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Fuzzy outranking relations in ELECTRE providing manageable disaggregation procedures

by

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Abstract

In ELECTRE methods, the construction of an outranking relation S amounts at validating or invalidating, for any pair of alternatives $(a, b) \in A$, an assertion aSb. This comparison is grounded on the evaluation vectors of both alternatives, and on additional information concerning the DM's preferences, accounting for two conditions: concordance and non-discordance.

In decision processes using these methods, the analyst should interact with DM(s) in order to elicit values for preferential parameters. This can be done either directly or through a disaggregation procedure that infers the parameters values from holistic judgements provided by the DM(s). Inference is usually performed through an optimization program that accounts for the aggregation model and minimizes an "error function". Although disaggregation approaches have been largely used in additive models, only few advances have been made towards a disaggregation approach for ELECTRE methods. This probably reflects the "optimization unfriendly" character of the most recent ELECTRE methods.

In this paper we are concerned with a slight adaptation of the fuzzy outranking relation used in the ELECTRE III and ELECTRE TRI that preserves the original ideas and is more optimization-friendly for parameter inference programs. Such modification is shown to preserve the original discordance concept. We show that the modified outranking relation makes it easier to solve inference programs.

Keywords: *Fuzzy Outranking Relations, Veto effect, Parameter Inference Procedures, ELECTRE*

Introduction

Let us consider a decision situation involving a finite set of alternatives $A = \{a_1, a_2, ..., a_l\}$ evaluated on n criteria $g_1, g_2, ..., g_n$[1], ($F = \{1, 2, ..., n\}$ denotes the set of criteria indices). In the field of Multiple Criteria Decision Aiding (MCDA), a class of methods ground the recommendations to the Decision Maker(s) (DM(s)) on the construction of one (or several) binary relation(s) representing the preference among pairs of alternatives (see [19]) rather than on the construction of a synthesizing utility function (see [7]). These methods are usually referred as outranking methods in the MCDA literature and belong to the so-called European school of MCDA (see [21]).

The construction of an outranking relation S amounts at validating or invalidating, for any pair of alternatives $(a, b) \in A^2$, an assertion aSb, whose meaning is "a is at least as good as b" or synonymously "a is not worse than b". This comparison is grounded on the evaluation vectors of both alternatives a and b, *i.e.*, $(g_1(a), g_2(a), ..., g_n(a))$ and $(g_1(b), g_2(b), ..., g_n(b))$, and on additional information concerning the DM's preferences. To validate a statement aSb, two basic conditions should be verified: concordance and non-discordance (or non-veto).

A criterion g_k is said to be concordant with the assertion aSb if a is at least as good as b with respect to criterion g_k. The concordance condition is fulfilled for the assertion aSb iff the subset of criteria concordant with aSb is "sufficiently" large. A criterion g_k is said to oppose a veto to the assertion aSb if the difference of evaluation $g_k(b) - g_k(a)$ is incompatible with the assertion aSb, whatever the evaluation on the other criteria. The non-discordance condition is fulfilled iff no criterion opposes a veto to the assertion aSb.

Several outranking methods using these concepts of concordance and non-discordance (see [16]) have been proposed in the literature (see [1], [19], [22], [15], [9], [14]) and put these concepts into a concrete form in different ways.

In a decision process using these methods, the analyst usually interacts with the DM(s) in order to elicit values for preference-related parameters. This can be done either directly or through a disaggregation procedure that infers the parameters values from holistic preferences provided by the DM(s) (see [6]). Inference is usually performed through an optimization program that accounts for the aggregation model and minimizes an "error function". This disaggregation approach has been largely used in additive models (e.g. see [5]). However, only few advances have been made towards a disaggregation approach for outranking methods (see [10], [12], [8]). A possible reason for this is that the outranking methods, namely the well-known Electre methods [20], originated from real-world applications and are rather "optimization unfriendly".

[1] We will assume without loss of generality that preferences are increasing with the value on g_j, *i.e.*, the greater $g_j(a)$ the better a.

In this paper we are concerned with some adaptations of the fuzzy outranking relation used in Electre III (see [18]) and Electre Tri (see [23], [20]) that preserves the original ideas and is more optimization-friendly for parameter inference programs. The modifications are also useful for robustness analysis [3] and other approaches [10]. The paper is organized as follows. The first section will describe how the Electre III and Electre Tri methods define the outranking relation S emphasizing the nature of the difficulties for disaggregation procedures. Modifications of Electre III's outranking relation in the way it accounts for veto phenomena are proposed in section 2. Such modifications are shown to preserve the original discordance concept. Section 3 shows that the modified outranking relations make it easier to solve inference programs, namely for the Electre Tri method.

1 The Fuzzy Outranking in Electre III

1.1 Outranking relations for a single criterion

Electre III builds a fuzzy outranking relation S_j restricted to a single criterion for each criterion g_j. $S_j(a,b)$ is defined by (1) on the basis of $g_j(a)$, $g_j(b)$ and two thresholds functions: indifference $q_j(g_j))$ and preference $p_j(g_j))$ $(0 \leq q_j(g_j)) < p_j(g_j))^2)$. $S_j(a,b)$ represents the degree to which alternative a outranks (is at least as good as) b (see Figure 1).

$$S_j(a,b) = \frac{p_j(g_j(a)) \ - \ min\{g_j(b) - g_j(a), p_j(g_j(a))\}}{p_j(g_j(a)) \ - \ min\{g_j(b) - g_j(a), q_j(g_j(a))\}} \qquad (1)$$

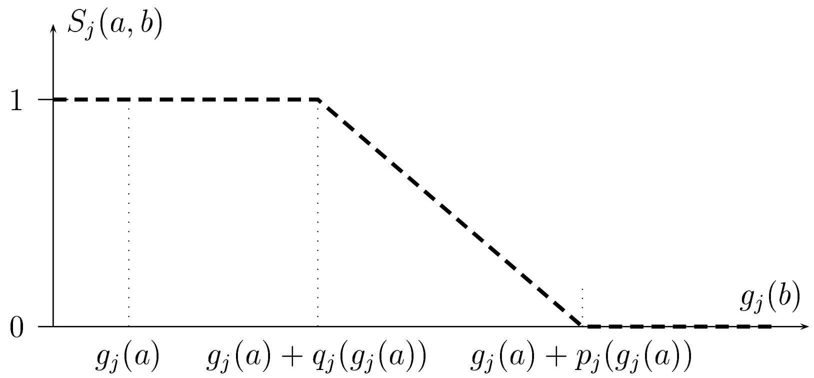

Figure 1: Partial valued outranking relation

1.2 Concordance relation

The fuzzy concordance relation $C(a,b)$ is grounded on the relations S_j $(j = 1, 2, ..., n)$ and represents the level of majority among the criteria in favor of the assertion "a is at least

[2] We will consider $p_j(g_j) < v_j(g_j)$, although Electre also consider the case $p_j(g_j) = v_j(g_j)$

as good as b". When computing this majority level, each criterion g_j has a weight $w_j \geq 0$ representing its voting power. Without any loss of generality, we will consider $\sum_{j=1}^{n} w_j = 1$. Therefore, $C(a, b)$ can be written as follows:

$$C(a,b) = \frac{1}{\sum_{j=1}^{n} w_j} \sum_{j=1}^{n} w_j.S_j(a,b) = \sum_{j=1}^{n} w_j.S_j(a,b) \tag{2}$$

1.3 Discordance relation for a single criterion

Electre III builds a fuzzy discordance relation d_j restricted to a single criterion for each criterion g_j. $d_j(a, b)$ is defined by (3) on the basis of $g_j(a)$, $g_j(b)$, a veto threshold function $v_j(g_j))$ and a preference threshold function $p_j(g_j))$ $(p_j(g_j)) > v_j(g_j))^3)$ (see Figure 2).

$$d_j(a,b) = 1 - \frac{v_j(g_j(a)) - min\{g_j(b) - g_j(a), v_j(g_j(a))\}}{v_j(g_j(a)) - min\{g_j(b) - g_j(a), p_j(g_j(a))\}} \tag{3}$$

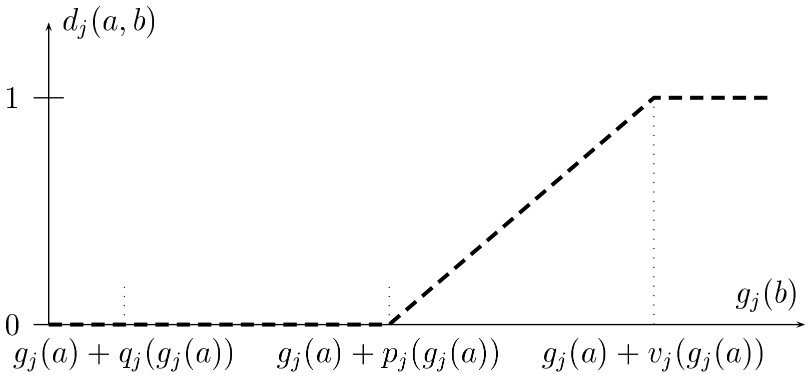

Figure 2: Partial valued outranking relation

1.4 Overall non-discordance relation

The fuzzy discordance relation $ND(a, b)$ is grounded on $C(a, b)$ and on the relations d_j, $j = 1, 2, ..., n$; it represents the degree to which the minority criteria (*i.e.*, such that bP_ja) collectively oppose a veto to the assertion "a is at least as good as b". A classical way of defining $ND(a, b)$ is given in (4). $ND(a, b) = 0$ corresponds to a situation where the minority criteria are totally opposed to aSb whereas $ND(a, b) = 1$ means that none of the criteria oppose a veto to aSb.

$$ND(a,b) = \prod_{j \in \overline{F}} \frac{1 - d_j(a,b)}{1 - C(a,b)} \quad \text{where } \overline{F} = \{j \in F / d_j(a,b) > C(a,b)\} \tag{4}$$

[3]We will consider $p_j(g_j) < v_j(g_j)$, although Electre also consider the case $p_j(g_j) = v_j(g_j)$

We give hereafter in (5)-(7) an equivalent formula to define $ND(a,b)$. Such definition will be helpful in order to compare $S(a,b)$ with the new outranking relation $S'(a,b)$ we define in section 2. Let us define $ND_j(a,b), j \in F, a,b \in A$ as

$$ND_j(a,b) = \begin{cases} 1 & \text{if} \quad d_j(a,b)) \leq C(a,b) \\ \frac{1-d_j(a,b)}{1-C(a,b)} & \text{if} \quad d_j(a,b)) > C(a,b) \end{cases} \tag{5}$$

Definition (5) is equivalent[4] to :

$$ND_j(a,b) = Min\left\{1, \frac{1-d_j(a,b)}{1-C(a,b)}\right\} \tag{6}$$

We can then formulate $ND(a,b)$ as in (7) (equivalent to (4)) in which the product considers each criterion in F.

$$ND(a,b) = \prod_{j \in F} ND_j(a,b) \tag{7}$$

1.5 Fuzzy outranking relation

Electre III combines the concordance and non-discordance relations in order to define the outranking relation S as shown in (8)

$$S(a,b) = C(a,b) \; . \; ND(a,b) \tag{8}$$

From the fuzzy outranking relation $S(a,b)$, it is possible to define a family of nested crisp outranking relation S_λ; these crisp relations correspond to λ-cuts of $S(a,b)$, where the cutting level $\lambda \in [0.5, 1]$ represents the minimum value for $S(a,b)$ so that $aS_\lambda b$ is true.

1.6 Discussion on the Electre III fuzzy outranking relation

1.6.1 Implementation of the non-discordance principle

Let us analyze the way the non-discordance condition is implemented through $ND(a,b)$. If $g_j(b) - g_j(a)$ exceeds $v_j(g_j(a))$ for at least one criterion then aSb is invalidated, *i.e.*, $\exists j \in F : d_j(a,b) = 1 \Rightarrow S(a,b) = 0$. Furthermore, the partial discordance indices $d_j(a,b)$ are defined in such a way that veto effects (*i.e.*, situations in which $d_j(a,b) > 0$) can occur even when $g_j(b) - g_j(a) < v_j(g_j(a))$. However, so as to avoid to account for low values of $d_j(a,b)$, the overall non-discordance relation defined in (4) consider the $d_j(a,b)$ only for criteria such that $d_j(a,b) > C(a,b)$.

[4] when $d_j(a,b) > C(a,b)$, then it holds $\frac{1-d_j(a,b)}{1-C(a,b)} < 1$, thus $ND_j(a,b) = \frac{1-d_j(a,b)}{1-C(a,b)}$; when $d_j(a,b) \leq C(a,b)$, then it holds $\frac{1-d_j(a,b)}{1-C(a,b)} \geq 1$, thus $ND_j(a,b) = 1$. Let us remark that we can state $C(a,b) < 1$, as the case $C(a,b) = 1$ corresponds to a situation where no discordant criterion exists

Another specific feature of $ND(a,b)$ lies in the fact that its value accounts both for the values of $d_j(a,b)$ and $C(a,b)$: the way $ND(a,b)$ accounts for $d_j(a,b)$ is amplified when $C(a,b)$ is low. The reason for this is that a veto situation should be accentuated when the concordance relation is not firmly established.

1.6.2 Difficulties with integrating $S(a,b)$ in inference programs

In order to elicit values for preference-related parameters (*i.e.*, w_j, $v_j(g_j)$, $p_j(g_j)$, $q_j(g_j)$, and limits of categories in Electre Tri) it is possible to proceed using a disaggregation procedure that infers the parameters values from holistic preferences provided by the DM(s). Hence, it is necessary to formalize $S(a,b)$ through an optimization program that minimizes an "error function". However, $S(a,b)$ is rather "optimization unfriendly". Difficulties arise mainly from the way the non-discordance condition is implemented, *i.e.*, the way $ND(a,b)$ is defined.

More precisely, two features of the non-discordance relation are concerned. First, the subset of criteria $\overline{F}$ (see (4)) is difficult to integrate into an optimization program. Second, the fact that $C(a,b)$ intervenes in the definition of $ND(a,b)$ implies that the optimization program will necessarily be non-linear, even when all the parameters are fixed except the weights.

Previous research [2] studying $S(a,b,T)$ under imprecise information (T being a polytope of possible values for the parameters involved in $S(a,b)$) has shown that this continuous, non-differentiable, non-linear function is quasi-concave in the domain where it is strictly positive, when a and b are fixed. One consequence of this result is that a constraint like $S(a,b,T) < \lambda$ does not define a convex set.

2 New Electre-like Fuzzy outranking relations

The definition of these outranking relations originated from previous unpublished work that has been presented in [11]. The modified outranking relations $S'(a,b)$ and $S''(a,b)$ aims at providing an easier way for $S'(a,b)$ and $S''(a,b)$ to be integrated in inference optimization programs. Therefore, $S'(a,b)$ and $S''(a,b)$ will differ from $S(a,b)$ only by its implementation of the discordance concept: new non-discordance relations are defined $ND'(a,b)$ and $ND''(a,b)$ (the outranking relations restricted to a single criterion $S_j(a,b)$ and the overall concordance relation $C(a,b)$ remain identical). Furthermore, we will define $S'(a,b) = C(a,b).ND'(a,b)$ and $S''(a,b) = C(a,b).ND''(a,b)$.

2.1 Definitions

2.1.1 Partial discordance indices $d'_j(a,b)$

$d'_j(a,b)$ is defined by (9) on the basis of $g_j(a)$, $g_j(b)$, a veto threshold function $v_j(g_j))$ and an additional threshold function $u_j(g_j))$ (such that $p_j(g_j)) \leq u_j(g_j)) < v_j(g_j))$[5]). $u_j(g_j(a))$ represents the difference of evaluation $g_j(b) - g_j(a)$ above which the discordance condition starts to weaken concordance $C(a,b)$ in the definition of $S(a,b)$. Hence, $d'_j(a,b)$ represents the degree to which criterion g_j opposes a veto to the assertion aSb (see Figure 3). The new threshold $u_j(g_j)$ can be considered either

- as an additional preferential parameter to be elicited either directly through an interaction with the DM, or indirectly using a disaggregation procedure, or
- as a fixed technical parameter (rather than a preference-related one) that defines the extent to which differences of evaluation $g_j(b) - g_j(a) < v_j(g_j(a))$ should (or should not) weaken the concordance $C(a,b)$ in the definition of $S(a,b)$ (a reasonable value for u_j depending on p_j and v_j is discussed in section 2.2).

$$d'_j(a,b) = 1 - \frac{v_j(g_j(a)) - min\{g_j(b) - g_j(a), v_j(g_j(a))\}}{v_j(g_j(a)) - min\{g_j(b) - g_j(a), u_j(g_j(a))\}} \tag{9}$$

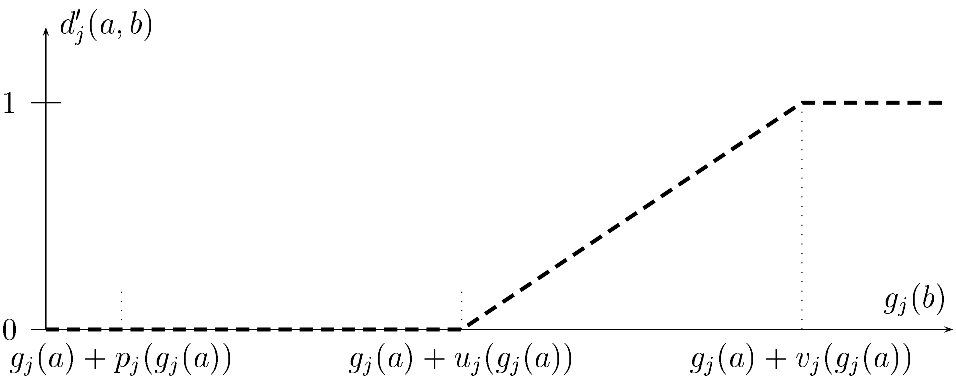

Figure 3: Partial discordance relation $d'_j(a,b)$

2.1.2 Overall non-discordance relation $ND'(a,b)$

The fuzzy non-discordance relation $ND'(a,b)$ is grounded on the relations $d'_j(a,b)$, $j = 1,2,...,n$. It is defined hereafter in (10). It should be noted that criteria that intervene in the product are not restricted to those for which $d'_j(a,b) > C(a,b)$, *i.e.*, small values of

[5] we consider $u_j(g_j) < v_j(g_j))$ although $u_j(g_j) = v_j(g_j))$ can also be considered as an extreme case in which discordance is effective only when the veto threshold is exceeded

$d'_j(a,b)$ will impact $ND'(a,b)$. Moreover, the concordance relation $C(a,b)$ does not intervene in the non-discordance implementation.

$$ND'(a,b) = \prod_{j \in F} 1 - d'_j(a,b) \tag{10}$$

This fuzzy non-discordance relation $ND'(a,b)$ can be defined equivalently as formulated in (11)-(12), where $\alpha_j \in [0,1[, j \in F$ is a parameter that should be defined such that $u_j = p_j + \alpha_j.(v_j - p_j)$. The values for α_j can be either equal for each $j \in F$, or distinct if the DM(s) has specific arguments to account for "weak veto situations" differently among criteria.

$$ND'(a,b) = \prod_{j \in F} ND'_j(a,b) \tag{11}$$

$$ND'_j(a,b) = min\left\{1, \frac{1 - d_j(a,b)}{1 - \alpha_j}\right\} \tag{12}$$

2.1.3 Overall non-discordance relation $ND''(a,b)$

The fuzzy non-discordance relation $ND''(a,b)$ is grounded on the relations $d'_j(a,b)$, $j = 1,2,...,n$ (9). It is defined hereafter in (13). It should be noted that criteria that intervene in the Min operator are not restricted to those for which $d'_j(a,b) > C(a,b)$. Moreover, the concordance relation $C(a,b)$ does not intervene in the non-discordance implementation.

$$ND''(a,b) = Min_{j \in F}\left(1 - d'_j(a,b)\right) \tag{13}$$

2.2 Comparative analysis

As already mentioned, $S(a,b)$, $S'(a,b)$ and $S''(a,b)$ differ only on the way the non-discordance condition is implemented. Let us recall that the modifications intervening in $S'(a,b)$ and $S''(a,b)$ aim at providing easier ways to integrate the outranking relation in disaggregation procedures. It should be emphasized that $S'(a,b)$ is closely related to $S(a,b)$; the similarity appears when comparing (12) and (6). As $S''(a,b)$ is very similar to $S'(a,b)$ ($S''(a,b)$ account for the "strongest" veto, whereas $S'(a,b)$ can account for several veto effects, as $S(a,b)$ does), we will mainly focus on the comparison between $S(a,b)$ and $S'(a,b)$. Two modifications have been introduced in S' as compared to S:

- $ND'(a,b)$ account for $d'_j(a,b)$ only. Hence a veto situation is not accentuated when the concordance relation is not firmly established as it is the case in $ND(a,b)$. Although disabling this feature removes some refinements, it also distinguishes more clearly the way the two concepts of concordance and non-discordance are implemented. Moreover, a low value for $C(a,b)$ still impact directly $S'(a,b)$, but not indirectly through $ND'(a,b)$ as it is the case with $S(a,b)$.

- As in $S(a,b)$, $S'(a,b)$ avoids to account for discordance situations that are not firmly established. Such feature is done in $S(a,b)$ by accounting in $ND(a,b)$ for discordant criteria such that $d_j(a,b) > C(a,b)$ only. $S'(a,b)$ also do not account for "weak veto": although $ND'(a,b)$ considers all discordant criteria ($ND'(a,b) = \prod_{j \in F} 1 - d'_j(a,b)$), the way $d'_j(a,b)$ are defined induce lower values (as compared to $d_j(a,b)$) for the same difference $g_j(b) - g_j(a)$, namely $d'_j(a,b) = 0, \forall a,b \in A$ such that $g_j(b) - g_j(a) \leq u_j(g_j(a))$. Figures 4 and 5[6] show how $ND(a,b)$ and $ND'(a,b)$ vary as a function of $g_j(b)$ when g_j is the only discordant criterion, *i.e.*, depict $ND_j(a,b)$ and $ND'_j(a,b)$.

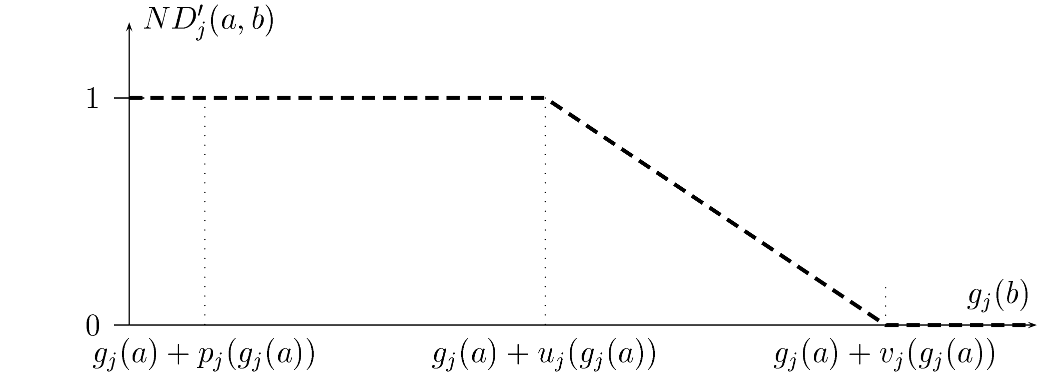

Figure 4: $ND'_j(a,b)$ as a function of $g_j(b)$

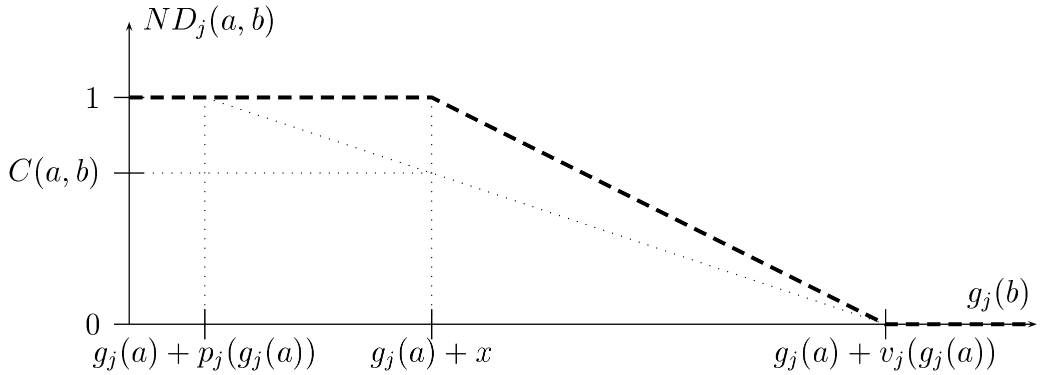

Figure 5: $ND_j(a,b)$ as a function of $g_j(b)$

One important question related to the way $S(a,b)$ and $S'(a,b)$ implement discordance is the following: if u_j is not considered as a preference parameter, how should it be defined in order for $S(a,b)$ and $S'(a,b)$ to be "as close as possible"?

Let us remark that, for $S(a,b)$ and $S'(a,b)$, it holds:

[6]in Figure 5, x corresponds to $p_j(g_j(a)) + C(a,b).(v_j(g_j(a)) - p_j(g_j(a)))$ and hence varies in the interval $[p_j(g_j(a)), v_j(g_j(a))]$ as $C(a,b)$ varies in $[0,1]$.

- discordance operates by weakening $C(a,b)$,
- for anyλ-cut S_λ (S'_λ, respectively) of $S(a,b)$ ($S'(a,b)$, respectively), the assertion $aS_\lambda b$ ($aS'_\lambda b$, respectively) can not hold for any (a,b) such that $C(a,b) < 0.5$.

It follows from these two preliminary remarks that it is sufficient so as to compare $S(a,b)$ and $S'(a,b)$ to restrict the analysis to the pairs (a,b) such that $C(a,b) \geq 0.5$.

In order for $S'(a,b)$ to be "close" to $S(a,b)$, $ND(a,b)$ should not differ too much from $ND'(a,b)$, *i.e.*, each $ND_j(a,b)$ should not be far from $ND'_j(a,b)$. Hence, we should define u_j so that $f(C(a,b), g_j(b)) = |ND_j(a,b) - ND'_j(a,b)|$ is as small as possible in average, when $C(a,b) \in [0.5, 1]$ and $g_j(b) \in [g_j(a) + p_j(g_j(a)), g_j(a) + v_j(g_j(a))]$ (Figures 6 and 7 represent ND_j and ND'_j as functions of $C(a,b)$ and $g_j(b)$). In order to do so we should set $\alpha_j = 0.75$, *i.e.*, $u_j = p_j(g_j(a)) + 0.75.(v_j(g_j(a)) - p_j(g_j(a)))$[7].

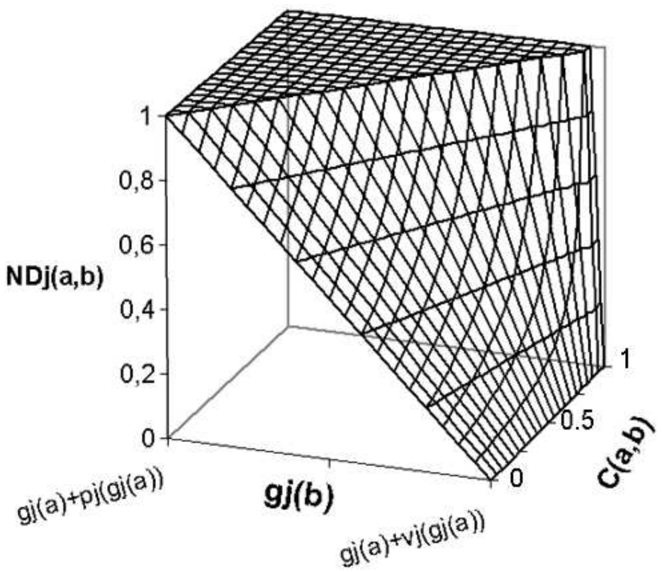

Figure 6: $ND_j(a,b)$ as a function of $g_j(b)$ and $C(a,b)$

[7]the value $\alpha_j = 0.75$ is obtained by minimizing the following expression $Min_{\alpha\in[0,1]} \left\{ \int_{0.5}^{1} \int_{g_j(a)+p_j(g_j(a))}^{g_j(a)+v_j(g_j(a))} f(C(a,b), g_j(b), \alpha)\; dC(a,b)\; dg_j(b) \right\}$

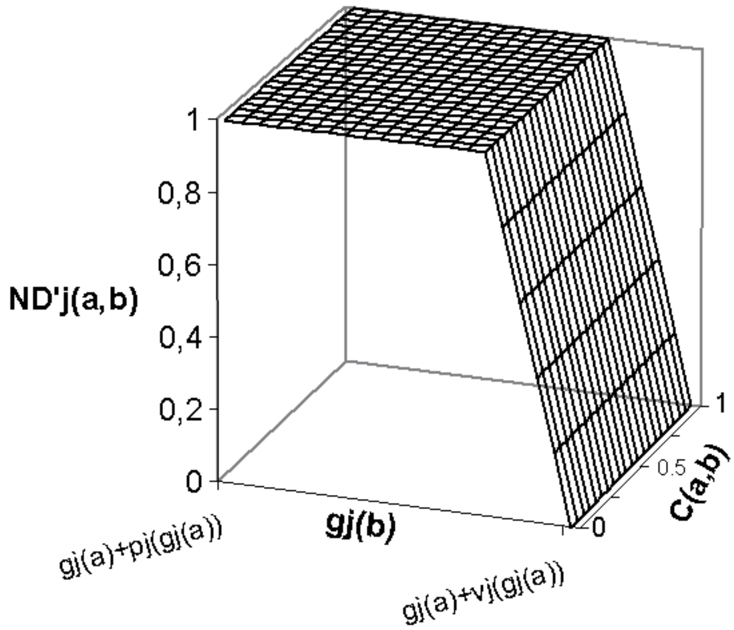

Figure 7: $ND'_j(a,b)$ as a function of $g_j(b)$ and $C(a,b)$

In order to appreciate the effective deviation between $S(a,b)$, $S'(a,b)$ and $S''(a,b)$, we have performed an empirical study on datasets stemming from real world case studies reported in the literature ([4] and [23]). More specifically, we observe how much $S'(a,b)$ and $S''(a,b)$ deviate from $S(a,b)$ on real data and analyze to what extend substituting $S'(a,b)$ (or $S''(a,b)$) for $S(a,b)$ affects the conclusions of these studies, which concerned assigning alternatives to categories using Electre Tri. Considering [4] and [23], we verified that the assignments of all alternatives are identical using S, S' and S''. The maximum deviation (*i.e.*, $|S(a,b) - S'(a,b)|$ and $|S(a,b) - S''(a,b)|$, $a \in A, b \in B$) is equal to 0.1. Although it is possible to design specific situations in which the assignment using $S'(a,b)$ (or $S''(a,b)$) and $S(a,b)$ are different, it seems that such cases do not occur frequently on real data.

3 Benefit of the revised index with respect to parameter inference programs

Assigning values to the parameters involved in the definition of S and S' is a difficult task for the DM(s). The disaggregation approach allows to infer preferential parameter values from holistic preferences. Such approach is usually performed using mathematical programs that minimize an "error function". By inference program, we refer to these mathematical programs aiming at determining values for preference parameters involved in $S(a,b)$ from holistic preferences provided by the DM(s). Such inference programs can either be partial if a subset of parameters only is being inferred (the values of the other parameters being fixed), or global if all parameters are to be inferred. In this section, we will illustrate the reduction in the complexity of the mathematical programs to be solved when substituting S' or S'' for S.

3.1 Inferring a fuzzy outranking relation from crisp outranking statements

3.1.1 Global inference program

In this section we will consider a decision process in which an outranking relation is used to model DM(s)' preferences. Furthermore, let us suppose that the DM(s) is/are not able (or not willing) to assign directly values to the preference-related parameters involved in the outranking relation, but can state crisp statements about this relation for some specific pairs of alternatives (a, b), *i.e.*, either aSb or $\neg aSb$. Our purpose is to define a fuzzy outranking relation and a cutting level that best account for the DM(s) statements.

Let us denote $S^+ = \{(a, b) \in A^2$ such that the DM(s) stated $aSb\}$ and $S^- = \{(a, b) \in A^2$ such that the DM(s) stated $\neg aSb\}$. Then, a combination of parameter values is able to restore the DM(s)' request iff $S(a, b) \geq \lambda,\ \forall (a, b) \in S^+$ and $S(a, b) < \lambda,\ \forall (a, b) \in S^-$, which may be written as $S(a, b) - \lambda \geq 0,\ \forall (a, b) \in S^+$ and $\lambda - S(a, b) + \varepsilon \geq 0,\ \forall (a, b) \in S^-$ (ε being a small positive value). The mathematical program given below (14)-(19) maximizes a common slack α for all these constraints, to obtain a relatively "central" combination of parameter values. Whenever the optimum value of α is negative, there is no combination of parameter values complying to all the constraints (the DM(s) provided inconsistent information).

$$
\begin{aligned}
& \alpha \quad \rightarrow \ max && (14)\\
s.t.\quad & \alpha \leq S(a, b) - \lambda, \quad \forall (a, b) \in S^+ && (15)\\
& \alpha \leq \lambda - S(a, b) + \varepsilon, \quad \forall (a, b) \in S^- && (16)\\
& \lambda \in [0.5, 1] && (17)\\
& v_j(g_j) \geq p_j(g_j) \geq q_j(g_j) \geq 0, \quad \forall j \in F && (18)\\
& \textstyle\sum_{j=1}^{n} w_j = 1; \quad w_j \geq 0,\ \forall j \in F. && (19)
\end{aligned}
$$

Some additional constraints can be added to this program, in order to integrate explicit statements of the DM(s) concerning the values of some parameters. From (4) and (8), it is obvious that this is a difficult nonlinear program when all the parameters are considered as variables (recall Section 1.6.2). A solution to circumvent this difficulty is to formulate partial inference programs, where only a subset of the parameters are considered as variables, while the remaining ones are fixed. In the context of a decision aiding process where the DM(s) interactively revise the information they provide and observe the results of the mathematical program, partial inference problems allow them to focus their attention on a subset of parameters at a time and to better understand the consequences of their modifications. Indeed, we believe that inference programs should not be considered as a problem to be solved once, but rather as problems to be solved many times in an interactive learning process. Among the partial inference problems, previous research on related problems has focused mainly on inferring the weights and the cutting level (see [13], [3], [10]). This is an important partial inference problem because the weights and the cutting level are the only parameters involving inter-criteria judgements (the remaining parameters do not interrelate the criteria).

3.1.2 Inferring the weights ($w_j, j \in F$) and cutting level (λ) only

If we consider the case where only the weights ($w_j, j \in F$) and cutting level (λ) are variables (all other parameters being fixed), then the constraints (15) and (16) can be rewritten in as in (20) and (21). These two constraints are obviously nonlinear, since they represent products of functions involving $C(a,b)$, which in turn involve the w_j variables (see (2)).

$$\alpha \leq C(a,b). \prod_{j \in F} Min \left\{ 1, \frac{1 - d_j(a,b)}{1 - C(a,b)} \right\} \; - \; \lambda, \quad \forall (a,b) \in S^+ \tag{20}$$

$$\alpha \leq \lambda \; - \; C(a,b). \prod_{j \in F} Min \left\{ 1, \frac{1 - d_j(a,b)}{1 - C(a,b)} \right\} + \varepsilon, \quad \forall (a,b) \in S^- \tag{21}$$

Let us now consider the same problem when $S(a,b)$ is substituted by $S'(a,b)$. In this case the constraints (15) and (16) become (22) and (23). Now, each $\prod_{j \in F}(1 - d'_j(a,b)) = ND'(a,b)$ is a fixed constant $\forall(a,b)$. The constraints (22) and (23) are hence linear, since $C(a,b)$ is an affine function of the weights.

$$\alpha \leq C(a,b). \prod_{j \in F} \left(1 - d'_j(a,b)\right) \; - \; \lambda, \quad \forall (a,b) \in S^+ \tag{22}$$

$$\alpha \leq \lambda \; - \; C(a,b). \prod_{j \in F} \left(1 - d'_j(a,b)\right) + \varepsilon, \quad \forall (a,b) \in S^- \tag{23}$$

Thus, considering $S'(a,b)$ instead of $S(a,b)$, the weights and the cutting level can be inferred by solving a linear program whose variables are α, w_1, ..., w_n, and λ, where (22) and (23) appear as (25) and (26):

$$\begin{aligned}
& \alpha \quad \rightarrow \; max & (24)\\
s.t. \quad & \alpha \leq \textstyle\sum_{j=1}^{n} \mathbf{w}_j.S_j(a,b).ND'(a,b) \; - \; \lambda, \quad \forall (a,b) \in S^+, & (25)\\
& \alpha \leq \lambda \; - \; \textstyle\sum_{j=1}^{n} \mathbf{w}_j.S_j(a,b).ND'(a,b) + \varepsilon, \quad \forall (a,b) \in S^-, & (26)\\
& \lambda \in [0.5, 1], & (27)\\
& \textstyle\sum_{j=1}^{n} \mathbf{w}_j = 1; \quad \mathbf{w}_j \geq 0, \; \forall j \in F. & (28)
\end{aligned}$$

If the maximum value of α is positive, then the values of w_1, ..., w_n, and λ at the optimum are able to restore all the statements defining S^+ and S^-. Otherwise, the inferred values provide suggestions for changing those examples. The DM(s) should ponder whether they want to change the the sets S^+ and S^-, or to analyse the non-discordance levels. Indeed, some of the differences among the current model and the DM(s)' requests may stem from inadequate non-discordance levels. Considering $S''(a,b)$ instead of $S'(a,b)$ leads to a similar linear program.

3.2 Inferring Electre tri parameters from assignment examples

3.2.1 Brief reminder on Electre tri

Electre Tri (see [23], [20]) assigns alternatives to pre-defined ordered categories. The assignment of an alternative $a \in A$ results from comparing it with the profiles defining the limits of the categories. Let $B = \{b_1, b_2, ..., b_p\}$ denote the set of profiles defining $p+1$ categories, b_h being the upper limit of category C_h and the lower limit of category C_{h+1}, $h = 1, 2, ..., p$ (profiles b_{p+1} and b_0 correspond to the ideal and anti-ideal alternatives, respectively). Let $K = \{C_1, C_2, ..., C_{p+1}\}$ be the set of categories that corresponds to B. Electre Tri assigns alternatives to categories following two consecutive steps :

- construction of a crisp outranking relation $S \subset (A \times B) \cup (B \times A)$; the definition of S corresponds to λ-cut of $S(a,b)$ described in section 1, except that the thresholds q_j, p_j and v_j are values attached to the profiles ($q_j(b_h)$, $p_j(b_h)$ and $v_j(b_h), \forall j \in F, \forall b_h \in B$), rather than dependent on the alternatives in A.
- exploitation of the crisp relation S in order to assign each alternative to a specific category using a pessimistic or optimistic procedure (in what follows we will restrict our analysis to the pessimistic procedure). The pessimistic procedure is defined as follows:

  a) compute $S(a, b_i)$ successively for i=p,p-1, ..., 0,
  b) b_h being the first profile such that $S(a, b_h) \geq \lambda$,
  assign a to category C_{h+1} ($a \to C_{h+1}$).

Hence, the pessimistic procedure of Electre Tri assigns alternative a to category C_h (b_{h-1} and b_h being the lower and upper profiles of C_h, respectively) iff $S(a, b_{h-1}) \geq \lambda$ and $S(a, b_h) < \lambda$ ($\lambda \in [0.5, 1]$ is the chosen cutting level).

3.2.2 Inference programs

Suppose the DM has specified a set of assignment examples, *i.e.*, a subset of $A^* \subset A$ such that each $a_k \in A^*$ is associated with $C^M(a_k)$ ($C^m(a_k)$, respectively) the maximum (minimum, respectively) category to which a should be assigned according to his/her holistic preferences. Hence $[C^m(a_k), C^M(a_k)]$ defines an interval of possible categories to which a_k can be assigned to. $C^m(a_k) = C^M(a_k) = C_{h_k}$ means that the DM wants a_k to be assigned to C_{h_k} precisely (we will note $a_k \rightarrow_{DM} C_{h_k}$ such statement), while $C^m(a_k) < C^M(a_k)$ corresponds to an imprecise statement ($a_k \rightarrow_{DM} [C^m(a_k), C^M(a_k)]$). Inferring all Electre Tri parameters can be formulated through the following mathematical program (29)-(35), see [12]. Note that all alternatives in A^* are assigned by Electre Tri (using the inferred parameters) consistently with the DM(s)' examples if and only if the optimal value objective function is positive.

$$
\begin{aligned}
& \alpha \quad \rightarrow \; max && (29)\\
s.t. \quad & \alpha \leq S(a_k, b_{C^m(a_k)-1}) \; - \; \lambda, \quad \forall a_k \in A^*, && (30)\\
& \alpha \leq \lambda \; - \; S(a_k, b_{C^M(a_k)}) + \varepsilon, \quad \forall a_k \in A^*, && (31)\\
& \lambda \in [0.5, 1], && (32)\\
& g_j(b_{h+1}) \geq g_j(b_h) + p_j(b_h) + p_j(b_{h+1}), \forall j \in F, \forall h \in B, && (33)\\
& v_j(b_h) \geq p_j(b_h) \geq q_j(b_h) \geq 0, \quad \forall j \in F \; \forall b_h \in B, && (34)\\
& \textstyle\sum_{j=1}^{n} w_j = 1; \quad w_j \geq 0, \; \forall j \in F. && (35)
\end{aligned}
$$

If we consider all preference parameters as variables, this mathematical program[8] is nonlinear due to constraints (30) and (31). Indeed, the constraints (30) and (31) are similar to the constraints (15) and (16) in the mathematical program presented in Sec. 3.1.1.

If we consider the case where only the weights ($w_j, j \in F$) and cutting level (λ) are variables (all other parameters being fixed), then the constraints (30) and (31) will be similar to (20) and (21), hence the mathematical program (29)-(35) remains nonlinear. Previous work ([13], [3]) presented linear programming formulations assuming that there was no discordance, i.e. $S(a, b) = C(a, b)$. One of the motivations for considering $S'(a, b)$ instead of $S(a, b)$ is that it becomes easy to infer the weights and the cutting level even when the veto-related parameters make $S(a, b) < C(a, b)$.

Indeed, when considering $S'(a, b)$, the weights and the cutting level can be inferred by solving a linear program whose variables are α, w_1, ..., w_n, and λ. The linear program for this partial inference problem is equal to (24)-(28) in Sec. 3.1.2, if we define:

$$S^+ = \left\{(a_k, b_{C^m(a_k)-1}) \in A^* \times B \; : \; a_k \rightarrow_{DM} [C^m(a_k), C^M(a_k)]\right\} \tag{36}$$

$$S^- = \left\{(a_k, b_{C^M(a_k)}) \in A^* \times B \; : \; a_k \rightarrow_{DM} [C^m(a_k), C^M(a_k)]\right\} \tag{37}$$

Considering $S''(a, b)$ instead of $S'(a, b)$ leads to a similar linear program.

[8]The constraint (33) is introduced to ensure the consistency of category definition, see [23]

3.3 Miettinen & Salminen Electre Like method [10]

[10] proposed a method using the Electre III fuzzy outranking relation that aims at "*providing the DM(s) descriptive information about the weighting vectors producing a specific alternative as the best*", *i.e.*, placed first in a preference ranking. Such procedure requires to solve inference programs. In [10], the preference rankings do not result from the same exploitation procedure as in Electre III (see [18]), but from the use of the "min" procedure (see [17]), *i.e.*, alternatives are ranked based on the minimum outranking degree $S(a, b)$ of each alternative $a, \forall b \in A$. In this approach, checking whether or not an alternative $a \in A$ can be ranked first for at least a weight vector amounts at verifying whether the constraints (38)-(39) define a consistent system. Such analysis is performed with fixed values for all the parameters except the weights w_j. Furtermore, the problem is rather difficult because the constraints (38)-(39) are nonlinear, unless the $v_j(g_j)$ are fixed sufficiently large so as no veto phenomenon to occur, *i.e.*, unless discordance is removed from the model.

$$min_{a' \in A-\{a\}} \ \{S(a, a')\} - min_{a' \in A-\{a\}} \{S(b, a')\} \ \geq \ \varepsilon, \ \ \forall b \in A - \{a\} \tag{38}$$
$$\varepsilon > 0, \ \ \textstyle\sum_{j \in F} w_j = 1, \ \ w_j \geq 0, \quad \forall j \in F \tag{39}$$

Different objective functions to optimize subject to (38)-(39) are proposed in [10]. Optimization programs are build in order to:

- determine ranges for w_j such that a is ranked first (max and min w_j s.t. (38)-(39)),
- rank a first in the most "robust" way (Max ε s.t. (38)-(39)),
- rank a first without omitting any criterion (Max Min w_j s.t. (38)-(39)).

If discordance is removed from the model, then these optimization programs can be solved using standard linear programming techniques[9]. However, if we replace $S(a, b)$ by $S'(a, b)$ in the preceding optimization programs, it becomes obvious that the restriction that [10] impose on discordance $(v_j(g_j(a)) > g_j(b) - g_j(a), \forall a, b \in A, \forall j \in F)$ is no longer necessary to simplify the problem. In fact, considering $S'(a, b)$, it is possible to solve their optimization programs using linear programming even when veto phenomena occur (the v_j functions being defined). The same statement can be made when considering $S''(a, b)$ instead of $S'(a, b)$.

Conclusion

This paper presents a slight adaptation of the fuzzy outranking relation used in the ELECTRE III (see [18]) and ELECTRE TRI (see [23], [20]); the modifications introduced concerns

[9] This requires to add supplementary variables to account for the min operator in (38)

the implementation of the non-discordance condition in the outranking relation. The two new outranking relations S' and S'' preserves the original ideas (namely the original discordance concept) and are designed to be more optimization-friendly for parameter inference programs. We show that the modified outranking relation makes it easier to solve inference programs. S' and S'' are equivalent as regards the complexity of inferring the weights and cutting level, but S'' is more friendly in what regards inferring veto thresholds.

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