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Local Groups in Free Burnside Groupoids

by

Alair Pereira do Lago[1]
alair@dimacs.rutgers.edu
alair@ime.usp.br

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ABSTRACT

The study of repeated sequences is the basis of the study of free Burnside groups and free Burnside semigroups, and a better knowledge on Burnside algebras can be very useful in the analysis of many properties of sequences containing repetitions. The main theorem announced here establishes an important connection between free Burnside groups and free Burnside semigroups. In order to establish this connection, we use interesting properties on graphs, categories, combinatorics on words to obtain the algebraic main theorem.

Let $\mathbf{G}$ be a (possibly infinite) strongly connected graph and let $\mathcal{T}$ be a set of monoid identities such that any monoid satisfying $\mathcal{T}$ is also a group. Let $\mathcal{B}$ be the free groupoid on $\mathbf{G}$ satisfying $\mathcal{T}$. Then, the local groups $\mathcal{B}_v$, for $v \in V(\mathbf{G})$, are all isomorphic to a free group satisfying $\mathcal{T}$. Furthermore, it is free over a generating set which can be effectively characterized and whose cardinality is the cyclomatic number of the graph $\mathbf{G}$.

1 Introduction

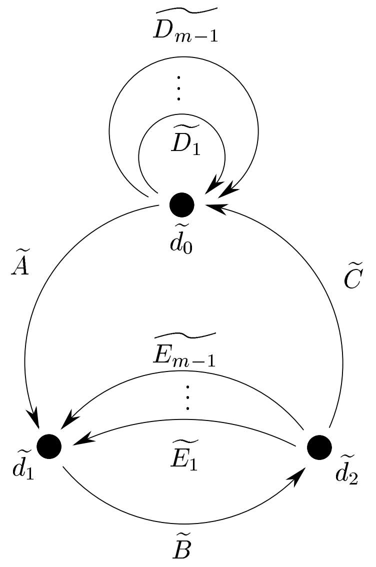
Brzozowski and Simon [3] began the investigation of category-related properties of finite semigroups with the characterization of locally testable semigroups. In this direction we can also include work by Thérien and Weiss [32]. The classical paper of Tilson [34] was the first one that systematized the investigation of categories as a generalization of monoids. Many other authors, like Straubing [31], used this technique to obtain important properties on monoids.

The main theorem of this paper follows the same direction. It generalizes a result that deals with free Burnside grupoids and was a cornerstone in our Ph.D. thesis [12] in order to establish an important connection between free Burnside groups and free Burnside semigroups. In that result, we define a special groupoid: the free Burnside groupoid satisfying $x^m = 1$ generated by a (possibly infinite) strongly connected graph. The result states that the local groups of this groupoid are all isomorphic to the same free Burnside group, satisfying $x^m = 1$. It also exhibits a generating set for this free group and show that its cardinality is the cyclomatic number of the generating graph.

Let A be an alphabet with k letters and let $n \geq 0$ and $m \geq 1$ be integers. Let π be the relation $\{(x^{n+m}, x^n) \mid x \in A^+\}$ and let $\sim$ be the smallest congruence that contains π. This is simply called *Burnside congruence*. Let $\widetilde{w}$ denote the congruence class of a word w and let $M(k, n, m)$, or simply $\mathcal{M}$, be the quotient $A^*/\sim$. We usually suppose $n \geq 1$ and this is the *free Burnside monoid on A satisfying $x^n = x^{n+m}$*. If $n = 0$, then $\mathcal{M}$ is a group and it is called the *free Burnside group on A satisfying $x^m = 1$*, denoted also by $B(k, m)$. We will suppose $k \geq 2$, unless stated in the contrary.

The problem of determining whether the free Burnside groups are finite or not is extremely complex and it has been studied since 1902. In particular, they are finite for $m \in \{1, 2, 3, 4, 6\}$ [4, 21, 29, 17]; and they are infinite for any $m \geq 8000$ or for any odd $m \geq 665$ [27, 28, 5, 1, 22, 23, 18, 19]. As for the theory of Burnside semigroups, it progressed in bursts of discoveries. By work of Thue [33] in 1912, these semigroups are infinite for most cases where $n \geq 2$. In the 50's Green and Rees [14] and McLane [26] advanced our knowledge for $n = 1$. In the early 70's some advances were made by the school of Brzozowski [2, 30] and then the problem became dormant for almost 20 years. After concluding the proof of the infiniteness of the free Burnside semigroups satisfying $x^n = x^{n+m}$ for $n \geq 2$, Brzozowski conjectured that some finiteness properties should remain true for the free Burnside semigroups. In particular, he conjectured in 1969 that, for $m = 1$, any congruence class should be recognizable. Motivated by Brzozowski's conjecture, de Luca and Varricchio [6, 7], McCammond [25], do Lago [9, 10, 11], and Guba [16, 15] produced a sequence of papers with impressive results in the area in the 90's (we refer an interested reader to [11]). This led to the discovery of many structural properties of the free Burnside semigroups which depend only on the so called stability of a specially defined rewriting system. This stability is proved for $n \geq 3$ and $m \geq 1$, it does not hold for $n = 2$ and $m \geq 2$ and it remains open for $n \geq 2$ and $m = 1$. Under the hypothesis of stability, Brzozowski's conjecture is verified, the word problem is solvable and all maximal subgroups are cyclic of order m, i.e. they are the free Burnside group satisfying $x^m = 1$ with only one generator.

We started our investigation in the almost unknown case $n = 2$ and discovered interesting properties on graphs, categories and combinatorics on words to obtain the algebraic main theorem of our Ph.D. thesis [12]. This theorem uses the one which is the object of this paper and states that the maximal subgroups in free Burnside semigroups satisfying $x^n = x^{n+m}$ are free Burnside groups satisfying $x^m = 1$, for $n \geq 1$ and $m \geq 1$. Furthermore, for $n = 2$, we exhibit examples (see figure 1) where the free Burnside group in question is



The transitions:		The $\mathcal{D}$-entrances:
A:	$a35ba$	d_0: $a33a$
B:	353	d_1: $35ba$
C:	$ab53a$	d_2: $ab53$
D_1:	$a373a$	
$\vdots$		
D_{m-1}:	$a35(ab)^{m-1}3a$	
E_1:	$ab535ba$	
$\vdots$		
E_{m-1}:	$ab53^{m-1}5ba$	

Where: 3 denotes bab
5 denotes $babab$
7 denotes $bababab$
$\widetilde{w}$ is the congruence class of w

Figure 1: This graph has cyclomatic number $2m - 1$ and is a subgraph of the strongly connected graph that generates a free Burnside groupoid. Hence, all local groups of this groupoid are isomorphic to the same free Burnside group satisfying $x^m = 1$ on a generating set with at least $2m - 1$ generators. They are isomorphic to any $\mathcal{H}$-class of the $\mathcal{D}$-class of $ababbaba$ in the free Burnside semigroup satisfying $x^2 = x^{2+m}$ on $\{a, b\}$.

free on a set with at least $2m-1$ generators. As a consequence, Brzozowski's conjecture (extended for any m) does *not* hold in a noncyclic free Burnside semigroup satisfying $x^2 = x^{2+m}$ for m such that the two-generated free Burnside group of exponent m is infinite [12, 13].

2 Local Groups in Free Groupoids

We refer the reader to check the terminology of category theory [24, 34]. It is interesting to notice that the concept of category generalizes the concept of monoid while that of groupoid generalizes the concept of group.

A category $\mathcal{C}$ can be defined as a graph endowed with a partial operation on the edges. The set of all edges that start in u and end in v is denoted by $\mathcal{C}_{u,v}$ and we denote by $\mathcal{C}_u$ the set $\mathcal{C}_{u,u}$. The subcategory $\mathcal{C}_u$ is also a monoid and is called *local monoid of u*.

Let $\mathbf{G}$ be a (possibly infinite) strongly connected graph whose set of vertices is V. We will frequently refer to a graph or category by its set of edges. For any edge $b \in \mathbf{G}$, let $\alpha(b)$ and $\omega(b)$ denote *the beginning of the edge* b and *the end of the edge* b respectively. Let $\mathcal{K} = \mathbf{G}^*$ be the free category generated by the graph $\mathbf{G}$. Loosely speaking, $\mathcal{K}$ is the set of the nonempty paths of $\mathbf{G}$ together with the empty paths 1_v, one for each vertex v. The multiplication in $\mathcal{K}$ is a partial multiplication defined only on consecutive paths. Notice that the elements of $\mathcal{K}$ are paths on $\mathbf{G}$ and edges in the category $\mathcal{K}$.

Let X be a set of unknowns and let $\mathcal{T} \subseteq X^* \times X^*$ be a set of monoid identities on X such that any monoid satisfying $\mathcal{T}$ is also a group. One example is $\mathcal{T} = \{(x^t, 1)\}$, for some $t \geq 1$. In fact, for any $\mathcal{T}$, one can verify that there is a minimal $t \geq 1$ such that the identity $x^t = 1$ is implied by $\mathcal{T}$. Let us fix t minimal with these properties. Besides, the groups are trivial iff t is 1.

We define the set $\mathcal{T}(\mathbf{G}) = \{(\varphi(u), \varphi(w)) \mid \forall (u, w) \in \mathcal{T},\ \forall v \in V,\ \forall \varphi : X \longrightarrow \mathcal{K}_v\}$ and the congruence $\cong$ on $\mathcal{K}$ to be the smallest congruence that contains $\mathcal{T}(\mathbf{G})$. Notice that the above definition implies that each congruence class is constituted of sets of pairwise coterminal paths. We define the category $\mathcal{B}$ to be $\mathcal{B} \stackrel{\text{def}}{=} \mathcal{K}/\!\cong$ and for each path $p \in \mathcal{K}$ we associate p with its canonical projection $\widehat{p} \in \mathcal{B}$, the congruence class of p relative to $\cong$. The objects of the category $\mathcal{B}$ are the same as those of the category $\mathcal{K}$ which are in turn the vertices of $\mathbf{G}$. Furthermore, the canonical projection from $\mathcal{K}$ onto $\mathcal{B}$ preserves the beginning and the end of every path. Since $\mathbf{G}$ is a strongly connected graph it immediately follows that $\mathcal{K}$ and $\mathcal{B}$ are strongly connected categories. Furthermore, we can verify that $\mathcal{B}$ is a groupoid (a category where every element is invertible). In fact, if $x \in \mathcal{B}$ is any element, we can choose a path $p \in \mathcal{K}$ such that $\widehat{p} = x$, we can choose a path $q \in \mathcal{K}$ from $\omega(p)$ to $\alpha(p)$ and we can define $r = q(pq)^{t-1}$. One can verify that $\widehat{r}$ is the inverse of x. It is easy to verify that each local monoid $\mathcal{B}_v$ is actually isomorphic to the same group. The groupoid $\mathcal{B}$ thus defined is the *free groupoid on* $\mathbf{G}$ *satisfying* $\mathcal{T}$. If $\mathcal{T}$ has only the identity $x^t = 1$ it is also called *free Burnside groupoid on* $\mathbf{G}$ *satisfying* $x^t = 1$.

Since $\mathcal{B}$ is a strongly connected groupoid, the local monoids $\mathcal{B}_v$ are isomorphic groups. As we have already said, we will see in Theorem 2.1 that these groups are isomorphic to the same free group satisfying $\mathcal{T}$. In the case $t = 1$, there is no substance to this statement because these groups are trivial since the congruence $\cong$ makes every element from $\mathcal{K}_v$ identical to 1_v, for every vertex v. Similarly, the free group satisfying $\mathcal{T}$ is trivial. Unless stated to the contrary, we will assume $t \geq 2$ from now on in this section.

Since the graph $\mathbf{G}$ is strongly connected, we may choose a spanning tree $\mathbf{T} \subseteq \mathbf{G}$ with beginning u. For every vertex v, we define p_v to be the unique path in $\mathbf{T}$ from u to v and we define $\gamma_v \stackrel{\text{def}}{=} \widehat{p_v} \in \mathcal{B}_{u,v}$. Then, the *normalization* $\lambda : \mathbf{G}^* \longrightarrow \mathcal{B}_u$ is a category morphism defined on the edges of $\mathbf{G}$ by

$$\lambda(b) \stackrel{\text{def}}{=} \gamma_{\alpha(b)}\, \widehat{b}\, {\gamma_{\omega(b)}}^{-1}.$$

One can prove that $\lambda(\mathbf{T}) = \{\widehat{1_u}\}$ and that only the edges of $\mathbf{T}$ are mapped onto $\widehat{1_u}$. One can also prove that λ establishes an injection on $\mathbf{G} \setminus \mathbf{T}$ since $t \geq 2$. Hence, it is natural to define the graph $\mathbf{H}$ by

$$\mathbf{H} = \lambda(\mathbf{G} \setminus \mathbf{T}).$$

That is, $\mathbf{H}$ is the subgraph of $\mathcal{B}_u$ induced by the edges $\lambda(b)$ for $b \in \mathbf{G} \setminus \mathbf{T}$ and having u as unique vertex. Since λ establishes a bijection from $\mathbf{G} \setminus \mathbf{T}$ onto $\mathbf{H}$, the cardinality of $\mathbf{H}$ is $|\mathbf{H}| = |\mathbf{G}| - |\mathbf{T}| =$ cyclomatic number of $\mathbf{G}$. This gives us the following diagram:

$$\begin{tikzcd} \mathbf{G} \setminus \mathbf{T} \arrow[r, tail, two heads, "\lambda"] & \mathbf{H} \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{T} \arrow[r, two heads, "\lambda"] & \widehat{1_u}. \end{tikzcd}$$

Let $\mathbf{H}^*$ be the free category generated by $\mathbf{H}$. Since all edges of $\mathbf{H}$ are consecutive, $\mathbf{H}^*$ is in fact the free monoid generated by $\mathbf{H}$. We define the map $\theta : \mathbf{G} \longrightarrow \mathbf{H}^*$ by

$$\theta(b) = \begin{cases} 1_u, & \text{if } b \in \mathbf{T}; \\ \lambda(b), & \text{if } b \in \mathbf{G} \setminus \mathbf{T}. \end{cases}$$

This map is a graph morphism because $\mathbf{H}^*$ has a unique vertex and θ can be uniquely extended to the category morphism $\theta : \mathcal{K} \longrightarrow \mathbf{H}^*$. The inclusion $\iota : \mathbf{H} \longrightarrow \mathcal{B}_u$ is a graph morphism and we can uniquely extend it to the morphism $\phi : \mathbf{H}^* \longrightarrow \mathcal{B}_u$, which, in turn, maps a path $p \in \mathbf{H}^*$ to the product in $\mathcal{B}_u$ of the sequence of edges of p. For each vertex $v \in V(\mathbf{G})$, let us choose a path $q_v \in \mathcal{K}_{v,u}$ such that $\widehat{q_v} = {\gamma_v}^{-1} = (\widehat{p_v})^{-1}$. We can now define the function $\beta : \mathbf{H} \longrightarrow \mathcal{K}_u$ that maps each $h \in \mathbf{H}$ onto $\beta(h) \overset{\text{def}}{=} p_{\alpha(b)}\, b\, q_{\omega(b)}$ where $b = \lambda^{-1}(h) \in \mathbf{G} \setminus \mathbf{T}$. This function can uniquely be extended to a morphism $\beta : \mathbf{H}^* \longrightarrow \mathcal{K}_u$. The domain of the morphisms λ and θ is $\mathcal{K}$, as for the morphism $\widehat{\ }$. Nevertheless, only the restriction of $\widehat{\ }$ to $\mathcal{K}_u$ has its image in $\mathcal{B}_u$. For this reason, we are more interested in the restrictions of the morphisms λ, θ and $\widehat{\ }$ to $\mathcal{K}_u$. Hence one can prove that the following diagrams commute:

$$\begin{tikzcd} \mathcal{K} \arrow[r, "\theta"] \arrow[dr, "\lambda"'] & \mathbf{H}^* \arrow[d, "\phi"] \\ & \mathcal{B}_u \end{tikzcd} \qquad \begin{tikzcd} \mathcal{K}_u \arrow[dr, "\lambda"] \arrow[dr, bend right, "\widehat{\ }"'] & \mathbf{H}^* \arrow[l, bend right, "\beta"'] \arrow[d, "\phi"] \\ & \mathcal{B}_u \end{tikzcd}.$$

In particular, the canonical projection from $\mathcal{K}$ to $\mathcal{B}$ and the normalization of paths are equivalent when restricted to paths of $\mathcal{K}_u$ and the normalization λ can be factorized through the morphisms θ and ϕ. As a consequence, the graph $\mathbf{H}$ is a graph generator of $\mathcal{B}_u$. This fact shows us the importance of the normalization in the study of $\mathcal{B}_u$. Moreover, we will see in Theorem 2.1 that **H** is the generating set on which $\mathcal{B}_u$ is proved to be a free group satisfying $\mathcal{T}$.

The main result relative to the free groupoid satisfying $\mathcal{T}$ is the following characterization of the local groups and generalizes a result in our Ph.D. thesis [12] where the case $\mathcal{T} = \{x^t = 1\}$ was considered. Recall that a free group satisfying $\mathcal{T}$ on a nonempty generating set is trivial iff $t = 1$.

Theorem 2.1 (Free Local Groups Theorem) *Let* $\mathbf{G}$ *be a strongly connected graph and let* $\mathcal{T}$ *be a set of monoid identities such that any monoid satisfying* $\mathcal{T}$ *is also a group. Let* $\mathcal{B}$ *be the free groupoid on* $\mathbf{G}$ *satisfying* $\mathcal{T}$*. Then, the local groups* $\mathcal{B}_v$*, for* $v \in V(\mathbf{G})$*, are all isomorphic to a free group satisfying* $\mathcal{T}$*. Furthermore, if* $t \geq 2$*, the group is free over* $\mathbf{H}$*, whose cardinality is the cyclomatic number of the graph* $\mathbf{G}$*.*

3 Main Applications

In this section we will see some applications of the Free Local Groups Theorem. We are particularly interested on introducing the main results that establish the connection between free Burnside groups and free Burnside semigroups and on giving some examples on how these graphs and free Burnside groupoids appear. Consider the Green relations $\mathcal{J}$, $\mathcal{D}$, $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{H}$ as defined in [20].

For simplicity, we will consider the example of the free Burnside monoid satisfying $x = x^3$ *on* $\{a, b\}$*. Hence* $A = \{a, b\}$ *and* $n = 1$ *and* $m = 2$*. We will also consider the* $\mathcal{D}$*-class of* $\widetilde{ab}$ *and one can verify that, in this case, any two words in* A^* *map in the same* $\mathcal{D}$*-class iff they use exactly the same set of letters.*

We would like to verify whether or not the words $u = aababbab$ *and* $v = aabbabab$ *map to the same element in* $\mathcal{M}$*. We would also like to exhibit the* $\mathcal{H}$*-class of* u*.*

Let w be a nonempty word and let D be the $\mathcal{D}$-class of $\widetilde{w}$ respectively. The $\mathcal{R}$-*entrance of* w, $\epsilon_R(w)$, is the shortest prefix of w which projects in D. The $\mathcal{L}$-*entrance of* w, $\epsilon_L(w)$, is the shortest suffix of w which projects in D. *A* $\mathcal{D}$-*entrance of* w is a minimal factor of w which projects in D and we simply say that w is a $\mathcal{D}$-*entrance* if w is a $\mathcal{D}$-entrance of w. We call w a *transition* if w itself is a minimal factor which has two $\mathcal{D}$-entrances of w as a factor.

Then, considering Figure 2:

1. *The* $\mathcal{D}$*-entrances of* u *are* y_0, y_1, y_2, y_3, y_4*.*

2. *The* $\mathcal{R}$*-entrance of* u *is* w_0*, the* $\mathcal{L}$*-entrance of* w *is* w_5*.*

3. *The transitions of* w *are* w_1, w_2, w_3, w_4*.*

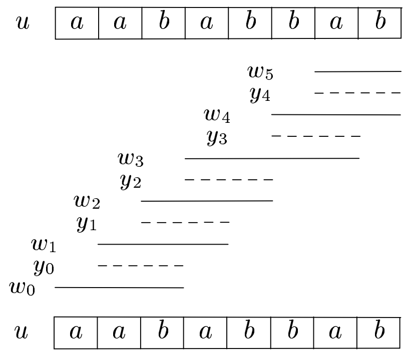


Figure 2: Entrances and transitions of *aababbab*

Let D be a $\mathcal{D}$-class of $\mathcal{M}$. The *fundamental graph* of D is the graph whose set of vertices and set of edges are

$$
\begin{aligned}
V(\mathbf{G}) &= \{\widetilde{d} \in D \mid d \text{ is a } \mathcal{D}\text{-entrance}\} \\
E(\mathbf{G}) &= \{\widetilde{t} \in D \mid t \text{ is a transition}\}
\end{aligned}
$$

and such that every edge $\widetilde{t}$ begins in $\widetilde{u}$ and ends in $\widetilde{v}$ where u and v are the proper prefix and suffix of t that are $\mathcal{D}$-entrances of t.

The fundamental graph of the $\mathcal{D}$-class of $\widetilde{ab}$ is shown in figure 3 and has cyclomatic number 3.

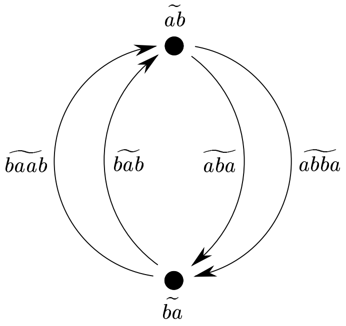


Figure 3: The fundamental graph of the $\mathcal{D}$-class of $\widetilde{ab}$ in the free Burnside monoid satisfying $x = x^3$

A word w can have many occurrences of $\mathcal{D}$-entrances and transitions as factors. The order of occurrences induces a natural order in the transitions. Consider a word w with: $\mathcal{R}$-entrance w_0; sequence of transitions $w_1, \dots, w_k$; $\mathcal{L}$-entrance w_{k+1}. The path $\widetilde{w_1}, \dots, \widetilde{w_k}$ from $\widetilde{\epsilon_L(w_0)}$ to $\widetilde{\epsilon_R(w_{k+1})}$, in the fundamental graph of the $\mathcal{D}$-class of w, is called the *fingerprint of* w and is denoted by $\sigma(w)$.

Consider the fundamental graph of $\widetilde{ab}$ in figure 3. Let p be the path $\widetilde{aba}, \widetilde{bab}$ and q be the path $\widetilde{abba}, \widetilde{bab}$. Both p and q are local to $\widetilde{ab}$. Hence $\sigma(u) = \widetilde{aba}, \widetilde{bab}, \widetilde{abba}, \widetilde{bab} = pq$ and $\sigma(v) = qp$.

Now we a ready to state the next most important results of [12].

Theorem 3.1 (Characterization Theorem) *Take a $\mathcal{D}$-class D of $\mathcal{M}$ with fundamental graph $\mathbf{G}$ and take the congruence $\cong$ associated with the canonical projection from $\mathbf{G}^*$ to the free Burnside groupoid on $\mathbf{G}$ satisfying $\mathcal{T} = \{x^m = 1\}$. Then, two given words projected in D are $\sim$-congruent iff:*
– *their $\mathcal{R}$-entrances are $\sim$-congruent;*
– *their $\mathcal{L}$-entrances are $\sim$-congruent;*
– *their fingerprints are $\cong$-congruent.*

By the Free Local Groups Theorem, the local group of $\widetilde{ab}$ is a free Burnside group satisfying $x^2 = 1$, which in turn is known to be commutative. Hence, $\sigma(u) = pq = qp = \sigma(v)$. Since the $\mathcal{R}$-entrances of u and v are aab and their $\mathcal{L}$-entrances are ba, using last theorem one can prove that u and v are $\sim$-congruent.

Theorem 3.2 (Maximal Groups Theorem) *Any maximal group in a free Burnside semigroup satisfying $x^{n+m} = x^n$ is a free Burnside group satisfying $x^m = 1$.
Moreover, this group is free over a generating set whose cardinality is the cyclomatic number of the fundamental graph of the $\mathcal{D}$-class that contains the group.*

Let r be the path $\widetilde{aba}, \widetilde{baab}$. Let B be $\{1_{\widetilde{ab}}, p, q, pq, r, pr, qr, pqr\}$ and let C be $\{ab, abab, abbab, ababbab, abaab, ababaab, abbabaab, ababbabaab\}$. By Free Local Groups Theorem, since the fundamental graph has cyclomatic number 3, the local group of $\widetilde{ab}$ is a free Burnside group satisfying $x^2 = 1$ with three generators. It is not difficult to verify that $\widehat{\ }$ establishes a bijection between B and the local group of $\widetilde{ab}$. The function σ establishes a bijection between C and B and by the proof of last theorem, one can prove that the $\mathcal{H}$-class of u is the image of the set aC by $\widetilde{\ }$.

4 Final Remarks

We would also like to state here some of other interesting results presented in [12]. First, we would like to remark that the graph in figure 1 is proved to be a *subgraph* of the fundamental graph of the $\mathcal{D}$-class of 55. Second, any irregular $\mathcal{D}$-class has a fundamental graph which has no edges and has only one vertex. Finally, for $n \geq 3$, the fundamental graph of any regular $\mathcal{D}$-class of $M(k, n, m)$ is a cycle and is characterized.

Suppose m is large enough. Consider $M(k, 2, m)$ and $M(k, 1, m)$ for $k \geq 2$. The following properties hold for both:

1. There are $\mathcal{D}$-classes whose maximal groups are free Burnside groups satisfying $x^m = 1$ with at least $2m-1$ generators.
2. These groups are infinite.
3. These free Burnside monoids are not finite $\mathcal{J}$-above.
4. Brzozowski's conjecture cannot be extended for these free Burnside monoids.

In the egg-box picture of the $\mathcal{D}$-class of $\widetilde{abc}$ in $M(k, 1, m)$, we have:

1. infinitely many $\mathcal{R}$-classes;
2. infinitely many $\mathcal{L}$-classes;
3. the $\mathcal{H}$-classes are free Burnside groups satisfying $x^m = 1$. (Hence these groups are infinite if they have at least two generators.) Indeed, these groups have infinitely many generators.

   N.B.: A complete version of this work is to appear on International Journal of Algebra and Computation [8].

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