six inequalities, that is, $n_i + n_j + n_k > N$ for some i, j, k, then there are at least two users from two immediate neighboring cells that have to use the same channel, which violates the assumption.

As to the sufficiency, we prove it by one construction method. We label every channel in the available channel list by $1, ..., N$, then we assign the channel 1 to channel n_1 to cell 1. We assign channel $n_1 + 1, ..., n_1 + n_2$, $n_1 + 1, ..., n_1 + n_4$, $n_1 + 1, ..., n_1 + n_6$ to cell $2, 4, 6$ respectively, and assign $N - n_3 + 1, ..., N$, $N - n_5 + 1, ..., N$, $N - n_7 + 1, ..., N$ to cell $3, 5, 7$ respectively. There is no overlapping in this assignment since n_i satisfy the six inequalities.

Now let's consider the cellular networks with more general channel assignment strategies, that is, there is no cluster concept when the channels are assigned to cells.

The triple conditions in Theorem 3.1 turn out to be only necessary conditions (Pallant and Taylor (1994)) by the following example:

Assume there are 19 cells in the cellular network shown in Figure 2, if $N = 2$ and

$$n_i = \begin{cases} 1 & \text{for } i = 2, 3, 4, 5, 6, 8, 16, 17, 18, 19. \\ 0 & \text{otherwise.} \end{cases}$$

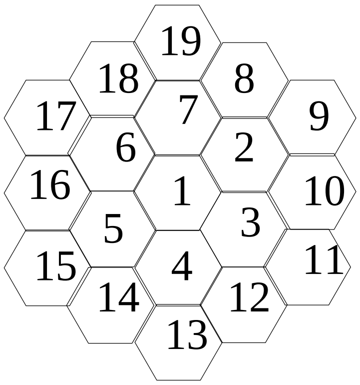

Figure 2: A Cellular Network with 19 Cells

the triple conditions are satisfied but there is no allocation of channels to calls which makes $\mathbf{n}$ feasible. To solve this problem, we introduce the following notations.

Let x_{ij} be the decision variable that cell i uses channel j, that is,

$$x_{ij} = \begin{cases} 1 & \text{if cell i uses channel j} \\ 0 & \text{otherwise.} \end{cases}$$

We can write down the following linear system:

$$\begin{array}{l} \sum_j x_{ij} = n_i, \quad \forall i = 1, 2, ..., C \\ x_{ji} + x_{ki} + x_{li} \leq 1, \quad \forall i = 1, 2, ..., N, \forall j, k, l. \end{array} \tag{16}$$

where cell j, k, l share a same vertex in the network.

The feasible solution of the above system is a feasible channel assignment for the cellular network.

Denote $\mathbf{x} = (x_{11}, x_{12}, ..., x_{1N}, x_{21}, ..., x_{2N}, ..., x_{CN})$, and

$$A_1 = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 1 \end{pmatrix}.$$

There are $C * N$ columns and C rows for matrix A_1. Notice that A_1 can be decomposed into C blocks while each block corresponds to one cell in the cellular network.

Assume there are altogether t triples in the network. We label triples by $k_1, ..., k_t$, respectively. Define A_2 the matrix which has $t * N$ rows and $C * N$ columns with the following property: each row corresponds to one pair of one triple and one channel, while each column corresponds to a pair of one cell and one channel. That is, if the two corresponding channels associated with row i and column j are the same and the cell number from column j is one of the cells which form the triple corresponding to row i, then $(A_2)_{i,j} = 1$, otherwise $(A_2)_{i,j} = 0$.

Hence, the above linear system can be written as follows:

$$\begin{aligned} A_1\mathbf{x} &= \mathbf{n} \\ A_2\mathbf{x} &\leq \mathbf{1} \end{aligned} \tag{17}$$

where $\mathbf{1}$ is a vector of size $t * N$ with all components equal to 1.

The relations (17) can be written in the forms:

$$\begin{aligned} A_1\mathbf{x} - \mathbf{n} &\geq \mathbf{0} \\ -A_1\mathbf{x} + \mathbf{n} &\geq \mathbf{0} \\ -A_2\mathbf{x} &\geq -\mathbf{1} \end{aligned} \tag{18}$$

Let

$$A = \begin{pmatrix} A_1 \\ -A_1 \\ -A_2 \end{pmatrix}$$

,

$$B = \begin{pmatrix} -I \\ I \\ \mathbf{0} \end{pmatrix}$$

,

$$b = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ -\mathbf{1} \end{pmatrix}$$

Then the above system can be written as

$$A\mathbf{x} + B\mathbf{n} \geq b. \tag{19}$$

In order to find the necessary and sufficient condition, i.e., the set for $\mathbf{n}$ to be feasible, we need to find the projection set of the above inequality system

$$L = \{\mathbf{n} | A\mathbf{x} + B\mathbf{n} \geq b\}. \tag{20}$$

Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ be a set of vectors that generate the convex polyhedral cone

$$C = \left\{\lambda | \lambda^T A = \mathbf{0}, \lambda \geq \mathbf{0}\right\}, \tag{21}$$

then after projection, we have

$$L = \left\{\mathbf{n} | \lambda_i^T B\mathbf{n} \geq \lambda_i^T b, \ i = 1, 2, ..., r\right\}, \tag{22}$$

which gives the necessary and sufficient conditions for $\mathbf{n}$ to be feasible. When we find the projection set, we assume that both $\mathbf{x}$ and $\mathbf{n}$ are continuous variables, actually they are both discrete variables. So this is just the first step we tackle this problem. It is extremely hard to find the projection set even in the continuous case.

4 Lower and Upper Bounds for the Probability That the Available Channels Can Serve the Call Demand

Let us associate the triple conditions in section 3 with the events $A_1, A_2, ...A_n$, where n is the total number of triple conditions. For instance, we associate the triple condition i, that is, $n_j + n_k + n_l \leq N$, with A_i where A_i is defined as the event that $X_j + X_k + X_l \leq N$.

How well the current available number of channels can serve the system can be approximated by $P(A_1 \cap ... \cap A_n)$. But it is computationally intensive to directly find $P(A_1 \cap ... \cap A_n)$. Below we describe the bounding technique we use to approximate $P(A_1 \cap ... \cap A_n)$. That is, we find the lower and upper bounds for $P(A_1 \cap ... \cap A_n)$, and if these two bounds are very close, then we can get the approximation for $P(A_1 \cap ... \cap A_n)$.

Instead of finding the lower and upper bound for $P(A_1 \cap ... \cap A_n)$, we find the bounds for $P(\bar{A}_1 \cup ... \cup \bar{A}_n)$, where $\bar{A}_i$ is the complementary event of A_i, $i = 1, 2, ..., n$. Then the lower (upper) bound for $P(A_1 \cap ... \cap A_n)$ is simply equal to $1-$ the obtained upper (lower) bound value of $P(\bar{A}_1 \cup ... \cup \bar{A}_n)$.

We only compute the individual probability for each $\bar{A}_i$ as well as some or all joint probabilities of up to m events, that is, $P(\bar{A}_{i1} \cap ... \cap \bar{A}_{im})$. Using this information, we give lower and upper bounds for $P(\bar{A}_1 \cup ... \cup \bar{A}_n)$.

Lower and upper bounds for the probability that at least one out of n events occurs, based on the knowledge of $S_1, ..., S_m$, were found by Bonferroni (1937). These bounds are not the best possible ones, i.e., not sharp. For the case of $m = 2$, the sharp lower bound for this probability was given by Dawson and Sankoff (1967). For the case of $m = 2$ the sharp upper boundwas given by Kwerel (1975a) and Sathe, Pradhan and Shah (1980). Kwerel (1975b) also derived the sharp lower and upper bounds for $m = 3$. For a general m, linear programs have been formulated and analyzed for the bounding problem, by Prékopa (1988, 1990). He also presented simple dual type algorithms to solve the problems. Boros and Prékopa (1989) utilized these results and presented closed form sharp bounds for the case of $m \leq 4$, and other closed form bounds. The sharp closed form bounds for $m \leq 4$ are the following:

Lower bound, using S_1, S_2 (Dawson, Sankoff (1967)):

$$P(A_1 \cup ... \cup A_n) \;\geq\; \frac{2}{h+1}S_1 \;-\; \frac{2}{h(h+1)}S_2, \tag{23}$$

where

$$h \;=\; 1 + \left\lfloor \frac{2S_2}{S_1} \right\rfloor.$$

Upper bound, using S_1, S_2 (Kwerel (1975a), Sathe, Pradhan and Shah (1980)):

$$P(A_1 \cup ... \cup A_n) \;\leq\; \min\{S_1 \;-\; \frac{2}{n}S_2,\; 1\}. \tag{24}$$

Lower bound, using S_1, S_2, S_3 (Kwerel (1975b), Boros and Prékopa (1989)):

$$P(A_1 \cup ... \cup A_n) \;\geq\; \frac{h+2n-1}{(h+1)n}S_1 \;-\; \frac{2(2h+n-2)}{h(h+1)n}S_2 \;+\; \frac{6}{h(h+1)n}S_3,$$

where

$$h \;=\; 1 \;+\; \left\lfloor \frac{-6S_3 + 2(n-2)S_2}{-2S_2 + (n-1)S_1} \right\rfloor . \tag{25}$$

Upper bound, using S_1, S_2, S_3 (Kwerel (1975b), Boros and Prékopa (1989)):

$$P(A_1 \cup ... \cup A_n) \;\leq\; \min\left(S_1 \;-\; \frac{2(2h-1)}{h(h+1)} S_2 \;+\; \frac{6}{h(h+1)} S_3,\; 1 \right), \tag{26}$$

where

$$h \;=\; 2 \;+\; \left\lfloor \frac{3S_3}{S_2} \right\rfloor .$$

Upper bound, using S_1, S_2, S_3, S_4 (Boros and Prékopa (1989)):

$$\begin{aligned} P(A_1 \cup ... \cup A_n) \;\leq\; \min\Bigg(S_1 \;&-\; \frac{2((h-1)(h-2) + (2h-1)n)}{h(h+1)n} S_2 \\ &+\; \frac{6(2h+n-4)}{h(h+1)n} S_3 \;-\; \frac{24}{h(h+1)n} S_4,\; 1 \Bigg), \end{aligned} \tag{27}$$

where

$$h \;=\; 1 \;+\; \left\lfloor \frac{(n-2)S_2 \;+\; 3(n-4)S_3 \;-\; 12S_4}{(n-2)S_2 \;-\; 3S_3} \right\rfloor .$$

A closed form lower bound for the last case also exists, but it is too complicated, therefore we disregard its presentation.

These bounds are obtained as optimum values of binomial moment linear programming problems (see Prékopa (1988,1990,1995)). Another type of LP we can use to find the bounds is the large scale Boolean probability bounding problem (see, e.g., Hailperin (1965) or Bukszar and Prékopa (2001)), but it may have huge size. Prékopa , Vizvári, Regös and Gao (2000) presented a third type of problem, which can be placed in between and can be regarded as aggregated problems, as compared to the Boolean problems, or disaggregated problems, as compared to the binomial moment problems. It can combine solvability and very good bounding performance, at least in many cases. The bounds are the following:

When $m = 2$,

$$P(A_1 \cup ... \cup A_n) \;\geq\; \sum_{i=1}^{n} \left(\frac{2}{h_i + 1} S'_{i1} \;-\; \frac{2}{h_i(h_i+1)} S'_{i2} \right), \tag{28}$$

where

$$h_i \;=\; 1 + \left\lfloor \frac{2S'_{i2}}{S'_{i1}} \right\rfloor, \quad i = 1, ..., n.$$

$$P(A_1 \cup ... \cup A_n) \;\leq\; \min\left(\sum_{i=1}^{n} (S'_{i1} \;-\; \frac{2}{n} S'_{i2}),\; 1 \right). \tag{29}$$

when $m = 3$,

$$P(A_1 \cup ... \cup A_n) \geq \sum_{i=1}^{n} \left(\frac{h_i + 2n - 1}{(h_i+1)n} S'_{i1} - \frac{2(2h_i + n - 2)}{h_i(h_i+1)n} S'_{i2} + \frac{6}{h_i(h_i+1)n} S'_{i3} \right), \qquad (30)$$

where

$$h_i \; = \; 1 \; + \; \left\lfloor \frac{-6S'_{i3} + 2(n-2)S'_{i2}}{-2S'_{i2} + (n-1)S'_{i1}} \right\rfloor, \quad i = 1, ..., n.$$

$$P(A_1 \cup ... \cup A_n) \; \leq \; \min \left(\sum_{i=1}^{n} \left(S'_{i1} \; - \; \frac{2(2h_i - 1)}{h_i(h_i+1)} S'_{i2} \; + \; \frac{6}{h_i(h_i+1)} S'_{i3} \right), \; 1 \right), \qquad (31)$$

where

$$h_i \; = \; 2 \; + \; \left\lfloor \frac{3S'_{i3}}{S'_{i2}} \right\rfloor, \quad i = 1, ..., n.$$

when $m = 4$,

$$P(A_1 \cup ... \cup A_n) \; \leq \; \min \left(\sum_{i=1}^{n} \left(S'_{i1} \; - \; 2\frac{(h_i - 1)(h_i - 2) + (2h_i - 1)n}{h_i(h_i+1)n} S'_{i2} \right.\right.$$
$$\left.\left. + \; 6\frac{2h_i + n - 4}{h_i(h_i+1)n} S'_{i3} \; - \; \frac{24}{h_i(h_i+1)n} S'_{i4} \right), \; 1 \right), \qquad (32)$$

where

$$h_i \; = \; 1 \; + \; \left\lfloor \frac{-12S'_{i4} + 3(n-4)S'_{i3} + (n-2)S'_{i2}}{(n-2)S'_{i2} - 3S'_{i3}} \right\rfloor, \quad i = 1, ..., n.$$

The bounds presented in Prékopa , Vizvári, Regös and Gao (2000) are even more general. However, we need only the above ones in the next section.

5 Numerical Examples

In this section we present numerical examples to show how the probability bounding technique mentioned in Section 4 can be used to find how well the supply of channels can satisfy the call demand in a cellular telecommunication network. In the examples we subdivide the collection of events $A_1, ..., A_n$ into two groups. For those events A_i which belong to the first group we create lower and upper bounds based on $S'_{i1}, S'_{i2}, S'_{i3}$. For those events A_j which are in the second group we use S'_{j1}, S'_{j2} to create the bounds. Below we discuss and present numerical results in connection with three subdividing strategies that we call order, greedy and passive. The order method means that we enlist each event that belongs to the first half of the sequence, written up in the original order, into the first group and all other events go to the second group. To describe the other two methods first we arrange the events in such a way that their probabilities form a decreasing sequence. The greedy method means

Table 1: Lower bounds for the example

Network	S_1, S_2	$S_1^{'}, S_2^{'}$	S_1, S_2, S_3	$S_1^{'}, S_2^{'}, S_3^{'}$	greedy	passive	order
1	0.934414	0.934414	0.9358725	0.9358726	0.9357356	0.934551	0.934551
2	0.9257033	0.9257033	0.9423515	0.9424524	0.9424309	0.9258247	0.9366772
3	0.8109305	0.8109305	0.8527109	0.8539949	0.8538911	0.8110344	0.8331048

Table 2: Upper bounds for the example

Network	S_1, S_2	$S_1^{'}, S_2^{'}$	S_1, S_2, S_3	$S_1^{'}, S_2^{'}, S_3^{'}$	greedy	passive	order
1	0.936002	0.9360009	0.9359372	0.9359372	0.9359604	0.9359777	0.9359777
2	0.9495501	0.949252	0.9486503	0.948498	0.9485024	0.9492476	0.9487634
3	0.874748	0.8715757	0.8735237	0.8707683	0.8707693	0.8715748	0.8711492

that the first (second) half of the events belongs to the first (second) group. The passive method does just the opposite, The first (second) half of the events belongs to the second (first) group.

Assume there are altogether 395 channels available in the network. We present three cellular telephone networks here.

Network 1: There are 7 cells in the network. In each cell i, the number of calls occurs according to a Poisson distribution with parameter λ_i. $\lambda_1 = 102$, $\lambda_2 = 108$, $\lambda_3 = 120$, $\lambda_4 = 115$, $\lambda_5 = 100$,$\lambda_6 = 136$, $\lambda_7 = 128$.

Network 2: There are 19 cells in the network. $\lambda_1 = 120$, $\lambda_2 = 100$, $\lambda_3 = 95$, $\lambda_4 = 125$, $\lambda_5 = 106$,$\lambda_6 = 132$, $\lambda_7 = 101$, $\lambda_8 = 99$, $\lambda_9 = 110$, $\lambda_{10} = 104$, $\lambda_{11} = 90$, $\lambda_{12} = 112$, $\lambda_{13} = 102$, $\lambda_{14} = 99$, $\lambda_{15} = 135$, $\lambda_{16} = 111$, $\lambda_{17} = 102$, $\lambda_{18} = 100$, $\lambda_{19} = 100$.

Network 3: There are 37 cells in the network. $\lambda_1 = 120$, $\lambda_2 = 100$, $\lambda_3 = 95$, $\lambda_4 = 125$, $\lambda_5 = 106$,$\lambda_6 = 132$, $\lambda_7 = 101$, $\lambda_8 = 99$, $\lambda_9 = 110$, $\lambda_{10} = 104$, $\lambda_{11} = 90$, $\lambda_{12} = 112$, $\lambda_{13} = 102$, $\lambda_{14} = 99$, $\lambda_{15} = 135$, $\lambda_{16} = 111$, $\lambda_{17} = 102$, $\lambda_{18} = 100$, $\lambda_{19} = 100$,$\lambda_{20} = 96$,$\lambda_{21} = 116$, $\lambda_{22} = 130$, $\lambda_{23} = 108$, $\lambda_{24} = 117$, $\lambda_{25} = 106$,$\lambda_{26} = 89$, $\lambda_{27} = 120$, $\lambda_{28} = 102$, $\lambda_{29} = 110$, $\lambda_{30} = 108$, $\lambda_{31} = 99$, $\lambda_{32} = 119$, $\lambda_{33} = 109$, $\lambda_{34} = 120$, $\lambda_{35} = 130$, $\lambda_{36} = 107$, $\lambda_{37} = 99$.

The lower bounds for the above three networks are presented in Table 1.

The upper bounds for the above three networks are presented in Table 2.

¿From Table 1 and Table 2, we know that the probabilities that a call demand is satisfied in networks 1, 2, and 3 are in the range from 0.9358726 to 0.9359372, from 0.9424524 to 0.948498, and from 0.8539949 to 0.8707683, respectively. The differences of the lower and upper bounds for networks 1, 2, and 3 are 0.0001, 0.006, and 0.02, respectively. In this way, we can approximate how well we can satisfy the demand for the whole network. We also observe that if we use the greedy method mentioned above to calculate the lower and upper bounds, we can get very close results compared to the results obtained by using S_1, S_2, S_3, while we save a lot of computation time.

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