Generalized Fitting Subgroups and p-Layers of Finite Groups

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This note was motivated by the desire to understand the similarities between the subgroups $F^*(G)$ and $E(G)$ defined by Bender [**B1**] and the p-layer $L_{p'}(G)$ and extended p-layer $L^*_{p'}(G)$ defined by Gorenstein and Walter [**GW**], as well as Bender's notion of p^*-groups [**B2**]. A slight variation on the last notion gives a common formulation for the first two theories. Unfortunately, it seems to have no other application.

All groups in this note are assumed to be finite.

Sylow functors

Definition 1. A **Sylow functor** $\mathcal{S}$ assigns to each finite group G a set $\mathcal{S}(G)$ of subgroups of G, such that the following properties hold:

(a) $\mathcal{S}(G)$ is a G-conjugacy class;
(b) If $\phi : G \to H$ is a surjection and $S \in \mathcal{S}(G)$, then $\phi(S) \in \mathcal{S}(H)$;
(c) If $N \triangleleft G$ and $S \in \mathcal{S}(G)$, then $S \cap N \in \mathcal{S}(N)$.

The Sylow functor $\mathcal{S}$ is **full** if and only if the following additional properties hold:

(d) If G is a group, $S \in \mathcal{S}(G)$, and $S \le H \le G$, then $S \in \mathcal{S}(H)$; and
(e) If $H \le G$, then every element of $\mathcal{S}(H)$ is contained in an element of $\mathcal{S}(G)$.

There are two examples of full Sylow functors, and perhaps no other interesting ones:

(1) $\mathcal{S}_0(G) = \{G\}$ for every G;
(2) For a fixed prime p, $\mathcal{S}_p(G) = \mathrm{S}yl_p(G)$ for every G.

Let $\mathcal{S}$ be a Sylow functor. Notice that by (b) of the definition, $\mathcal{S}(G)$ is $\mathrm{A}ut(G)$-invariant. Hence each $S \in \mathcal{S}(G)$ is intravariant, that is, the Frattini argument holds:

$$\text{(A)} \qquad \text{If } N \triangleleft G \text{ and } S \in \mathcal{S}(N), \text{ then } G = NN_G(S).$$

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Another simple consequence is that for direct products,

$$\mathcal{S}(G_1 \times G_2) = \{S_1 \times S_2 \mid S_1 \in \mathcal{S}(G_1) \text{ and } S_2 \in \mathcal{S}(G_2)\}. \tag{B}$$

Indeed if $S \in \mathcal{S}(G_1 \times G_2)$, then both the projection of S on G_i and $S \cap G_i$ are in $\mathcal{S}(G_i)$, so they are equal, which implies (B).

The groups in $\bigcup_G \mathcal{S}(G)$, as G varies over all groups, are called the $\mathcal{S}$-groups. They are thus the groups X for which $\mathcal{S}(X) = \{X\}$. Clearly normal subgroups and quotients of $\mathcal{S}$-groups are $\mathcal{S}$-groups.

$\mathcal{S}$-flat groups

Definition 2. Let $\mathcal{S}$ be a Sylow functor and G a group. Let H and K be normal subgroups of G such that $K \leq H$. The section H/K is **$\mathcal{S}$-neutral in** G if and only if for every $S \in \mathcal{S}(H/K)$, we have

$$\underset{G}{\mathrm{A}ut}(S) = \underset{H/K}{\mathrm{A}ut}(S) \tag{C}$$

Here whenever X is a group upon which the group H acts by automorphisms, and whenever $Y \leq X$, we write $\mathrm{A}ut_H(Y)$ for the image in $\mathrm{A}ut(Y)$ of $N_H(Y)$, the stabilizer of Y in H. Thus $\mathrm{A}ut_H(Y) \cong N_H(Y)/C_H(Y)$.

We make three simple remarks. First by Definition 1a), all elements of $\mathcal{S}(H/K)$ are H/K-conjugate and so to verify that H/K is $\mathcal{S}$-neutral in G it suffices to check (C) for one element of $\mathcal{S}(H/K)$. Second, condition (C) is trivial if $H = G$. Finally, condition (C) is equivalent to

$$G = HC_G(S). \tag{C$'$}$$

Indeed (C) is equivalent to $N_G(S) = N_B(S)C_G(S)$, and hence to (C') by a Frattini argument.

Definition 3. Let $\mathcal{S}$ be a Sylow functor. A group G is **$\mathcal{S}$-flat** if and only if each of its chief factors is $\mathcal{S}$-neutral in G.

Proposition 4. *Let $\mathcal{S}$ be a Sylow functor and G a group. Then*

(a) *If G is $\mathcal{S}$-flat, then so are every normal subgroup and quotient group of G.*

(b) *If G is the product of two normal $\mathcal{S}$-flat subgroups, then G is $\mathcal{S}$-flat.*

The proof depends on the following elementary lemmas.

Lemma 5. *Let $N \lhd G$ and let $C = B/A$ be a chief factor of G with $B \leq N$. Then there exist chief factors $F_1, \ldots, F_n$ of N such that $C \cong_N F_1 \times \cdots \times F_n$.*

The notation means that the two groups are isomorphic as groups with operators, the operator set being N. We could use the notation $C|_F$ in place of C.

Proof. There exists a chief factor B/D of N with $A \leq D$. Then $A = \cap_{g \in G} D^g$ since B/A is a chief factor of G. Consequently B/A, considered as a group with operator set N, is a subdirect product of the groups B/D^g, all of which are chief factors of N. The proof is completed by the familiar fact that a subdirect product of simple groups with operators is isomorphic (as a group with operators) to a direct product of some of those simple groups. $\boxed{\text{QED}}$

LEMMA 6. *Let $N \lhd G$ and let $C = B/A$ be a chief factor of G with $B \le N$. Let $F_1, \dots, F_n$ be the chief factors of G within C. Then*

(a) *If C is $\mathcal{S}$-neutral within G, then it is $\mathcal{S}$-neutral within N.*

(b) *C is $\mathcal{S}$-neutral within N if and only if each F_i is $\mathcal{S}$-neutral within N.*

PROOF. Let $S \in \mathcal{S}(C)$. Since $\mathrm{A}ut_C(S) \le \mathrm{A}ut_N(S) \le \mathrm{A}ut_G(S)$, (a) is trivial. Identifying C with $F_1 \times \cdots \times F_n$ as an N-group, and letting $S_i \in \mathcal{S}(F_i)$, we have $S = S_1 \times \cdots \times S_n$. Since N normalizes each F_i, $\mathrm{A}ut_N(S)$ normalizes each S_i. Consequently we may regard

$$\begin{aligned}
&\underset{N}{\mathrm{A}ut}(S) \le \underset{N}{\mathrm{A}ut}(S_1) \times \cdots \times \underset{N}{\mathrm{A}ut}(S_n), \text{ with} \\
\text{(D)} \qquad &\underset{C}{\mathrm{A}ut}(S) = \underset{C}{\mathrm{A}ut}(S_1) \times \cdots \times \underset{C}{\mathrm{A}ut}(S_n) \\
&\qquad\quad = \underset{F_1}{\mathrm{A}ut}(S_1) \times \cdots \times \underset{F_n}{\mathrm{A}ut}(S_n).
\end{aligned}$$

Furthermore, we assert that $\mathrm{A}ut_N(S)$ projects onto each direct factor $\mathrm{A}ut_N(S_i)$. Indeed for any i, any automorphism in $\mathrm{A}ut_N(S_i)$ is induced by some element of $N = BN_N(S)$ normalizing S_i, and hence by an element of $N_B(S_i)N_N(S)$. Hence to prove our assertion it is enough to show that the projection of $\mathrm{A}ut_N(S)$ on $\mathrm{A}ut_N(S_i)$ contains $\mathrm{A}ut_C(S_i)$; but this is clear by the second line of (D).

Now if C is $\mathcal{S}$-neutral in N, then the left sides in (D) coincide, and projecting onto each direct factor shows that each F_i is $\mathcal{S}$-neutral in N. Conversely if the right sides coincide, then so do the left sides as $\mathrm{A}ut_C(S) \le \mathrm{A}ut_N(S)$. $\boxed{\text{QED}}$

PROOF OF PROPOSITION 4. It is trivial that a quotient of an $\mathcal{S}$-flat group is $\mathcal{S}$-flat. Furthermore if $N \lhd G$ and G is $\mathcal{S}$-flat, then every chief factor F of N occurs between two terms of a chief series of G within N, i.e. within some chief factor C of G within N. Since G is $\mathcal{S}$-flat, C is $\mathcal{S}$-neutral in N by Lemma 6a, whence F is as well by Lemma 6b. As F was arbitrary, N is $\mathcal{S}$-flat by definition. This proves (a).

As for (b), suppose that $G = G_1G_2$ with $G_i \lhd G$ and G_i $\mathcal{S}$-flat. Let C be a chief factor of G and $S \in \mathcal{S}(C)$. Then C is isomorphic to a chief factor B/A such that either $G_1 \cap G_2 \le A \le B \le G_i$ ($i = 1$ or 2), or $B \le G_1 \cap G_2$.

In the first case, since every chief factor of G_i within C is $\mathcal{S}$-neutral within G_i, Lemma 6 yields that C is $\mathcal{S}$-neutral within G_i. Moreover $[B, G_j] \le G_1 \cap G_2 \le A$ (for $j \ne i$) and so $G_j \le C_G(S)$. As $G = G_1G_2$, these facts yield $\mathrm{A}ut_G(S) = \mathrm{A}ut_{G_i}(S) = \mathrm{A}ut_C(S)$ so C is $\mathcal{S}$-neutral in G, as desired.

Suppose then that $B \le G_1 \cap G_2$. By the Frattini argument $G_i = BN_{G_i}(S)$ for each i, so $G = G_1G_2 = BN_{G_1}(S)N_{G_2}(S)$. Hence $N_G(S) = N_B(S)N_{G_1}(S)N_{G_2}(S)$ and so to complete the proof it suffices to show that $\mathrm{A}ut_{G_i}(S) \le \mathrm{A}ut_C(S)$ for each i, i.e., C is $\mathcal{S}$-neutral in each G_i. But this again follows from Lemma 6b and the fact that G_i is $\mathcal{S}$-flat. The proof is complete. $\boxed{\text{QED}}$

$\mathcal{S}$-cores and Bender's theorem

Part (b) of Proposition 4 allows us to make the definition:

DEFINITION 7. Let $\mathcal{S}$ be a Sylow functor and G a group. Then $G_{\mathcal{S}}$ is the largest normal $\mathcal{S}$-flat subgroup of G.

Obviously $G_{\mathcal{S}}$ *chr* G.

THEOREM 8 (BENDER). *Let $\mathcal{S}$ be a Sylow functor and G a group. Let $S \in \mathcal{S}(G_{\mathcal{S}})$. Then $C_G(S) \le G_{\mathcal{S}}$.*

PROOF. By the Frattini argument $G = G_{\mathcal{S}}N_G(S)$, so if we put $C = G_{\mathcal{S}}C_G(S)$, then $C \triangleleft G$. We have $G_{\mathcal{S}} \le C$ and must prove equality. So suppose that $G_{\mathcal{S}} < C$ and let $D = B/G_{\mathcal{S}}$ be a chief factor of G with $B \le C$. Thus $B = G_{\mathcal{S}}C_B(S)$. We argue that B is $\mathcal{S}$-flat, which immediately will give the contradiction $B \le G_{\mathcal{S}}$.

Accordingly let F be a chief factor of B; we verify that F is $\mathcal{S}$-neutral in B. If F lies between B and $G_{\mathcal{S}}$, then observe that $B/G_{\mathcal{S}}$, being a chief factor of G, is isomorphic to the direct product of simple groups. Thus F is a top chief factor of B, so is $\mathcal{S}$-neutral in B, as remarked above. Finally if $F = H/K$ with $H \le G_{\mathcal{S}}$, we apply the following lemma, with $G_{\mathcal{S}}$ and B in the roles of N and G, to complete the proof. $\boxed{\text{QED}}$

LEMMA 9. *Let $N \triangleleft G$ and $S \in \mathcal{S}(N)$, and suppose that $G = NC_G(S)$. If N is $\mathcal{S}$-flat, then every chief factor of G within N is $\mathcal{S}$-neutral in G.*

PROOF. Let $C = H/K$ be a chief factor of G with $H \le N$. Set $S_C = S \cap H/S \cap K$, so that $S_C \in \mathcal{S}(C)$. Clearly $C_G(S) \le C_G(S_C)$ so $G = NC_G(S_C)$ and thus $\mathrm{A}ut_G(S_C) = \mathrm{A}ut_N(S_C)$. Moreover by the $\mathcal{S}$-flatness of N and Lemma 6b, C is $\mathcal{S}$-neutral in N, so $\mathrm{A}ut_N(S_C) = \mathrm{A}ut_C(S_C)$, completing the proof. $\boxed{\text{QED}}$

Reduced $\mathcal{S}$-cores; commutator-centralizer factorization

From this point on, we assume that $\mathcal{S}$ is a full Sylow functor.

DEFINITION 10. $\pi(\mathcal{S}) = \bigcup_G \bigcup_{S \in \mathcal{S}(G)} \pi(S)$, the first union over all finite groups G. Here $\pi(H)$ is the set of prime divisors of $|H|$, when H is a finite group.

DEFINITION 11. For any group G, $O^{\mathcal{S}}(G) = O^{\pi(\mathcal{S})}(G)G^{(\infty)}$, and $G^o_{\mathcal{S}} = O^{\mathcal{S}}(G_{\mathcal{S}})$.

Thus $G/O^{\mathcal{S}}(G)$ is the largest quotient of G which is a solvable $\pi(\mathcal{S})$-group.

THEOREM 12. *Let $\mathcal{S}$ be a full Sylow functor. Suppose that the $\mathcal{S}$-group P act on G. If $G = G^o_{\mathcal{S}}$, then*

$$G = (C_G(P))^o_{\mathcal{S}}[G, P].$$

In the proof we use the following lemmas.

LEMMA 13. *Suppose that $N \triangleleft G$, $S \in \mathcal{S}(N)$ and $G = NC_G(S)$. If N and G/N are both $\mathcal{S}$-flat, then G is $\mathcal{S}$-flat.*

PROOF. Let C be a chief factor of G. If C is a chief factor of N, then it is $\mathcal{S}$-neutral in G by the $\mathcal{S}$-flatness of G/N. If C lies in N, on the other hand, then C is $\mathcal{S}$-neutral in G by Lemma 9. Hence G is $\mathcal{S}$-flat by definition.

LEMMA 14. *Suppose that $G = NY$, where $N \triangleleft G$ and $Y \le G$. Let $S \in \mathcal{S}(N)$ and assume that $[Y, S] = 1$. Then $Y_{\mathcal{S}} \le G_{\mathcal{S}}$.*

PROOF. Since $NY_{\mathcal{S}} \triangleleft NY = G$, we may assume that $Y = Y_{\mathcal{S}}$. Let X be the subnormal closure of Y in G. Thus $X = \langle Y^X \rangle \triangleleft\triangleleft G$. Since $H_{\mathcal{S}}$ chr H for any group H, we have $X_{\mathcal{S}} \le G_{\mathcal{S}}$, so it is enough to show that $Y \le X_{\mathcal{S}}$. Moreover $X = (N \cap X)Y$, and as $X \triangleleft\triangleleft G$ we have $N \cap X \triangleleft\triangleleft N$ and thus $S \cap X \in \mathcal{S}(N)$. But

$[Y, S \cap X] = 1$ so the hypotheses are satisfied with X, $N \cap X$ and $S \cap X$ in place of G, N and S. Consequently we may assume that $G = X$, i.e, that $G = \langle Y^G \rangle$. We must then prove that $G = G_{\mathcal{S}}$.

Since Y is $\mathcal{S}$-flat by assumption, so is $G/N \cong Y/N \cap Y$, and so every chief factor of G above N is $\mathcal{S}$-neutral in G. Moreover $G = NY = NC_G(S)$ so by Lemma 9, every chief factor of G within N is $\mathcal{S}$-neutral in G. Thus G is $\mathcal{S}$-flat and the proof is complete. QED

Lemma 15. *Let $N \lhd G$ and $S \in \mathcal{S}(N)$ and assume that $G = NC_G(S)$ and G/N is $\mathcal{S}$-flat. Then $C_G(S)$ is $\mathcal{S}$-flat.*

Proof. First observe that

$$Z(S) \in \mathcal{S}(C_N(S)). \tag{E}$$

Namely Definition 1d) implies that $S \in \mathcal{S}(SC_N(S))$, and intersecting with the normal subgroup $C_N(S)$ of $SC_N(S)$ we get $Z(S) \in \mathcal{S}(C_N(S))$, as required.

Let C be a chief factor of $C_G(S)$. If C lies within $Z(S)$, then as $Z(S) \leq Z(C_G(S))$, C is $\mathcal{S}$-neutral in $C_G(S)$. If C lies within $C_N(S)/Z(S)$, then we use (E) to conclude that $\mathcal{S}(C_N(S)/Z(S)) = \{1\}$ and hence $\mathcal{S}(C) = \{1\}$, whence C is trivially $\mathcal{S}$-neutral in $C_G(S)$. Finally if C is a chief factor of $C_G(S)/C_N(S) \cong G/N$, then C is $\mathcal{S}$-neutral in $C_G(S)$ since G/N is $\mathcal{S}$-flat. Therefore $C_G(S)$ is $\mathcal{S}$-flat by definition. QED

Lemma 16. *Suppose that $N \lhd G$ and G is $\mathcal{S}$-flat. If $N \leq H \leq G$ and H/N is $\mathcal{S}$-flat, then H is $\mathcal{S}$-flat.*

Proof. Inductively we may assume that N is a chief factor of G. Then $G = NC_G(S)$ for $S \in \mathcal{S}(N)$, whence $H = NC_H(S)$. Since G is $\mathcal{S}$-flat, so is N; now Lemma 13 completes the proof. QED

Proof of Theorem 12. Form the semidirect product GP. Let N be a minimal normal subgroup of GP within G. By Definition 1e, there exist $U \in \mathcal{S}(C_N(P)P)$ and $T \in \mathcal{S}(GP)$ such that $P \leq U \leq T$. Set $S = T \cap N$. Thus $S \in \mathcal{S}(N)$ and S is P-invariant.

By induction $G/N = [P, G/N](C_{G/N}(P))^o_{\mathcal{S}}$. Let C be the complete preimage of the second factor, so that $G = [P, G]C$. Since $G = O^{\mathcal{S}}(G)$, we have $G = [P, G]O^{\mathcal{S}}(C)$. Moreover C and $O^{\mathcal{S}}(C)$ are $C_G(P)$-invariant and so $C_{O^{\mathcal{S}}(C)}(P)^o_{\mathcal{S}}$ is as well, whence $C_{O^{\mathcal{S}}(C)}(P)^o_{\mathcal{S}} \leq C_G(P)^o_{\mathcal{S}}$. If we can prove the factorization for $O^{\mathcal{S}}(C)$, we will then be done. By Lemma 16, C is $\mathcal{S}$-flat. Replacing G by $O^{\mathcal{S}}(C)$ we may therefore assume that $[P, G] \leq N$. Since N is a chief factor we may assume that $[P, G] = N$.

Since $G = G_{\mathcal{S}}$, we have $G = NC_G(S)$, and both factors are P-invariant. By Proposition 4a and Lemma 15, both factors are $\mathcal{S}$-flat. Indeed since $G = G^o_{\mathcal{S}}$, we get $G = NM$ where $M = O^{\mathcal{S}}(C_G(S))$, with N and M both P-invariant and $\mathcal{S}$-flat.

Suppose that $M < G$. Then by induction $G = NM = N[P, M](C_M(P))^o_{\mathcal{S}} = N(C_M(P))^o_{\mathcal{S}}$, and we need to show that $(C_M(P))^o_{\mathcal{S}} \leq C_G(P)^o_{\mathcal{S}}$. We set $X = C_G(P)$ and $Y = C_M(P)$ and apply Lemma 14 with X in the role of G there. By the Dedekind law, $X = (X \cap N)Y$, so $X \cap N$ will play the role of N in Lemma 14. By construction S contains an element of $\mathcal{S}(X \cap N)$, which is then centralized by Y. Now Lemma 14 implies that $Y_{\mathcal{S}} \leq X_{\mathcal{S}}$. Then $Y = O^{\mathcal{S}}(Y) \leq X^o_{\mathcal{S}}$, as desired.

Thus we may assume that $M = G$, whence $S \leq Z(G)$. If $S \neq 1$, then $S = N \leq Z(G)$ and $N \cong Z_p$ for some $p \in \pi(\mathcal{S})$. Put $\overline{G} = G/Z$, so that $\overline{G} = \overline{G}^o_{\mathcal{S}}$. Now $[G, P] \leq Z$, so for each $x \in P$ the mapping $y \mapsto [y, x]$ is a homomorphism from G to Z. Since $p \in \pi(\mathcal{S})$ and $G = O^{\mathcal{S}}(G)$ it follows that this mapping is trivial and so $G = C_G(P)$, and we are done.

Finally suppose that $S = 1$. Expanding P to $Q \in \mathcal{S}(NP)$ we have $Q \cap N \in \mathcal{S}(N) = \{1\}$. Therefore $Q = P(Q \cap N) = P$, so $P \in \mathcal{S}(NP)$. Moreover $NP \lhd GP$ as $[P, G] \leq N$. Consequently $GP = NPN_{GP}(P) = NN_G(P)P$ so $G = NN_G(P) = NC_G(P)$. As $S = 1 \in \mathcal{S}(N)$, we have $1 \in \mathcal{S}(C_N(P))$, so $C_N(P)$ is $\mathcal{S}$-flat. So is $C_G(P)/C_N(P) \cong G/N$, so $C_G(P)$ is $\mathcal{S}$-flat by Lemma 13. Therefore $G = NC_G(P)_{\mathcal{S}} = NC_G(P)^o_{\mathcal{S}}$, the last step as $G = G^o_{\mathcal{S}} = O^{\mathcal{S}}(G)$.

$\mathcal{S}$-balance

THEOREM 17 ($\mathcal{S}$-BALANCE). *Suppose that $\mathcal{S}$ is a full Sylow functor and all $\mathcal{S}$-groups are nilpotent. Then for any group G and any $\mathcal{S}$-subgroup P acting on G, $(C_G(P))^o_{\mathcal{S}} \leq G^o_{\mathcal{S}}$.*

PROOF. Forming PG as in Theorem 12, we may find $P \leq T \in \mathcal{S}(PG)$ and put $S = T \cap G$, so that $S \in \mathcal{S}(G)$ and S is P-invariant. Let $S_0 = S \cap G_{\mathcal{S}} \in \mathcal{S}(G_{\mathcal{S}})$. We shall show that

$$C_G(P)^o_{\mathcal{S}} \leq C_G(S_0)[S_0, G]. \tag{F}$$

By Theorem 8, the first term lies in $G_{\mathcal{S}}$, and the second obviously does. Therefore $C_G(P)^o_{\mathcal{S}} \leq G_{\mathcal{S}}$, whence $C_G(P)^o_{\mathcal{S}} = O^{\mathcal{S}}(C_G(P)_{\mathcal{S}}) \leq G^o_{\mathcal{S}}$.

We may assume that T was chosen to contain an element of $\mathcal{S}(PG_{\mathcal{S}})$, and this element is then PS_0. By assumption this group is nilpotent. Put $Q = N_{S_0}(P)$. Then $Q \lhd\lhd S_0P$ by nilpotence, and so Q is an $\mathcal{S}$-group. Now Q acts on $C = C_G(P)^o_{\mathcal{S}}$ so Theorem 12 gives

$$C = [Q, C]C_C(Q)^o_{\mathcal{S}}.$$

The first term lies in $[S_0, G]$, as required for (F). Moreover $C_C(Q) \lhd C_{C_G(P)}(Q) = C_G(PQ)$ so the second term lies in $C_G(PQ)^o_{\mathcal{S}}$. Now let $Q_1 = N_{S_0}(PQ)$ and apply Theorem 12 to the action of Q_1 on $C_G(PQ)^o_{\mathcal{S}}$, to obtain that the latter group lies in the product of $[S_0, G]$ and $C_G(PQ_1)^o_{\mathcal{S}}$. Continuing, and using the nilpotence of PS_0, we find that $C \leq [S_0, G]C_G(PS_0)^o_{\mathcal{S}}$. This implies (F) and the proof is complete. $\boxed{\text{QED}}$

Applications

When interpreted for the examples $\mathcal{S}_0$ and $\mathcal{S}_p$, our results are familiar. It is not difficult to prove the following characterizations.

THEOREM 18. *The group G is $\mathcal{S}_0$-flat if and only if G is the central product of a nilpotent group and quasisimple groups. Moreover, $O^{\mathcal{S}_0}(X) = X^{(\infty)}$.*

Consequently for any group G,

$$G_{\mathcal{S}_0} = F^*(G) \text{ and } G^o_{\mathcal{S}_0} = E(G).$$

Thus Theorem 8 is Bender's F^*-Theorem, and Theorem 12 is the identity $G = E(C_G(P))[G, P]$ for an arbitrary group acting on a group $G = E(G)$.

The application to p-layers has the usual defect of using the fact that G has the "Glauberman property":

(G_p)

For every nonabelian simple section K of G, and every $P \in \mathrm{Syl}_p(K)$, $C_{\mathrm{Aut}K}(P)$ has a normal p-complement.

For $p = 2$, this property holds by a theorem of Glauberman [**Gl**]. For $p > 2$ it also holds, but to date this has been proved only as a consequence of the classification of the finite simple groups (Theorem 7.1.1 of [**GLS3**]).

Theorem 19. *The group G is $\mathcal{S}_p$-flat if and only if $G/O_{p'}(G)$ is the product of commuting normal quasisimple groups ("components") with a p-nilpotent group centralizing a Sylow p-subgroup of each of the components. Moreover, $O^{\mathcal{S}_p}(X) = O^p(X)$.*

Thus, in the language of [**GW**], $G^o_{\mathcal{S}_p} = L^*_{p'}(G)$. Theorem 14 is $L^*_{p'}$-balance and has $L_{p'}$-balance as a corollary, by taking the subgroups of both sides generated by all their p-elements. (See [**GLS2**], p. 29.) Theorem 8 is then a generalization of p-constraint.

There is of course an analogous situation in the theory of linear algebraic groups. If we consider groups and "chief" series which we insist must consist of connected groups, and let $\mathcal{S}(G)$ be the set of maximal tori of G, then every connected linear algebraic group is "$\mathcal{S}$-flat"; this fits with the structure theorem for connected reductive groups (ignoring for the moment the question of classifying the simple groups).

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