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On Hamiltonian Cycles in Strong Products of Graphs

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ABSTRACT

We prove that the strong product $G_1 \times \cdots \times G_n$ contains a hamiltonian cycle for $n \geq \Delta$ whenever all G_i are connected graphs of maximum degree at most Δ; in particular $G^{\Delta(G)}$ contains a hamiltonian cycle. For large Δ we prove the same statement for $n \approx c\Delta$ for any $c > \ln(25/12) + 1/60$.

1 Introduction

Let the vertex set of the graph G be denoted by $V(G)$ and let its edge set be denoted by $E(G)$. The strong product of two graphs G and H is the graph $G \times H$ with the vertex set $V(G) \times V(H)$. Its two distinct vertices $[u_1, v_1]$ and $[u_2, v_2]$ are joined by an edge iff $u_1 = u_2 \vee u_1u_2 \in E(G)$ and $v_1 = v_2 \vee v_1v_2 \in E(H)$. A cycle containing all the vertices of the graph is called a hamiltonian cycle; each vertex is contained in such a cycle exactly once. We write G^k for the strong product of k copies of G and we call graphs containing a hamiltonian cycle hamiltonian graphs for the brevity.

Zaks asked in [3] whether there exists $k(G)$ for any connected graph G with at least two vertices such that $G^{k(G)}$ is hamiltonian. Bermond, Germa and Heydemann proved in [1] the existence of the number $k(G)$ and they proved that if G^k is hamiltonian, then also G^h is hamiltonian for all $k \leq h$ using some results of Rosenfeld and Barnette contained in [2]. They did not give any upper bound on $k(G)$ in terms of maximum degree of the graph G, but they conjectured that G^Δ should be hamiltonian for any connected graph G with at least two vertices and of maximum degree Δ (for $\Delta \geq 2$). We prove their conjecture. We also prove that G^Δ is not the smallest power of G containing a hamiltonian cycle for large value of Δ.

We introduce notation used in the paper and we give basic properties of strong graph products in Section 2. We study structure of strong products of graphs in Section 3. We prove the original conjecture by Bermond, Germa and Heydemann in Section 4. We give the proofs of our hamiltonian results for large value of Δ in Section 5.

2 Definitions and Basic Properties

All the graphs considered in this paper contain at least two vertices. Whenever we mention a product of graphs, we always mean the strong product of graphs; we are not going to use other definitions of the graph product in this paper. We write C_n for the cycle containing n vertices, P_n for the path containing n vertices and S_n for the star of order n, i.e. the star with n edges ($S_n = K_{1,n}$). We write $\leq k$ for the set of all the positive integer numbers less or equal to k; we use $\geq k$, $< k$ and $> k$ in a similar manner. Let I be a set of integer numbers, then S_I denotes the set of all the stars S_i such that $i \in I$; C_I and P_I are used in the similar manner. If A and B are sets of graphs, then we write $A \times B$ for $\{G \times H \mid G \in A \wedge H \in B\}$ and A^2 for $A \times A$, A^3 for $A \times A \times A$, etc.

We assign the vertices of P_n and S_n numbers and a star ($*$); this makes proofs dealing with paths or stars more clear. We understand the set $V(P_n)$ as $\{0, \ldots, n-1\}$; its two vertices are connected iff their difference is exactly one. We understand the set $V(S_k)$ as $\{*, 0, \ldots, k-1\}$; the edges of S_k are only between $*$ and the number vertices. Let S be a product of n stars; we write $V_i(S)$ for the set of vertices of S whose exactly i coordinates are equal to $*$. Let $a = [a_0, \ldots, a_{n-1}]$ be any vertex of S. We mean by x mod y the number between 0 and $y-1$ which is congruent to x modulo y, e.g. 9 mod 7 is 2. We use some useful notation in the paper: We write $\sum a$ for $\left(\sum_{i=0}^{n-1} a_i\right) \mod n$; we consider $*$ to be counted as

zero in this sum. We write $a[k \to s]$ for the vertex b obtained from a by substituting s for its k-th entry, i.e. $b_{k \bmod n} = s$ (we take s mod k instead of s if necessary) and $b_i = a_i$ for $i \neq k \bmod n$.

We address the question of existence of a hamiltonian cycle in products of graphs. The key theorem proved in [1] using ideas from [2] is the following one:

Theorem 1 *Let G be a connected graph of maximum degree at most Δ. Then any graph of $C_{\geq\Delta} \times G$ is hamiltonian.*

The immediate corollary of this theorem is that if G^k is hamiltonian then G^h is hamiltonian for all $h \geq k$. We write $h(G)$ for the smallest value of h such that G^h is hamiltonian; the existence of this value follows from the results of [1]. Let $h_{\max}(\Delta)$ be $\max\{h(G)|\Delta(G) \leq \Delta\}$ where $\Delta(G)$ is maximum degree of G. The existence of $h_{\max}(\Delta)$ follows from the results of Section 4 and Section 5. Zaks addressed in [3] the question of lower bounds for $h_{\max}(\Delta)$, too (he used different notation in his paper). He proved the following theorem:

Theorem 2 *If S_n^k is hamiltonian, then the following inequality holds:*

$$k \geq \frac{\ln 2}{\ln(1 + \frac{1}{n})} \approx n \ln 2$$

A covering of the graph G by the set of graphs I is a set J of subgraphs of G such that each vertex of G is contained exactly in one of the graphs of J and each graph of J is isomorphic to some of the graphs of I. Note that we cover the vertices of the graph, we do not cover its edges. The following theorem trivially holds (it is enough to consider all the subgraphs of $G \times H$ equal to the product of a subgraph covering G and each single vertex of H):

Theorem 3 *Let G and H be any graphs and let I be any set of graphs. If G can be covered by I, then $G \times H$ can be covered by I.*

3 Coverings of Graph Products by Stars and Cycles

We first prove some theorems about covering graphs by stars and about covering their products by small degree stars.

Lemma 1 *Each connected graph G of maximum degree at most Δ can be covered by $S_{\leq\Delta}$.*

Proof: We assume w.l.o.g. that G is a tree. The proof proceeds by induction on the number of vertices of G. The statement is trivial for $|V(G)| = 2$. Let us consider a leaf v of G for $|V(G)| \geq 3$. There exists a covering of $G \setminus \{v\}$ by $S_{\leq\Delta}$ due to the induction hypothesis; let us consider this covering. Let w be the only neighbour of v in G (remember that G is a tree and v is its leaf). We distinguish several cases:

- **The vertex w is covered by S_1.**
  We extend the star covering w by v to S_2.

- **The vertex w is covered by S_k ($k \geq 2$) and w is its center.**
  We extend the star covering w by v to S_{k+1} (note that $k+1 \leq \Delta$).

- **The vertex w is covered by S_k ($k \geq 2$) and w is its peripheral vertex.**
  We remove w from that star (obtaining S_{k-1}) and we create a new star S_1 consisting only of v and w.

It is easy to check that the obtained covering of G is a covering of G by $S_{\leq\Delta}$. ■

Theorem 4 *For all $k \geq 1$, $l \geq 2$, any graph of $S^k_{\leq kl}$ can be covered by $S_{\leq l}$.*

Proof: Let $S \in S^k_{\leq kl}$ be $S_{n_0} \times S_{n_1} \times \cdots \times S_{n_{k-1}}$ where $1 \leq n_i \leq kl$. We find a spanning subgraph T of S such that T has no isolated vertices and $\Delta(T) \leq l$. There exists a covering of T (and thus also of S) by $S_{\leq l}$ due to Lemma 1 used separately on each connected component of T.

Consider a vertex $a \in V(S) \setminus V_0(S)$. We say that coordinate i ($0 \leq i \leq k-1$) is substitutable iff there exists some $0 \leq b \leq n_i - 1$ such that $\sum a[i \to b] = i$. We call all b satisfying this condition possible substitutions for fixed i. Because $n_i \leq kl$ it is clear that there are at most l possible substitutions. Let $0 \leq i_1 < i_2 < \cdots < i_m \leq k-1$ be all indices such that $a_i = *$. Before constructing a spanning subgraph T of S we construct another spanning subgraph of S — we call it T'. We distinguish several cases when choosing edges adjacent to a in T':

- **At least one of the coordinates $i_1,\ i_2, \ldots,\ i_m$ of a is substitutable.**
  Let i be the smallest such substitutable coordinate and let $b_1,\ b_2, \ldots,\ b_p$ ($1 \leq p \leq l$) be all its possible substitutions. We join all the vertices $a[i \to b_j]$, $j = 1, \ldots,\ p$, to a in T'. We call the vertices $a[i \to b_j]$ for $j = 1, \ldots,\ p$ children of a and a their parent.

- **None of the coordinates $i_1,\ i_2, \ldots,\ i_m$ is substitutable.**
  If there exists some $p < i_1$ such that $a_p = 0$ and p is not substitutable, then we choose j to be the smallest such p and we join a and $b = a[j \to *]$ in T'. Otherwise, we join a and $b = a[i_1 \to 0]$ in T'. We call b a sibling of a. It is easy to see that if $b \notin V_0(S)$ then also a is a sibling of b. It is obvious that every vertex has at most one sibling and the siblings make pairs of the vertices which have no children.

All the edges of T' are determined by these two rules. It is easy to check that every vertex $a \in V_0(S)$ has exactly one parent, namely $a[\sum a \to *]$, and every other vertex $b \in V(S)$ has at most one parent — the only possible candidate is $b[\sum b \to *]$. All the vertices of S has at most l children due to the upper bound of the maximum number of possible substitions. It is also easy to check that vertex $a \in V(S) \setminus V_0(S)$ has a sibling iff it has no children, T' has no isolated vertices and its maximum degree is at most $l+1$ (l children and a parent). We change some edges of T' to get the desired subgraph T in two consecutive steps:

- We remove one of the edges connecting $a \in V(S) \setminus (V_0(S) \cup V_1(S))$ to its children if the degree of a is $l+1$. The degree of the child of a ¡ remains at least one, since it has either some children or a sibling.

- Let $a \in V_1(S)$ be a vertex of degree $l+1$. If there is a son of a, say s, which has a sibling, we remove the edge sa. Otherwise let s be arbitrary son a and let f be the parent of a. We replace edges sa and af by the edge sf.

The degree of vertices $a \in V_0(S)$ can be increased during this procedure to at most 2 and the degree of all the other vertices is decreased or remains below l. The spanning subgraph T of S obtained from T' through these changes has clearly the desired properties. ■

Theorem 5 *Let $G_0, \ldots, G_{n-1}$ be graphs of maximum degree at most Δ, $2 \leq l$ and $\Delta \leq nl$. Then $G_0 \times \cdots \times G_{n-1}$ can be covered by $S_{\leq l}$.*

Proof: By Lemma 1 each G_i can be covered by $S_{\leq nl}$ hence $G_0 \times \cdots \times G_{n-1}$ can be covered by $S^n_{\leq nl}$ and every graph of $S^n_{\leq nl}$ can be covered by $S_{\leq l}$ due to Theorem 4. ■

The following technical lemmas will prove their value in Section 5:

Lemma 2 *The graph $S_k \times S_l$ can be covered by $S_{\{\lfloor \frac{l}{2} \rfloor,\ \lfloor \frac{l-1}{2} \rfloor,\ \lfloor \frac{k+1}{2} \rfloor,\ \lfloor \frac{k}{2} \rfloor\}}$ for all $k \geq 2$, $l \geq 3$.*

Proof: We construct a covering of $S_k \times S_l$ with the desired properties. Centers of stars of the covering are the vertices $[a, b]$ such that at least one of a, b is $*$. The covering consists of several types of stars:

- The center vertex $[a, *]$ for odd a is joined to the following peripheral vertices $[a, 1]$, $[a, 3], \ldots,\ [a, 2\left\lfloor \frac{l}{2} \right\rfloor - 1]$; order of this star is $\left\lfloor \frac{l}{2} \right\rfloor$.

- The center vertex $[a, *]$ for even a is joined to the following peripheral vertices $[a, 2]$, $[a, 4], \ldots,\ [a, 2\left\lfloor \frac{l-1}{2} \right\rfloor]$; order of this star is $\left\lfloor \frac{l-1}{2} \right\rfloor$.

- The center vertex $[*, b]$ for odd b is joined to the following peripheral vertices $[0, b]$, $[2, b], \ldots,\ [2\left\lfloor \frac{k-1}{2} \right\rfloor, b]$; order of this star is $\left\lfloor \frac{k+1}{2} \right\rfloor$.

- The center vertex $[*, b]$ for even b is joined to the following peripheral vertices $[1, b]$, $[3, b], \ldots,\ [2\left\lfloor \frac{k}{2} \right\rfloor - 1]$; order of this star is $\left\lfloor \frac{k}{2} \right\rfloor$.

- The center vertex $[*, *]$ is joined to all the remaining peripheral vertices, namely $[0, 0],\ [2, 0], \ldots,\ [2\left\lfloor \frac{k-1}{2} \right\rfloor, 0]$; order of this star is $\left\lfloor \frac{k+1}{2} \right\rfloor$.

It is a routine to check that all the vertices of $S_k \times S_l$ are used exactly once and thus the construction gives a covering of $S_k \times S_l$ by $S_{\{\lfloor \frac{l}{2} \rfloor,\ \lfloor \frac{l-1}{2} \rfloor,\ \lfloor \frac{k+1}{2} \rfloor,\ \lfloor \frac{k}{2} \rfloor\}}$. ■

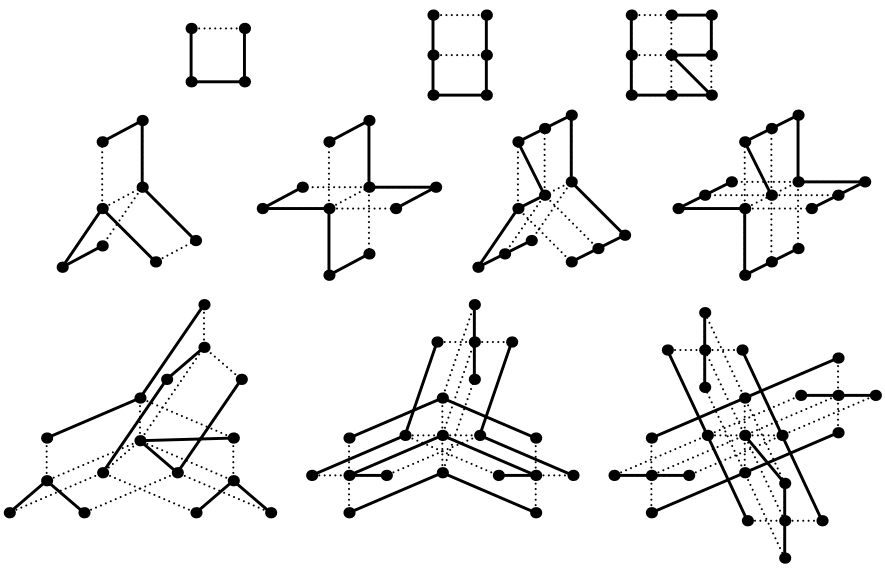


Figure 1: Path coverings of $S_1 \times S_1$, $S_1 \times S_2$, $S_2 \times S_2$, $S_1 \times S_3$, $S_1 \times S_4$, $S_2 \times S_3$, $S_2 \times S_4$, $S_3 \times S_3$, $S_3 \times S_4$ and $S_4 \times S_4$

Lemma 3 *Any graph of $S_{\leq 2n} \times S_{\leq 2n+1}$ can be covered by $S_{\leq n}$.*

Proof: It is clear due to Theorem 4 that all the graphs of $S_{\leq 2n} \times S_{\leq 2n}$ can be covered by $S_{\leq n}$. The graphs of $S_{\leq n} \times S_{\leq 2n+1}$ can be covered by $S_{\leq n}$ due to Theorem 3. Thus it is enough to prove that for every $n < k \leq 2n$ $S = S_k \times S_{2n+1}$ can be covered by $S_{\leq n}$. This is a direct corollary of Lemma 2 when setting $l = 2n + 1$. ■

Lemma 4 *Any graph of $S_{\{4,5,6,7,8\}} \times S_{\{5,6,7,8,9\}}$ can be covered by $S_{\{2,3,4\}}$.*

Proof: The statement immediately follows from Lemma 2. ■

Lemma 5 *Any graph of $S_{\leq 4} \times S_{\leq 4}$ can be covered by $P_{\geq 3}$.*

Proof: It is straightfoward to check that the following coverings exist (see Figure 1):

- $S_1 \times S_1$ can be covered by one copy of P_4.
- $S_1 \times S_2$ can be covered by one copy of P_6.
- $S_1 \times S_3$ can be covered by two copies of P_4.
- $S_1 \times S_4$ can be covered by two copies of P_5.
- $S_2 \times S_2$ can be covered by one copy of P_9.

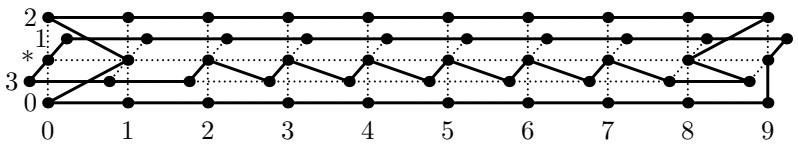


Figure 2: Hamiltonian cycle in $P_{10} \times S_4$

- $S_2 \times S_3$ can be covered by one copy of P_{12}.
- $S_2 \times S_4$ can be covered by one copy of P_7 and one copy of P_8.
- $S_3 \times S_3$ can be covered by four copies of P_3 and one copy of P_4.
- $S_3 \times S_4$ can be covered by five copies of P_3 and one copy of P_5.
- $S_4 \times S_4$ can be covered by seven copies of P_3 and one copy of P_4.

■

Lemma 6 *Any graph of $P_{\geq 4} \times S_4$ is hamiltonian.*

Proof: We just construct a hamiltonian cycle in $P_k \times S_4$ for $4 \leq k$. The following sequence of the vertices of $P_k \times S_4$ is a hamiltonian cycle (see Figure 2):

$$[0,0],[1,0],\ldots,[k-2,0],[k-1,0],[k-1,*],[k-1,1],[k-2,1],\ldots,[1,1],[0,1],[0,*],$$

$$[0,3],[1,3],[2,3],[2,*],[3,3],[3,*],[4,3],[4,*],\ldots,[k-3,3],[k-3,*],[k-2,3],$$
$$[k-1,3],[k-2,*],[k-1,2],[k-2,2],\ldots,[1,2],[0,2],[1,*],[0,0]$$

■

The proof of the following lemma uses results of Section 4; the lemma itself is used only in Section 5.

Lemma 7 *Let n_0, n_1, n_2 be integers between 1 and 4 (inclusively) and let k be an integer such that $k \leq \min\{(n_0+1)(n_1+1)(n_2+1), 15\}$. Then $S_{n_0} \times S_{n_1} \times S_{n_2} \times S_k$ can be covered by $C_{\geq l}$ where l is equal to $\min\{(n_0+1)(n_1+1)(n_2+1), 15\} \cdot k$.*

Proof: Let us assume $n_0 \leq n_1 \leq n_2$. If $n_2 \leq 3$ then $S_{n_0} \times S_{n_1} \times S_{n_2}$ is hamiltonian due to Theorem 6 and $S_{n_0} \times S_{n_1} \times S_{n_2} \times S_k$ is hamiltonian due to Theorem 1 since $k \leq (n_0+1)(n_1+1)(n_2+1)$. Simple calculation gives that the length of this hamiltonian cycle is sufficient.

Let us consider the case that $n_2 = 4$. Due to Lemma 5 $S_{n_0} \times S_{n_1}$ can be covered by $P_{\geq 3}$. The product of $P_{\geq 4}$ and S_{n_2} is hamiltonian due to Lemma 6 and its hamiltonian cycle contains at least 20 vertices. The product of cycle of length at least 20 and S_k is hamiltonian due to Theorem 1 and thus the product of any of $P_{\geq 4}$ with $S_{n_2} = S_4$ and S_k can be covered by

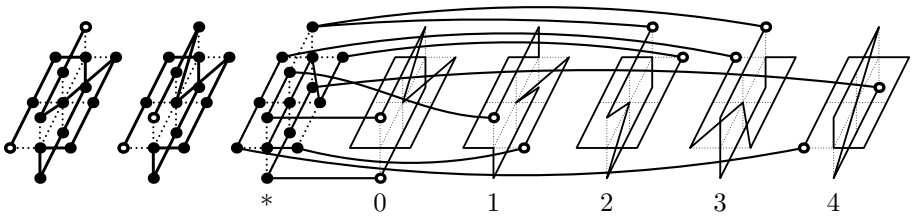


Figure 3: Hamiltonian paths in $P_3 \times S_4$ and a hamiltonian cycle in $P_3 \times S_4 \times S_5$

$C_{\geq 20(k+1)}$. The proof that $P_3 \times S_4 \times S_k$ is hamiltonian would finish the proof of the whole lemma. Simple calculation gives that the lengths of the found cycles are sufficient.

We prove now that $P_3 \times S_4 \times S_k$ is hamiltonian. Consider two possible types of hamiltonian paths in $P_3 \times S_4$ shown at the left in Figure 3. These paths in the different copies of $P_3 \times S_4$ corresponding to the numbered vertices of S_k can be joined together through the copy of $P_3 \times S_4$ corresponding to the vertex $*$ of S_k as shown in Figure 3 for $k = 5$; it is a straightforward work to check that the hamiltonian paths can be joined for all $1 \leq k \leq 15$. Since $k \leq 15$ due to the assumption of the lemma, we have found a hamiltonian cycle of $P_3 \times S_4 \times S_k$. ■

4 The Proof of the Conjecture

We prove Bermond, Germa and Heydemann's conjecture in this section. We need to introduce several definitions and new notation to make our life easier. We are going to deal with coverings of undirected graphs consisting of directed cycles — we simply take undirected cycles and consistently orient their edges, i.e. we orient the edges such that the in-degrees and out-degrees of all the vertices of the cycles are one; we do not demand that the orientation has to satisfy any additional conditions. We write u^+ for the successor and u^- for the predecessor of the vertex u in the cycle.

If G is $G_0 \times \cdots \times G_{m-1}$, we understand the vertex $v \in V(G)$ as $[v_0, \ldots, v_{m-1}]$ where $v_i \in G_i$ for $0 \leq i \leq m-1$. We say that v is k–constant iff $v_k = v_k^+$. If all G_i are stars, we say that v is simple, if $v_i \in \{*, 0\}$ for all i. We write $*v$ for the maximal i such that $v_i = *$. We write $*^m$ for a vertex of G whose all coordinates are stars. If $r = [r_0, \ldots, r_{k-1}, r_{k+1}, \ldots, r_{m-1}] \in G_0 \times \cdots G_{k-1} \times G_{k+1} \times \cdots \times G_{m-1}$ and $w \in G_k$, then we write $r[w]$ for $[r_0, \ldots, r_{k-1}, w, r_{k+1}, \ldots, r_{m-1}] \in G_0 \times \cdots \times G_{m-1}$; the value of k will be clear from the context.

We first prove the conjecture for stars. Stars seem to be the worst case and we use later Theorem 6 for stars to prove the conjecture for general graphs.

Theorem 6 *Any graph of $S^{\Delta}_{\leq \Delta}$ is hamiltonian for $\Delta \geq 2$.*

Proof: The proof proceeds by induction on Δ. The statement is clear for $\Delta = 2$, since the product of two paths is certainly hamiltonian. Let $S = S_{n_0} \times \cdots \times S_{n_{\Delta-1}}$ be any graph

of $S^{\Delta}_{\leq\Delta}$ and Δ be at least 3. We assume w.l.o.g. that $n_0 \geq n_1 \geq n_2, \ldots, n_{\Delta-1}$, i.e. we assume that n_0 and n_1 are the two largest integers between $n_0, \ldots, n_{\Delta-1}$. If $n_1 < \Delta$, then $S_{n_1} \times \cdots \times S_{n_{\Delta-1}}$ is hamiltonian due to the induction hypothesis and S is hamiltonian due to Theorem 1 used for S_{n_0} and the hamiltonian cycle of $S_{n_1} \times \cdots \times S_{n_{\Delta-1}}$. Let n_0 and n_1 be equal to Δ in the rest of the proof.

The construction of the hamiltonian cycle consists of several steps. First, we find a set $\mathcal{Z}$ of directed cycles covering all the vertices of $V_0(S)$. Next, we construct a long cycle L in S and we use it to join all the small cycles of $\mathcal{Z}$ covering $V_0(S)$. Finally, we insert the remaining vertices of S to the obtained cycle.

The edge xy is a directed edge of a cycle of $\mathcal{Z}$ iff it satisfies one of the following rules:

- $x \in V_0(S)$, $y \in V_1(S)$ and $y = x[\sum x + 1 \to *]$ ($\Leftrightarrow x = y[*y \to *y - \sum y - 1]$)
- $x \in V_1(S)$, $y \in V_0(S)$ and $x = y[\sum y \to *]$ ($\Leftrightarrow y = x[*x \to *x - \sum x]$)
- $x \in V_1(S)$, $y \in V_2(S)$, $y = x[0 \to *]$ and x has a predecessor but no successor defined by the first two conditions ($\Leftrightarrow n_{*x} < \Delta$ and $(*x - \sum x) \bmod \Delta = n_{*x}$)
- $x \in V_2(S)$, $y \in V_1(S)$, $x = y[0 \to *]$ and y has a successor but no predecessor defined by the first two conditions ($\Leftrightarrow n_{*y} < \Delta$ and $*y - \sum y = 0$)

Before we prove that these rules define directed cycles, we give (as an example) a part of a cycle defined by these rules in $S_3 \times S_3 \times S_2$:

$$[0,0,1] \to [0,0,*] \to [*,0,*] \to [2,0,*] \to [2,0,0] \to [*,0,0] \to [0,0,0] \to \ldots$$

Step 1: Every vertex of S has either both in-degree and out-degree one or both zero, i.e. $\mathcal{Z}$ consists of directed cycles covering $V_0(S)$.
It is clear that the in-degrees and out-degrees for vertices of $V_0(S)$, $V_1(S)$ and $V_{\geq 3}(S)$ are equal and at most 1 and that all the vertices of $V_0(S)$ are covered by $\mathcal{Z}$. Let $x \in V_2(S)$. If $x_0 \neq *$ or $x_{*x} = \Delta$ then x has both in-degree and out-degree zero. Otherwise it must hold that $x^- = x[0 \to *x - n_{*x} - \sum x]$ and $x^+ = x[0 \to *x - \sum x]$, thus the both in-degree and out-degree of x is exactly one.

Step 2: For each directed cycle $C \in \mathcal{Z}$ there is a vertex $u \in C \cap V_0(S)$ such that $u_1 = 0$ and $\sum u = 0$. We choose any such u to be a representant of C and we write $r(C)$ for it.
Let C be fixed for the proof of this statement. Let a be a vertex of $C \cap V_0(S)$; we call a vertex b the first non–star successor of a if $b \in C \cap V_0(S)$ and there is no vertex of $C \cap V_0(S)$ between a and b in the cycle C; note that either $b = a^{++}$ or $b = a^{++++}$. We call a vertex c the second non–star successor of a, if it is the first non–star successor of the first non–star successor of a; we define in this fashion the i–th non–star successor of a. Let a be any vertex of $C \cap V_0(S)$ and b its first non–star successor; the following three claims are true for a and b:

- $\sum b = (\sum a + 1) \bmod \Delta$

- $b_i = a_i$ for $i \neq 0$ and $i \neq (\sum a + 1) \bmod \Delta$
- $b_i = (a_i + 1) \bmod n_i$ for $i = (\sum a + 1) \bmod \Delta$

If $b = a^{++}$, then $b = a[\sum a + 1 \to a_{(\sum a+1) \bmod \Delta} + 1]$ and the claims are true. If $b = a^{++++}$, then $b = a[\sum a + 1 \to 0][0 \to a_0 + a_{(\sum a+1) \bmod \Delta} + 1]$ and the claims are true, too.

Let a be any vertex of $C \cap V_0(S)$; let b be its $(\Delta - \sum a)$-th non–star successor; $\sum b$ is zero due to the first claim. Let $b^0 = b$ and b^i be the $(i\Delta)$-th non–star successor of b for $1 \leq i < \Delta$. The sum of entries of any b^i is zero due to the first claim. Due to the second and the third claim, it holds that $b_1^{i+1} = (b_1^i + 1) \bmod \Delta$. Thus exactly one of b^i $(0 \leq i < \Delta)$ satisfies the above conditions and can be choosen as a representant of the cycle C.

Step 3: There is a directed cycle L in S which contains all the vertices of the set $\{u \in V_2(S) | u_0 = u_1 = *\}$ and no vertices of cycles of $\mathcal{Z}$.
Let W be the set $\{u \in V_2(S) | u_0 = u_1 = *\}$ and let $u^0, u^1, \ldots, u^{m-1}$ be a sequence of its vertices sorted lexicographically; consider u^m to be u^0. We first deal separately with the case $|W| = 1$, i.e. $n_2 = \ldots = n_{\Delta-1} = 1$. We L to be a cycle consisting of two vertices: $[*, *, 0, \ldots, 0], [0, *, \ldots, *], [*, *, 0, \ldots, 0]$; this is not actually a cycle, but this does not affect the proof. If $|W| > 1$, fix i an integer between 0 and $m-1$ and let k be the first coordinate in which the vertex u^i and the vertex u^{i+1} differ. Let $\tilde{u}^i = [-\sum u^{i+1}, *, u_2^i, \ldots, u_{k-1}^i, *, \ldots, *] = [-\sum u^{i+1}, *, u_2^{i+1}, \ldots, u_{k-1}^{i+1}, *, \ldots, *]$ (note that all $\tilde{u}^i$ are mutually different). No vertex u^i or $\tilde{u}^i$ is contained in any cycle of $\mathcal{Z}$ and there are edges $u^i\tilde{u}^i$ and $\tilde{u}^i u^{i+1}$ in S. Thus we can set L to be $u^0, \tilde{u}^0, u^1, \tilde{u}^1, \ldots, u^{m-1}, \tilde{u}^{m-1}, u^m = u^0$.

Step 4: There is a directed cycle L' in S which contains all the vertices of the cycles of $\mathcal{Z}$ and all the vertices of the cycle L.
For each cycle C of $\mathcal{Z}$ let $r(C)$ be its representant and let $l(C)$ be $r(C)[0 \to *][1 \to *]$. We write in this step r for $r(C)$ and l for $l(C)$. We replace edges rr^+ and l^-l by edges l^-r^+ and rl. We need to check that l^-r^+ and rl are really edges of S and that l is a different vertex of W for different cycles of $\mathcal{Z}$:

- The entries of l and r are equal except for the first two entries which are in case of l equal to $*$; thus there is an edge rl in S.
- All the coordinates of l^- are either equal to the coresponding coordinates of r or they are equal to $*$ with a possible exception for the coordinate indexed by zero. $\sum r = 0$ due to the selection of the representant of C, $\sum l = (-r_0 - r_1 + \sum r) \bmod \Delta = (-r_0) \bmod \Delta$ due to the selection of l as $r[0 \to *][1 \to *]$. Since $l_0^- = -\sum l$ due to the construction of L, it must hold that $l_0^- = r_0$ and there is the edge l^-r in S. Since $\sum r = 0$, it holds that $r^+ = r[1 \to *]$ and thus (since there is the edge l^-r in S) there is the edge l^-r^+.
- Suppose that $l = l(C')$ for two different cycles C and C'. It follows that $r_i = r(C')_i$ for $i \geq 2$ from the equality $l = l(C')$, $r_1 = r(C')_1 = 0$ due to the selection of the representants of the cycles. But since $\sum r = \sum r(C') = 0$ it must be also $r_0 = r(C')_0$ and thus $r = r(C')$. Therefore the cycles C and C' are actually the same cycle. Thus the vertices l are different for different cycles $C \in \mathcal{Z}$.

Step 5: S is hamiltonian.
The cycle L' contains all the vertices of $V_0(S)$. Let $u \notin L'$, we can insert this vertex to L' as follows: Let $\tilde{u}$ be any vertex of L' obtained from u by substituting some of the coordinates of u equal to $*$ by arbitrary numbers; note that if we substitute all the coordinates equal to $*$ by numbers, we get a vertex of $V_0(S)$, thus a vertex of L', i.e. it is always possible to get a vertex $\tilde{u} \in L'$. We can replace the edge $\tilde{u}\tilde{u}^+$ of L' with the edges $\tilde{u}u$ and $u\tilde{u}^+$ (it is easy to check that these are really edges of S). We can also insert u to L' by replacing the edge $\tilde{u}^-\tilde{u}$ with the edges $\tilde{u}^-u$ and $u\tilde{u}$. In this manner we can insert to L' all the vertices not contained in L'. The exact procedure of insertion of these vertices to the cycle L' is up to us and we will adjust this part of the proof of Theorem 6 in the proofs of Lemma 8 and Lemma 9. ■

We describe the properties of the hamiltonian cycle constructed in the proof of Theorem 6 in the following three claims. Since we are going to deal with several modifications of the last step of the construction of the hamiltonian cycle later in this section, we get a precise description of all the vertices inserted to the cycle in the last step in the first of these claims. We also describe some basic properties of the cycle L' used in the proof. The proofs of the claims are straightforward and thus omitted.

Claim 1 *Any vertex v contained in the cycle L' in the proof of Theorem 6 satisfies one of the following conditions:*

- $v \in V_0(S)$, *i.e. v contains no coordinate equal to* $*$
- $v \in V_2(S)$ *and* $v_0 = v_1 = *$
- $v \in V_2(S)$, $v_0 = *$ *and* $n_{*v} < \Delta$
- $v \in V_1(S)$, *i.e. v contains one coordinate equal to* $*$
- *There exists* $i_0 \geq 2$, *such that* $v_i = *$ *iff* $i \in \{1, i_0, i_0 + 1, \ldots, \Delta - 2, \Delta - 1\}$.

Note that all the vertices satisfying any of the first three conditions are contained in L', i.e. the first three conditions are sufficient for vertices to be included in L'.

Claim 2 *If v is a vertex of L' containing at least one star and $v_1 \neq *$, then v^- and v are 1–constant in L'. (Note that such v has to be already contained in some of the cycles of $\mathcal{Z}$.)*

Claim 3 *If v is a vertex of L' containing at least two stars and $v_1 = *$, then v is 1–constant in L'.*

The immediate corollary of the last theorem is the following:

Corollary 1 *The product of any at least Δ connected graphs of maximum degree Δ can be covered by $C_{\geq\Delta}$ for $\Delta \geq 2$.*

Proof: Due to Lemma 1 each of the graphs of the product can be covered by $S_{\leq\Delta}$ and due to Theorem 6 the products of these stars are hamiltonian. Since each of these star products contains at least 2^Δ vertices, their hamiltonian cycles are a covering of the product of the graphs by $C_{\geq 2^\Delta} \subseteq C_{\geq\Delta}$. ■

The last theorem states that the product of Δ stars of order at most Δ contains a hamiltonian cycle; the following two lemmas describe properties of these hamiltonian cycles in more depth. We will prove these lemmas together since their proofs are almost identical.

Lemma 8 *Let $\Delta \geq 3$, let $k \neq l$ be integers between 0 and $\Delta - 1$ and let $n_0, \ldots, n_{k-1}$, $n_{k+1}, \ldots, n_{\Delta-1}$ be integers between 1 and Δ. Let s be either $*^\Delta$ or $*^\Delta[l \to 0]$. Then there exists a subset R the vertices of $S_{n_0} \times \cdots \times S_{n_{k-1}} \times S_{n_{k+1}} \times \cdots \times S_{n_{\Delta-1}}$ such that $\Delta - 1 \leq |R|$ and for any choice of n_k between 1 and Δ there is a hamiltonian cycle C in $S = S_{n_0} \times \cdots \times S_{n_{\Delta-1}}$ satisfying the following conditions:*

- *The vertex $r[t]$ is k–constant for all $r \in R$ and $t \in S_{n_k}$ with $\deg_{S_{n_k}} t < \Delta$.*
- *s is l–constant, s^+ (the succestor of s in C) is simple and s is different from $r[*]$ for all $r \in R$.*

Lemma 9 *Let $\Delta \geq 3$, let $0 \leq k \leq \Delta - 1$ and let $n_0, \ldots, n_{k-1}, n_{k+1}, \ldots n_{\Delta-1}$ be integers between 1 and Δ. Then there exists a subset R of the vertices of $S_{n_0} \times \ldots \times S_{n_{k-1}} \times S_{n_{k+1}} \times \ldots \times S_{n_{\Delta-1}}$ such that $\Delta - 1 \leq |R|$ and for any choice of n_k between 1 and Δ there is a hamiltonian cycle C in $S = S_{n_0} \times \ldots \times S_{n_{\Delta-1}}$ satisfying the following conditions:*

- *The vertex $r[t]$ is k–constant for all $r \in R$ and $t \in S_{n_k}$ with $\deg_{S_{n_k}} t < \Delta$.*
- *R contains at least two simple vertices.*

Proof: We prove the previous two lemmas together, since their proofs are very similar. The proof proceeds by induction on Δ. We prove even more, namely that there is always $r_0 \in R$ such that r_0 is simple, at least one of its coordinates is equal to $*$ and $r_0[t]$ is k–constant for all $t \in S_{n_k}$. We assume w.l.o.g. that $k = \Delta - 1$ and $n_0 \geq n_1 \geq \ldots \geq n_{\Delta-2}$. We distinguish several cases in the proof:

- **$n_0 < \Delta$, the proof of Lemma 8 and Lemma 9**
  We first consider the case $\Delta = 3$ and $n_0 = n_1 = 1$. We can assume w.l.o.g. that $l = 0$ in Lemma 8, thus s is either $[*, *, *]$ or $[0, *, *]$. We set R to $\{[*, 0], [0, 0]\}$ and r_0 to $[*, 0]$. The desired hamiltonian cycles in $S_1 \times S_1 \times S_1$, $S_1 \times S_1 \times S_2$ and $S_1 \times S_1 \times S_3$ for different choices of s are shown in Figure 4.

  There is either $\Delta \geq 4$ or $n_0 \geq 2$ in the remaining case. There is a hamiltonian cycle C' in $S_{n_0} \times \cdots \times S_{n_{\Delta-2}}$ due to Theorem 6. The length of this cycle is even at least 2Δ (since $\Delta \geq 4$ or $n_0 \geq 2$). Let $v^0, \ldots, v^m$ be the vertices of C' ($v^{i+1} = (v^i)^+$; note that $m \geq 2\Delta - 1$ and note that at least four of these vertices are simple) and let

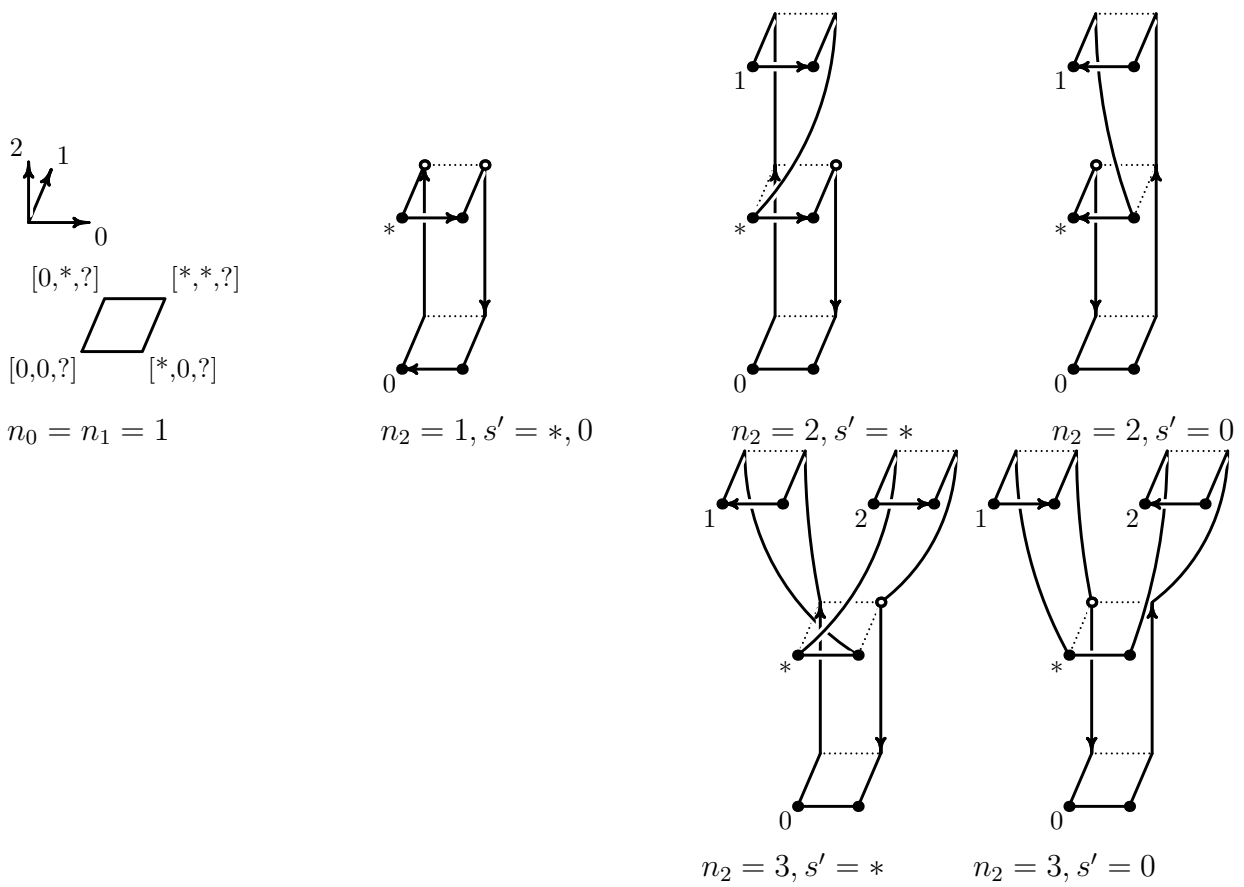


Figure 4: The desired hamiltonian cycles in $S_1 \times S_1 \times S_{n_2}$ for $n_2 = 1, 2, 3$ and for $s = [s', *, *], s' \in \{*, 0\}$. The set R is $\{[*, 0], [0, 0]\}$. The vertex s is drawn as an empty circle, other important vertices are drawn as full circles.

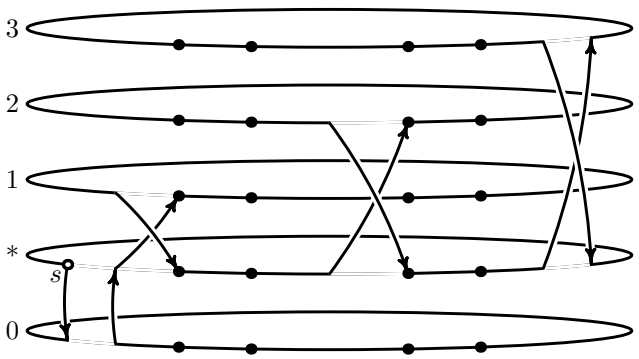


Figure 5: Joining different cycles in the product of a cycle C and a star $S_{n_{\Delta-1}}$ for $n_{\Delta-1} = 4$ in the proof of the case $n_0 < \Delta$; the vertex s and the vertices $r[t]$ for $r \in R$ and $t \in \{*, 0, \ldots, 3\}$ are drawn as empty circles

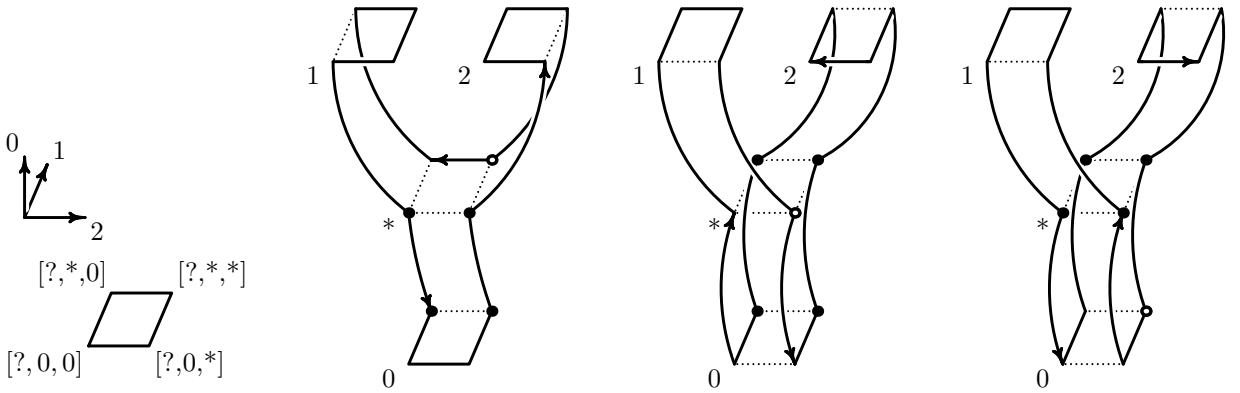


Figure 6: The hamiltonian cycles in $S_3 \times S_1 \times S_1$; the vertex s is drawn as an empty circle, the other important vertices are drawn by full circles.

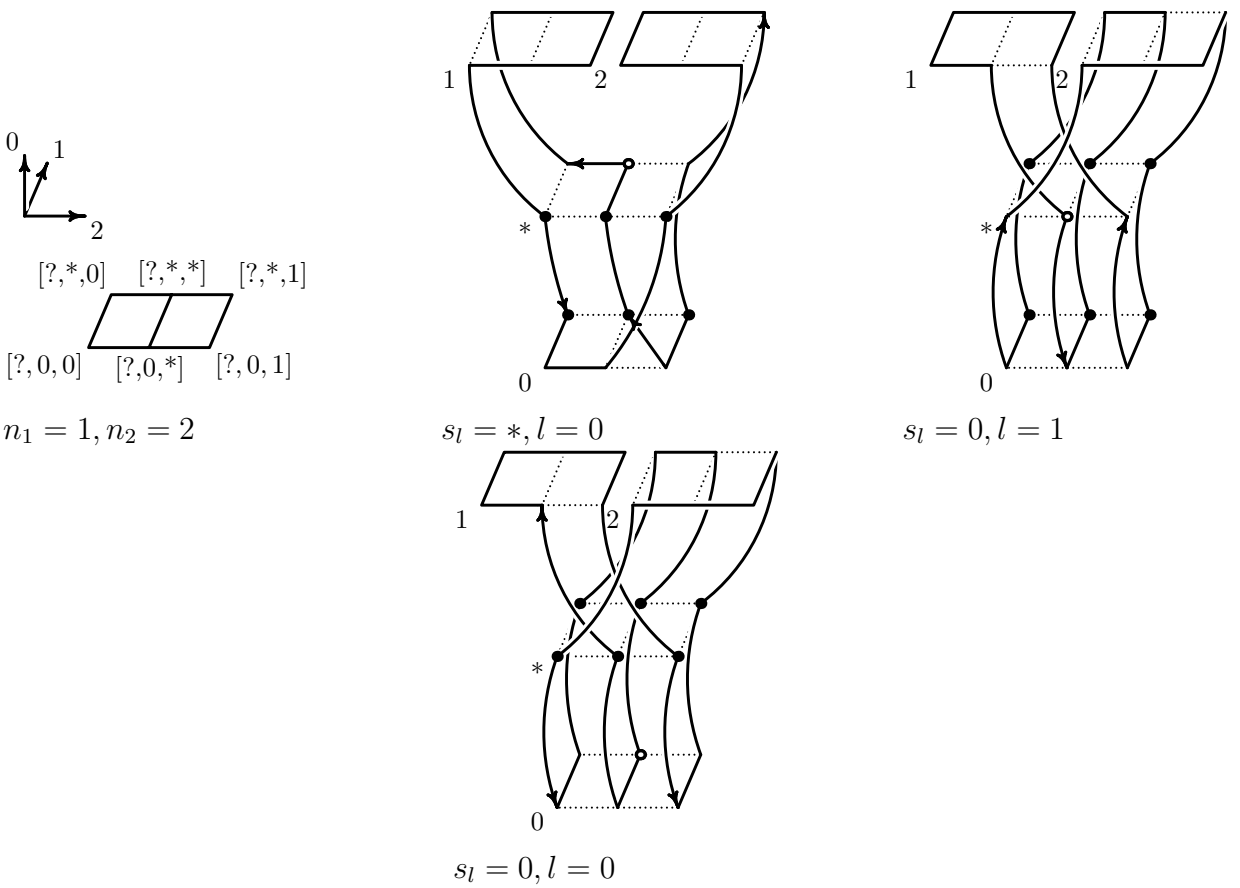
w.l.o.g. $v^0[*] = s$ (in case of Lemma 8). Let R be any set of vertices of C' of size $\Delta - 1$ which contains at least two simple vertices and which does not contain either v^0 or v^1 and let $r_0 \neq [0, \ldots, 0]$ be any simple vertex of R. We use a construction which is very similar to that used in the proof of Theorem 1 — see Figure 5. We use edges from and to vertices v^0 and v^1 to join the cycles corresponding to the vertices $*$ and to vertices 0 of $S_{n_{\Delta-1}}$ and we use edges from the vertices of C' different from v^0 and different from the vertices of R to join the cycles corresponding to the vertex $*$ and vertices $1, \ldots, n_{\Delta-1} - 1$ of $S_{n_{\Delta-1}}$. It is clear that $s^+ = v^0[*]^+$ is simple, s is l–constant (for any $0 \leq l \leq \Delta - 2$) and it is also clear that all the vertices $r[t]$ for $r \in R$ and $t \in \{*, 0, \ldots, n_{\Delta-1}\}$ are $(\Delta-1)$–constant. Thus the just constructed hamiltonian cycle C satisfies the conditions of Lemma 8 and Lemma 9.

- $n_0 = \Delta > n_1$**,** $\Delta = 3$**, the proof of Lemma 8 and Lemma 9**
  We set R to a two–element subset of $\{[0, *], [*, 0], [*, *]\}$ such that $r[*] \neq s$ for all $r \in R$ and we set r_0 to any member of R. If $n_2 \in \{1, 2\}$, then the hamiltonian cycles in $S_{n_0} \times S_{n_1} \times S_{n_2}$ are shown in Figure 6, in Figure 7, in Figure 8 and in Figure 9. The vertices of R are represented in the figures by full circles and the vertex s is represented by an empty circle.

  If $n_2 = 3 = \Delta$, we use a construction of a hamiltonian cycle based on the proof of Theorem 6. We use the construction of the proof for $n'_0 = n_0 = \Delta$, $n'_1 = n_2 = \Delta$ and $n'_2 = n_1 < \Delta$, i.e. for $S_3 \times S_3 \times S_1$ and $S_3 \times S_3 \times S_2$. We set k equal to 1 in the rest of the proof of this case to be consistent with the proof of Theorem 6. The cycle L' in the product $S_3 \times S_3 \times S_1$ looks as follows:

$$[0,0,0], [*,*,0], [0,*,*], [0,*,0], [0,1,0], [0,1,*], [*,1,*], [1,1,*], [1,1,0],$$

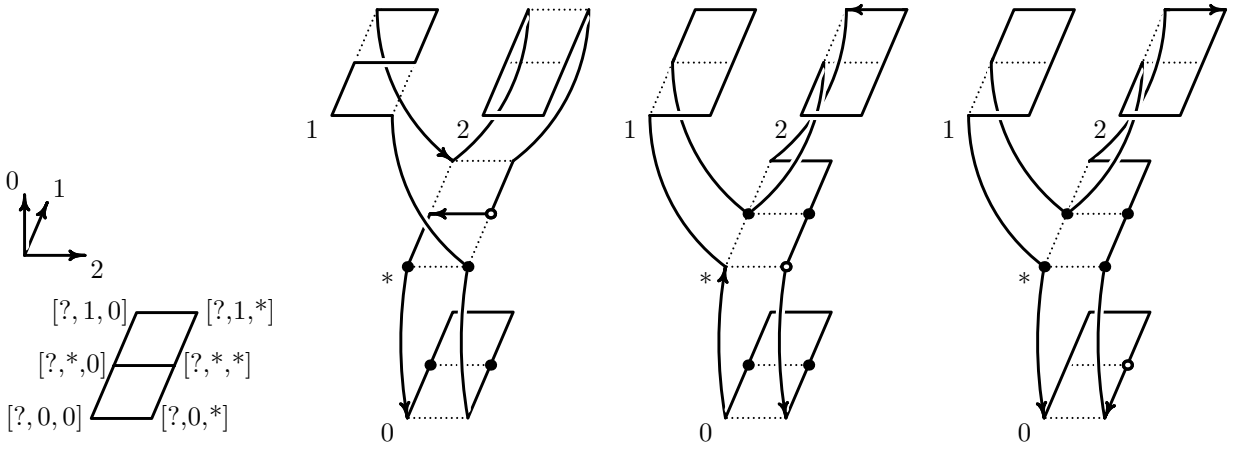
$$[*,1,0], [2,1,0], [2,*,0], [2,2,0], [2,2,*], [*,2,*], [0,2,*], [0,2,0], [*,2,0],$$



$n_1 = 1, n_2 = 2$ $s_l = *, l = 0$ $s_l = 0, l = 1$

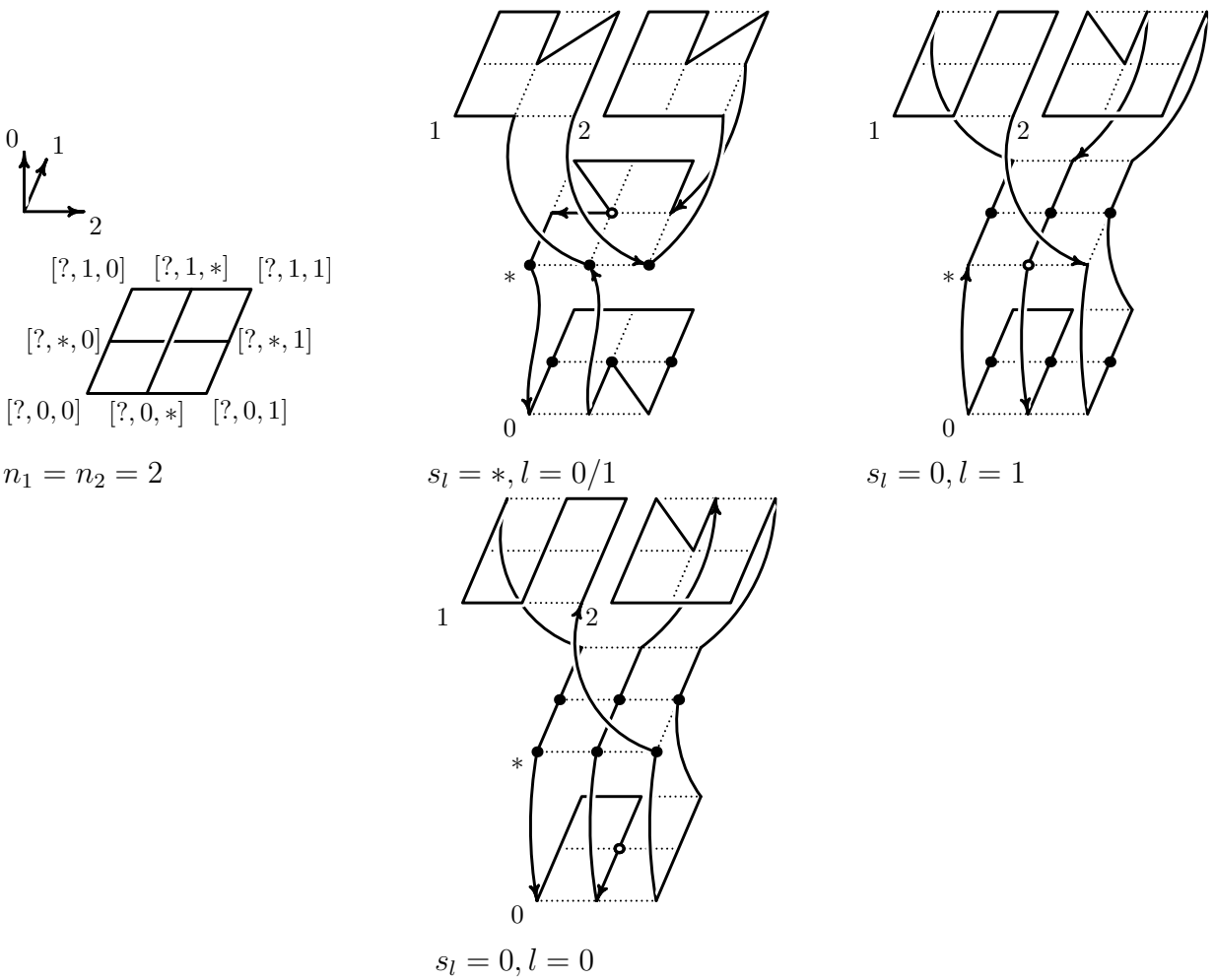
$s_l = 0, l = 0$

Figure 7: The hamiltonian cycles in $S_3 \times S_1 \times S_2$; the vertex s is drawn as an empty circle, the other important vertices are drawn by full circles.



$n_1 = 2, n_2 = 1$ $s_l = *, l = 0/1$ $s_l = 0, l = 1$ $s_l = 0, l = 0$

Figure 8: The hamiltonian cycles in $S_3 \times S_2 \times S_1$; the vertex s is drawn as an empty circle, the other important vertices are drawn by full circles.



$n_1 = n_2 = 2$

$s_l = *, l = 0/1$

$s_l = 0, l = 1$

$s_l = 0, l = 0$

Figure 9: The hamiltonian cycles in $S_3 \times S_2 \times S_2$; the vertex s is drawn as an empty circle, the other important vertices are drawn by full circles.

$$[1,2,0],[1,*,0],[1,0,0],[1,0,*],[*,0,*],[2,0,*],[2,0,0],[*,0,0](,[0,0,0])$$

The vertices $[*,*,*]$ and $[0,0,*]$ remain for the last (insertion) step of the construction (consult Claim 1 in case of $[*,*,*]$). The cycle L' in the product $S_3 \times S_3 \times S_2$ looks as follows (if we have choosen $[0,0,0]$ to be the representant):

$$[0,0,0],[*,*,0],[1,*,*],[*,*,1],[0,*,*],[0,*,0],[0,1,0],[0,1,*],$$

$$[0,1,1],[*,1,1],[1,1,1],[1,*,1],[1,2,1],[1,2,*],[*,2,*],[0,2,*],$$

$$[0,2,0],[*,2,0],[1,2,0],[1,*,0],[1,0,0],[1,0,*],[1,0,1],[*,0,1],$$

$$[2,0,1],[2,*,1],[2,1,1],[2,1,*],[*,1,*],[1,1,*],[1,1,0],[*,1,0],$$

$$[2,1,0],[2,*,0],[2,2,0],[2,2,*],[2,2,1],[*,2,1],[0,2,1],[0,*,1],$$

$$[0,0,1],[0,0,*],[*,0,*],[2,0,*],[2,0,0],[*,0,0](,[0,0,0])$$

The vertex $[*,*,*]$ remains to to the last (insertion) step of the construction (consult Claim 1). We reverse the orientation of the cycle L' in some of the cases. Check that $r[t]$ for $r \in \{[0,*],[*,0],[*,*]\}$ and $t \neq *$ are k–constant both before and after the reversion of the orientation of L'; this requires carefull inserton of $[0,0,*]$ in case of $S_3 \times S_3 \times S_1$. Check also that $[0,*][*] = [0,*,*]$ and $[*,0][*] = [*,*,0]$ are k–constant before the reversion of the cycle and that $[0,*][*] = [0,*,*]$ is k–constant after the reversion. We distinguish several cases:

- $s = [*,*,*]$, thus $R = \{[0,*],[*,0]\}$ and we set $r_0 = [0,*]$
  We considered the reversed cycle in this case. We insert in the last step the vertex $[*,*,*]$ just after the vertex $[*,0,*]$, i.e. it holds that $[*,*,*]^+ = [*,0,*]$ after the reversion of the cycle. Both s and s^+ are simple and s is l–constant (l is either 0 or 2, since it has to be different from k).
- $s = [0,*,*]$ (i.e. $l = 0$), thus $R = \{[*,*],[*,0]\}$ and we set $r_0 = [*,0]$
  We considered the original cycle in this case. Both s and $s^+ = [0,*,0]$ are simple and s is l–constant.
- $s = [*,*,0]$ (i.e. $l = 2$), thus $R = \{[*,*],[0,*]\}$ and we set $r_0 = [0,*]$
  We consider the reversed cycle in this case. Both $s = [*,*,0]$ and $s^+ = [0,0,0]$ are simple and s is l–constant. We insert $[*,*,*]$ just after $[*,*,0]$, i.e. $[*,*,*]^+ = [*,*,0]$ after the reversion of the cycle.

The detailed check that the constructed cycles satisfy the conditions of Lemma 8 and Lemma 9 is left to the reader, since it is very straightforward.

- $n_0 = \Delta > n_1$, $\Delta \geq 4$, **the proof of Lemma 8 and Lemma 9**
  We are going to use the induction hypothesis for the product of $\Delta - 1$ stars: we set $n'_i = n_{i+1}$ for $0 \leq i \leq \Delta - 3$, $k' = \Delta - 2$ and in case of Lemma 8 we also set $s' = [s_1, \ldots, s_{\Delta-1}]$, $l' = l - 1$ for $l \geq 1$ and $l' = 0$ for $l = 0$. There exists a set

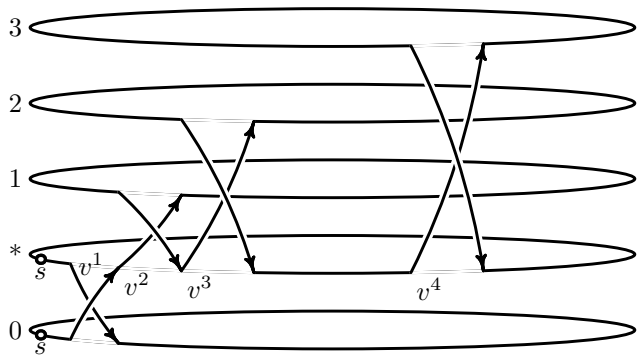


Figure 10: Joining different cycles in the product of a star S_{n_0} and a cycle C' for $n_0 = 4$ in the proof of the case $n_0 = \Delta > n_1$; the vertex s is drawn as an empty circle

$R' \subseteq S_{n'_0} \times \ldots \times S_{n'_{\Delta-3}} = S_{n_1} \times \ldots \times S_{n_{\Delta-2}}$ and a vertex $r'_0 \in R'$ satisfying the conditions of the lemmas for $n'_0, \ldots, n'_{\Delta-3}, k', s', l'$. We set R to $\{z \times r'_0 | z \in \{*, 0, \ldots, \Delta - 1\} = V(S_{n_0}) = V(S_\Delta\})$ and r_0 to be either $* \times r'_0$ or $0 \times r'_0$ in such manner that $r_0[t] \neq s$ for any choice of t. Note that there cannot be a vertex $r \in R$ such that $r[*] = s$, since this would imply that $r'_0[*] = s'$. We are going to show the existence of suitable hamiltonian cycles in $S_{n_0} \times \cdots \times S_{n_{\Delta-1}}$ for different values of $n_{\Delta-1}$. We distinguish two cases: $n_{\Delta-1} < \Delta$ and $n_{\Delta-1} = \Delta$.

If $n_{\Delta-1} < \Delta$, let C' be the hamiltonian cycle from the induction used for $n'_{\Delta-2} = n_{\Delta-1}$. The size of the cycle C' is at least $\Delta\ n_{\Delta-1} \geq 2\Delta$; let $v^1, \ldots, v^\Delta$ be any vertices of the cycle C' different from s'. We are going to use a construction which is very similar to that used in the proof of Theorem 1 — see Figure 10. We use edges from the vertices $* \times v^1, \ldots, * \times v^\Delta$ to join the copy of the cycle C' corresponding to the vertex $*$ of S_{n_0} to other copies of the cycle C'. We check that the just obtained hamiltonian cycle in $S_{n_0} \times \cdots \times S_{n_{\Delta-1}}$ satisfies all the conditions. Let t be any vertex of $S_{n_{\Delta-1}}$ and let r be any vertex of R; $r[t]$ is $(\Delta - 1)$–constant, since r'_0 is constant in the corresponding entry in C' and the only entries of $r[t]$ and $r[t]^+$ possibly affected by joining the cycles are their first entries (i.e. $r[t]_0$ and $r[t]_0^+$). The vertex r_0 is simple and contains a star among its entries because the same holds for r'_0. The vertex s is l–constant: in case $l = 0$ it is 0–constant due to the joining procedure (s is different from all $* \times v^1, \ldots, * \times v^\Delta$) and in case $l > 0$ it is l–constant since s' is constant in the corresponding entry. The vertex s^+ is simple, since it is equal to $s_0 \times s'^+$ and s'^+ is simple due to the induction.

If $n_{\Delta-1} = \Delta$, we cannot use the induction hypothesis, since $n_{\Delta-1} > \Delta - 1$. We use the construction of the proof of Theorem 6 for the star product $S_{n_0} \times S_{n_{\Delta-1}} \times S_{n_1} \times \cdots \times S_{n_{\Delta-2}}$, thus $k = 1$ in the following discussion. We slightly modify the construction used in the proof. We choose the vertex $[0, \ldots, 0]$ to be the representant of the cycle of $\mathcal{Z}$ in which it is contained; since the choice of representants was up to us (under some conditions which are clearly satisfied for the vertex $[0, \ldots, 0]$), this can be done. We insert the vertex $*^\Delta = [*, \ldots, *]$ in the last (insertion) step of the proof of Theorem 6

just before the vertex $[*, 0, *, \ldots, *]$. If $r_0[*]$ is not contained in L', we insert $r_0[*]$ just before any vertex v such that $v_1 = *$. If $s \neq *^\Delta$ (i.e. $s = *^\Delta[l \to 0]$), and if $l > 0$ we insert the vertex $*^\Delta[l+1 \to 0]$ (note that the entries are shifted by one because $S_{n_{\Delta-1}}$ is the second star in the product) just before the vertex $[0, *, 0, \ldots, 0]$. All the vertices to insert remain to the last step due to Claim 1. We check that this cycle has the desired properties in the next paragraph.

We need to check that all the vertices $r[t]$ for $r \in R$, $t \in \{0, \ldots, \Delta\}$, and the vertex $r_0[*]$ are k–constant. We adjust the insertion step: The vertex v not contained in L' is inserted in L' before the vertex w such that $v_1 = w_1$. All the vertices v containing $*$ such that $v_1 \neq *$ in the cycle L' constructed in the proof of Theorem 6 are 1–constant due to Claim 2; this property is kept through the insertion step (due to our adjustment). But $r[t]_1 = t \neq *$ (since r'_0 contains a star, r contains a star too) and thus all the vertices $r[t]$ contained in L' for all $r \in R$ and $t \in \{0, \ldots, \Delta - 1\}$ are k–constant ($k = 1$); the remaining vertices $r[t]$ (i.e. those not contained in L') are k–constant due to the adjusted insertion of them to the hamiltonian cycle in the last construction step. If $r_0[*]$ has not been contained in L', then it is k–constant due to the manner of its insertion. If $r_0[*]$ is contained in L', then it is k–constant in L' due to Claim 3 (note that r_0 contains at least one star) and this property is kept through the insertion step (due to our adjustment of it). In case of Lemma 8 we also need to check that the vertices s and s^+ have the desired properties. In case that $l = 0$ and $s = [0, *, *, \ldots, *]$, then $s^+ = [0, *, 0, \ldots, 0]$ is simple and s is l–constant. If $s = *^\Delta$ and $l > 0$, then $s^+ = [*, 0, *, \ldots, *]$ and thus s is l–constant. If $s = *^\Delta[l + 1 \to 0]$, then $s^+ = [0, *, 0, \ldots, 0]$ and thus s is l–constant.

- $n_0 = n_1 = \Delta$, **the proof of Lemma 8**
  We can assume w.l.o.g that $l \neq 1$. We set the vertex r_0 to be the vertex $[*, 0, 0, \ldots, 0]$ and R to the set $\{[*, 0, 0, \ldots, 0], [*, 1, 0, \ldots, 0], \ldots, [*, \Delta - 2, 0, \ldots, 0]\}$. We slightly modify the proof of Theorem 6 and then we use the construction described in it to get the desired hamiltonian cycle. We choose the vertex $[0, \ldots, 0]$ to be the representant of the cycle of $\mathcal{Z}$. In the last (insertion) step of the proof of Theorem 6, we insert the vertex $*^\Delta$ just before $*^\Delta[1 \to 0]$ and in case that $l \neq 0$ we insert the vertex $*^\Delta[l \to 0]$ just before $[0, \ldots, 0, *]$. Both the vertices $*^\Delta$ and $*^\Delta[l \to 0]$ (for $l \neq 0$; note that in this case Δ is at least 4, since $l \neq 1$, $l \neq k$ and $k = \Delta - 1$) remain to the last insertion step during the construction of the hamiltonian cycle (see the Claim 1). If $n_{\Delta-1} = \Delta$, then also the vertices $[*, t, 0, \ldots, 0, *]$ for all $t \in \{*, 0, \ldots, \Delta - 1\}$ remain to the last step and we insert them just before the vertices $[(-t) \bmod \Delta, t, 0, \ldots, 0, *]$.

  We now check that the just constructed hamiltonian cycle has the desired properties. Let $t' \in \{*, 0, \ldots, n_{\Delta-1} - 1\}$ be arbitrary. We need to check that for any $r \in R$ the vertex $r[t']$ is $(\Delta - 1)$-constant — this is clear, since $[*, t, 0, \ldots, 0, t']^+ = [(-t - t') \bmod \Delta, t, 0, \ldots, 0, t']$ for $t' \neq *$; if $t' = *$, then $[*, t, 0, \ldots, 0, t']^+ = [(-t - 1) \bmod \Delta, t, 0, \ldots, 0, *]$. Since we are proving Lemma 8, we need to check also that the vertices s and s^+ have also the desired properties. If $l = 0$ and $s = [0, *, *, \ldots, *]$,

then $s^+ = [0, *, 0, \ldots, 0]$ is simple and s is l–constant. Otherwise the vertex s^+ is simple, since it is either $*^{\Delta}[1 \to 0] = [*, 0, *, \ldots, *]$ or $[0, \ldots, 0, *]$ and s is l–constant due to the manner of insertion of s to the cycle in the last construction step.

- $n_0 = n_1 = \Delta$, **the proof of Lemma 9**
  We set the vertex r_0 to be the vertex $[*, 0, 0, \ldots, 0]$ and the set R to be the set $\{[0, *, 0, \ldots, 0], [*, 0, 0, \ldots, 0], [*, 1, 0, \ldots, 0], \ldots, [*, \Delta-3, 0, \ldots, 0]\}$. We modify the last insertion step of the proof of Theorem 6 and then we use the construction of this proof to get the desired hamiltonian cycle. We insert the vertex $*^{\Delta}$ just after the vertex $[0, *, 0, \ldots, 0, *]$ and in case that $n_{\Delta-1} = \Delta$ we insert the vertices $[*, t, 0, \ldots, 0, *]$ for $t \in \{*, 0, \ldots, n_{\Delta-1} - 1\}$ just before the vertices $[0, t, 0, \ldots, 0, *]$; all these vertices remain to the last insertion step due to Claim 1. We now check that the just constructed hamiltonian cycle has the desired properties. The size of R is $\Delta - 1$ and it contains two simple vertices. We finish the proof of this case by checking that the vertices $r[t']$ are $(\Delta - 1)$–constant for all $r \in R$ and for all $t' \in \{*, 0, \ldots, n_{\Delta-1} - 1\}$:

  - $t' \neq *$ and $r = [0, *, 0, \ldots, 0]$
    It holds that $r[t']^+ = r[t'][1 \to -\sum r - t' + 1]$ and thus the vertex $r[t']$ is $(\Delta - 1)$–constant.
  - $t' \neq *$ and $r = [*, t, 0, \ldots, 0]$
    It holds that $r[t']^+ = r[t'][0 \to -\sum r - t']$ and thus the vertex $r[t']$ is $(\Delta - 1)$–constant.
  - $t' = *$ and $r = [0, *, 0, \ldots, 0]$
    The vertex $r[t'] = [0, *, 0, \ldots, 0, *]$ is $(\Delta - 1)$–constant, since the vertex $*^{\Delta}$ has been inserted just after it.
  - $t' = *$, $r = [*, t, 0, \ldots, 0]$ for some t and $n_{\Delta-1} < \Delta$
    It holds that $r[t']^+ = [*, t, 0, \ldots, 0, *]^+ = [(-t) \bmod \Delta, t, 0, \ldots, 0, *]$ due to the construction of the cycles in the set $\mathcal{Z}$ (consult the proof of Theorem 6) and thus $r[t']$ is $(\Delta - 1)$–constant.
  - $t' = *$, $r = [*, t, 0, \ldots, 0]$ for some t and $n_{\Delta-1} = \Delta$
    The vertex $r[t'] = [*, t, 0, \ldots, 0, *]$ is $(\Delta - 1)$–constant, since it has been inserted just before the vertex $[0, t, 0, \ldots, 0, *]$.

■

We are now almost ready to prove the original conjecture of Bermond et al. We find a covering of each graph in the product by stars, then we find hamiltonian (directed) cycles in the products of these stars and finally, we join them together. Lemma 8 and Lemma 9 give us a tool for finding suitable hamiltonian directed cycles in the products of stars, but we need to develop some tools for joining them together. These missing tools are contained in the following two lemmas:

Lemma 10 *Let G be any graph, let $\mathcal{C}$ be a covering of G with directed cycles and let M be a set of edges of G satisfying the following conditions:*

- *No two edges of M have a common vertex.*
- *For each edge $xy \in M$, there are edges xy^+ and yx^+ in G.*
- *The spanning subgraph of G containing exactly edges of the cycles of $\mathcal{C}$ and the edges of the set M is connected.*

Then G contains a hamiltonian cycle.

Proof: Assume w.l.o.g. that no proper subset of M satisfies the conditions of the lemma; this means that in the spanning subgraph from the third condition, no edge of M is contained in any cycle of that spanning subgraph. We replace the edges xx^+ and yy^+ with xy^+ and yx^+ for all edges $xy \in M$; this is possible, since the edges of M are disjoint. We change the directed cycles of $\mathcal{C}$ to one connected oriented cycle, since the in–degree and out–degree of all the vertices is preserved and the resulting subgraph is really connected due to the third condition in the lemma and assumed minimality of M. ■

We call a subgraph G' of $G = G_0 \times \cdots \times G_{\Delta-1}$ a copy of G_i in G if there exists $v_0 \in G_0, \ldots, v_{i-1} \in G_{i-1}, v_{i+1} \in G_{i+1}, \ldots, v_{\Delta-1} \in G_{\Delta-1}$ such that $G = v_0 \times \cdots \times v_{i-1} \times G_i \times v_{i+1} \times \cdots v_{\Delta-1}$.

Lemma 11 *Let $T_0, \ldots, T_{\Delta-1}$ be trees (possibly consisting of a single vertex), let a and b be two different leaves of T_0. Then there exists a spanning tree T in $T_0 \times \cdots \times T_{\Delta-1}$ satisfying the following conditions:*

- *T consists of copies of the trees $T_0, \ldots, T_{\Delta-1}$, i.e. for each edge e of T there exists T' a copy of T_i in T, $e \in T'$ and $T' \subseteq T$.*
- *If $v \in T$ and $v_0 \neq a, b$, then the edges incident to v in T are only edges of a copy of T_0.*
- *If $v \in T$ and $v_0 = a$ or $v_0 = b$, then the edges incident to v in T are edges of a copy of T_0 and of a copy of at most one another tree T_i.*

Proof: The proof proceeds by the induction on Δ. We prove a bit stronger statement, namely: There is T satisfying the conditions of the lemma and two vertices $v^1, v^2 \in T$ such that $v_0^1, v_0^2 \in \{a, b\}$ and the edges incident to the vertices v^1 and v^2 are only edges of copies of T_0. If $\Delta = 1$, the statement is trivial — let T be T_0, let v^1 be a and let v^2 be b. Let Δ be at least 2, now. We use the induction: Let T' be the spanning tree of $T_0 \times \ldots \times T_{\Delta-2}$ and let u^1 and u^2 be the two special vertices of T'. If $T_{\Delta-1}$ is a single vertex w, we simply set T to $T' \times w$, v^1 to $u^1 \times w$ and v^2 to $u^2 \times w$. Let w^1 and w^2 be any two vertices of $T_{\Delta-1}$. We set T to $\left(\cup_{w \in T_{\Delta-1}} T' \times w\right) \cup v^{1\prime} \times T_{\Delta-1}$, v^1 to $u^2 \times w^1$ and v^2 to $u^2 \times w^2$ — see Figure 11. It is routine to check that T has the desired properties. ■

We are now ready to prove the conjecture of Bermond et al.:

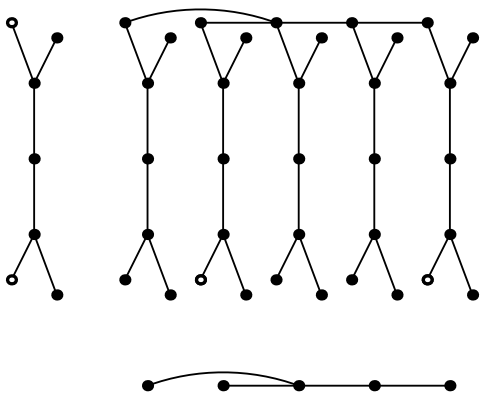


Figure 11: Creating T from T', $v^{1'}$, $v^{2'}$ and $T_{\Delta-1}$; the tree T' is in the left, the tree $T_{\Delta-1}$ is in the bottom; the vertices $v^{1'}$, $v^{2'}$, v^1 and v^2 are drawn as empty circles

Theorem 7 *Let $G_0, \ldots, G_{\Delta-1}$ be connected graphs with maximum degree at most Δ. Then $G_0 \times \ldots \times G_{\Delta-1}$ is hamiltonian.*

Proof: The statement is trivial for $\Delta = 2$, since in this case both the graphs G_0 and G_1 are paths (with at least two vertices) and thus their product is hamiltonian. Let us suppose that Δ is at least 3, now. We can find coverings $\mathcal{C}_i$ of G_i by $S_{\leq\Delta}$ for $0 \leq i \leq \Delta - 1$ due to Lemma 1. If all the coverings consist of exactly one star each, then the product $G_0 \times \cdots \times G_{\Delta-1}$ is hamiltonian due to Theorem 6. Otherwise let us suppose w.l.o.g. that $\mathcal{C}_0$ contains at least two stars. Let E_i be the minimal set of edges of G_i such that the spanning subgraph of G_i with edge–set equal to the union of the edges used in $\mathcal{C}_i$ and the edges of E_i is connected — this set exists, since G_i is connected; call the graph with this edge–set $\tilde{G}_i$. Let $c_i : E_i \to \{0, \ldots, \Delta-2\}$ be colouring of edges of E_i in $\tilde{G}_i$ by colours $\{0, \ldots, \Delta-2\}$ such that no two incident edges have the same colour — such a colouring exists, since each vertex v of $\tilde{G}_i$ is incident to at most $\Delta - 1$ edges of E_i (the degree of v in G_i is at most Δ and at least one edge incident to v is a part of a star in a covering $\mathcal{C}_i$) and the subgraph of $\tilde{G}_i$ induced by edges E_i is a forest. Let T_i be the graph obtained from $\tilde{G}_i$ by contracting each star of the covering of $\mathcal{C}_i$ to one vertex; note that the edge–set of T_i is precisely E_i and T_i is a tree (possibly consisting of a single vertex), since E_i is a minimal edge–set to make $\tilde{G}_i$ connected. Since $\mathcal{C}_0$ contains at least two stars, T_0 contains at least two vertices and thus also at least two leaves — call these leaves a and b. Let $\tilde{a}$ ($\tilde{b}$) be the vertex of G_0 incident with the only edge of E_0 incident with the star corresponding to a (b). Note that we can assume w.l.o.g. that the colours assigned by c_0 to the edges incident to $\tilde{a}$ and $\tilde{b}$ are only 0 or 1. Let $n_i(v)$ for $v \in T_i$ be the order of the star which has been contracted to the vertex v of T_i and let $f_i^v : V(G_i) \to V(S_{n_i(v)}) = \{*, 0, \ldots, n_i(v) - 1\}$ be an isomorphism mapping the vertices of the star contracted to the vertex $v \in T_i$ to the original star $S_{n_i(v)}$. We assume w.l.o.g. that the vertices $\tilde{a}$ and $\tilde{b}$ in $\tilde{G}_0$ are mapped by f_0^a and f_0^b to $*$ or 0. Let T be a spanning tree of $T_0 \times \ldots \times T_{\Delta-1}$ with two special vertices of T_0 equal to a and b with properties described in Lemma 11. We call the subgraph of $G_0 \times \ldots \times G_{\Delta-1}$ corresponding to a vertex of T

a cluster; the vertex $[v_0, \ldots, v_{\Delta-1}]$ of T corresponds to $S_{n_0(v_0)} \times \ldots \times S_{n_{\Delta-1}(v_{\Delta-1})}$ and there is a mapping[1] $f^{[v_0,\ldots,v_{\Delta-1}]} = f_0^{v_0} \times \ldots \times f_{\Delta-1}^{v_{\Delta-1}}$ of this cluster to the corresponding product of stars. We find a directed hamiltonian cycle in each cluster using Lemma 8 or Lemma 9 and we find a suitable set of edges M corresponding to edges of T between clusters. This procedure is described in more detail in the following two paragraphs.

We first find hamiltonian cycles in the clusters corresponding to vertices incident to edges of two different trees. We also find edges of M corresponding to edges between clusters of T in copies of $T_1, \ldots, T_{\Delta-1}$. Look at a fixed copy of T_k in T for $1 \leq k \leq \Delta - 1$, i.e. we have fixed k and $v_0 \in T_0, \ldots, v_{k-1} \in T_{k-1}, v_{k+1} \in T_{k+1}, \ldots, v_{\Delta-1} \in T_{\Delta-1}$ such that $v_0 \times \ldots \times v_{k-1} \times T_k \times v_{k+1} \times \ldots \times v_{\Delta-1} \subseteq T$. We use Lemma 8 for $n_i = n_i(v_i)$ $(0 \leq i \leq \Delta - 1, i \neq k)$, $l = 0$ and $s = [t, *, \ldots, *]$, there t is $f_0^{v_0}(w)$ (either $*$ or 0) where w is the vertex ($\tilde{a}$ or $\tilde{b}$) connecting the star corresponding to v_0 in $\tilde{G}_0$ to the rest of the graph. There exists a set $R \subseteq S_{n_0} \times \ldots \times S_{n_{k-1}} \times S_{n_{k+1}} \times \ldots \times S_{n_{\Delta-1}}$ of size at least $\Delta - 1$ with properties described in the statement of the Lemma 8. Let $r_0, \ldots, r_{\Delta-2}$ be elements of R. We place a copy (under the inverse mapping to $f^{[v_0,\ldots,v_{\Delta-1}]}$) of a cycle C from Lemma 8 to each cluster corresponding to $[v_0, \ldots, v_{k-1}, v_k, v_{k+1}, \ldots, v_{\Delta-1}] \in T$ for $v_k \in T_k$. Let xy be an edge of E_k corresponding to an edge $v_k^x v_k^y$ in the fixed copy of T_k we deal with. We add to the set M an edge $\tilde{x}\tilde{y}$ such that $f^{[v_0,\ldots,v_{k-1},v_k^x,v_{k+1},\ldots,v_{\Delta-1}]}(\tilde{x}) = r_{c_k(xy)}[f_k^{v_k^x}(x)]$ and $f^{[v_0,\ldots,v_{k-1},v_k^y,v_{k+1},\ldots,v_{\Delta-1}]}(\tilde{y}) = r_{c_k(xy)}[f_k^{v_k^y}(y)]$. The existence of edges $\tilde{x}\tilde{y}^+$ and $\tilde{y}\tilde{x}^+$ is ensured by the k–constantness of $\tilde{x}$ and $\tilde{y}$: Since $\tilde{x}_k = \tilde{x}_k^+$, since $\tilde{x}_i = \tilde{y}_i$ for $i \neq k$ and since there is an edge between $\tilde{x}_i$ and $\tilde{x}_i^+$ in G_i (because $\tilde{x}\tilde{x}^+$ is an edge), there is an edge between $\tilde{y}$ and $\tilde{x}^+$ in $G_0 \times \ldots \times G_{\Delta-1}$. The proof of the existence of the edge $\tilde{x}\tilde{y}^+$ is actually the same. We do the above described procedure for all copies of $T_1, \ldots, T_{\Delta-1}$ in T.

We deal with the copies of T_0 in T now. Each vertex of $v \in T$ is contained in exactly one copy of T_0. The only edges incident to v are the edges of the copy of T_0 if $v_0 \notin \{a, b\}$. Let $v_1 \in T_1, \ldots, v_{\Delta-1} \in T_{\Delta-1}$ be fixed. We use Lemma 9 for $k = 0$ and $n_i = n_i(v_i)$ for $1 \leq i \leq \Delta - 1$. There exists a set $R \subseteq S_{n_1} \times \ldots \times S_{n_{\Delta-1}}$ of size at least $\Delta - 1$ with the properties described in the statement of the Lemma 9. Let $r_0, \ldots, r_{\Delta-2}$ be elements of R and assume that r_0 and r_1 are the two simple vertices which R has to contain. Let $v_0 \in T_0$; we place a copy (under the inverse mapping to $f^{[v_0,\ldots,v_{\Delta-1}]}$) of a cycle C from Lemma 9 (the cycle for $n_0 = n_0(v_0)$) to each cluster corresponding to $[v_0, \ldots, v_{\Delta-1}] \in T$ to which the hamiltonian cycle has not been placed in the previous paragraph; this happens iff the edges incident to $[v_0, \ldots, v_{\Delta-1}]$ in T are only the edges of the copy of T_0. Let xy be an edge of E_0 corresponding to an edge $v_0^x v_0^y$ in the fixed copy of T_0 we deal with, such that both x and y are different from $\tilde{a}$ and $\tilde{b}$. We add to the set M an edge $\tilde{x}\tilde{y}$ such that $f^{[v_0^x,v_1,\ldots,v_{\Delta-1}]}(\tilde{x}) = r_{c_0(xy)}[f_0^{v_0^x}(x)]$ and $f^{[v_0^y,v_1,\ldots,v_{\Delta-1}]}(\tilde{y}) = r_{c_0(xy)}[f_0^{v_0^y}(y)]$. The existence of edges $\tilde{x}\tilde{y}^+$ and $\tilde{y}\tilde{x}^+$ is ensured by the 0–constantness of $\tilde{x}$ and $\tilde{y}$ (for the same reason we used at the end of the previous paragraph).

It remains to add edges to M corresponding to the edges of T_0 incident to a and b:

[1]Let $f^1 : A_1 \to B_1$ and $f^2 : A_2 \to B_2$, then the mapping $f^1 \times f^2 : A_1 \times A_2 \to B_1 \times B_2$ is defined as $f^1 \times f^2([a_1, a_2]) = [f^1(a_1), f^2(a_2)]$.

- ab **is an edge of** T_0**.** We add to the set M the edge $\tilde{x}\tilde{y}$ such that $f^{[a,v_1,\ldots,v_{\Delta-1}]}(\tilde{x}) = [f_0^a(\tilde{a}), *, \ldots, *]$ and $f^{[b,v_1,\ldots,v_{\Delta-1}]}(\tilde{y}) = [f_0^b(\tilde{b}), *, \ldots, *]$. Both of the vertices $\tilde{x}$ and $\tilde{y}$ are 0–constant due to Lemma 8 and thus there exist edges $\tilde{x}\tilde{y}^+$ and $\tilde{y}\tilde{x}^+$ in $G_0 \times \ldots \times G_{\Delta-1}$ for the same reason we used at the end of the previous paragraph.

- ab **is not an edge of** T_0**, i.e. there are edges** ay **and** xb **in** T_0**.** Let us deal only with the edge ay; the case of the edge xb is actually same. Let $\tilde{a}z$ be the edge of E_0 corresponding to the edge ay of T_0. We add to the set M the edge $\tilde{x}\tilde{y}$ such that $f^{[a,v_1,\ldots,v_{\Delta-1}]}(\tilde{x}) = [f_0^a(\tilde{a}), *, \ldots, *]$ and $f^{[y,v_1,\ldots,v_{\Delta-1}]}(\tilde{y}) = r_{c_0(\tilde{a}z)}[f_0^y(z)]$. Both of the vertices $\tilde{x}$ and $\tilde{y}$ are 0–constant due to Lemma 8 and Lemma 9; the vertices $\tilde{x}$, $\tilde{x}^+$ and $r_{c_0(\tilde{a}z)}$ are simple. Thus there exist edges $\tilde{x}\tilde{y}^+$ and $\tilde{y}\tilde{x}^+$ in $G_0 \times \ldots \times G_{\Delta-1}$ for the similar reason we used at the end of the previous paragraph.

We do the above described procedure for all copies of T_0 in T.

We need to check that the set M has the properties demanded by Lemma 10. We have already checked that there are edges xy^+ and yx^+ for all the edges $xy \in M$. Since each edge in M corresponds to an edge of T, M is minimal set of edges such that the subgraph of $G_0 \times \ldots \times G_{\Delta-1}$ with the edge–set consisting of the edges of hamiltonian cycles of the clusters and the edges of M is connected. We are going to check that the edges of M are vertex disjoint.

Let xy and $x'y'$ be two non–disjoint edges of M corresponding to edges of a copy of T_i and $T_{i'}$; assume that $x = x'$ and $i \geq i'$. The edges of M corresponding to edges of T of two different copies of T_j and T_k for $j, k > 0$ (not necessarily $i \neq k$) are vertex disjoint, since the edges of T corresponding to them join different clusters; thus the same reason is true also for the edges corresponding to edges of T of different copies of T_0. It must hold that $i = i'$ or $i' = 0$.

We first deal with the case that $i = i'$. Let v be the cluster containing $x = x'$. The edges xy and $x'y'$ of G correspond to the edges x_iy_i and $x'_iy'_i$ of E_i. Since $x_i = x'_i$, it holds that $c_i(x_iy_i) \neq c_i(x'_iy'_i)$. The vertices x and x' have been chosen to satisfy that $f^v(x) = r_{c_i(x_iy_i)}[f_i^{v_i}(x_i)]$ and $f^v(x') = r_{c_i(x'_iy'_i)}[f_i^{v_i}(x'_i)]$, but since $c_i(x_iy_i) \neq c_i(x'_iy'_i)$ it cannot be $x = x'$.

We deal with the remaining case that $i > i' = 0$. Let v be the cluster containing $x = x'$. It holds that $s = f^v(x')$ due to the construction of M and thus $f_j^{v_j}(x_j) = *$ for all $j > 0$, especially $f_i^{v_i}(x_i) = *$. Then it must hold $f^v(x) = r_c[f_i^{v_i}(x_i)]$ for some c due to the construction of M; but $r_c[*]$ is different from s for all choices of c — this contradicts our assumption that the edges xy and $x'y'$ are not disjoint.

We have just proved that the edges of M are vertex disjoint and thus we have finished the proof of the theorem. ■

5 Strong Products of Graphs with Large Maximum Degree

Lemma 12 *Any graph of* $S^3_{\leq 8} \times S^3_{\leq 9} \times S_{\leq 15}$ *can be covered by* $C_{\geq 120}$.

Proof: Let $G = S_{n_0} \times \ldots \times S_{n_6}$ be a graph in $S^3_{\leq 8} \times S^3_{\leq 9} \times S_{\leq 15}$; we assume that $n_0 \leq \ldots \leq n_6$. From the assumption of the lemma follows that $n_0, n_1, n_2 \leq 8$, $n_3, n_4, n_5 \leq 9$ and $n_6 \leq 15$. We distinguish several cases:

- **All n_i are at most 7 ($n_6 \leq 7$).**

  In this case G is hamiltonian due to Theorem 6. Since G contains at least $2^7 = 128$ vertices, the length of its hamiltonian cycle is at least 128 and this cycle covers G.

- **At least two of n_i are at most 3 and at least one of n_i is at least 8 ($n_1 \leq 3 \wedge n_6 \geq 8$).**

  Due to Theorem 5 $S_{n_3} \times S_{n_4} \times S_{n_5}$ can be covered by $S_{\leq 3}$. The product of the first six stars can be covered by $C_{\geq 16}$ due to Theorem 1 applied to S_{n_2} and the hamiltonian cycles in $S_{n_0} \times S_{n_1} \times S_{\leq 3}$. We conclude by application of Theorem 1 to the products of these cycles and S_{n_6} that G can be covered by $C_{\geq 128}$.

- **At most one of n_i is at most 3 and at least one of n_i is at least 8 ($n_1 \geq 4 \wedge n_6 \geq 8$).**

  If $n_3 \leq 4$, then $S_{n_0} \times S_{n_1} \times S_{n_2} \times S_{n_3}$ is hamiltonian due to Theorem 6. We conclude by repeated application of Theorem 1 in the manner similar to the previous case that G can be covered by $C_{\geq 128}$.

  If $n_3 \geq 5$, then $S_{n_0} \times S_{n_3}$ can be covered by $S_{\{1,2,3,4\}}$ due to Lemma 3 and both $S_{n_1} \times S_{n_4}$ and $S_{n_2} \times S_{n_5}$ can be covered by $S_{\{2,3,4\}}$ due to Lemma 4. Thus due to Lemma 7 applied to the product of this covering by $S_{\{1,2,3,4\}} \times S_{\{2,3,4\}} \times S_{\{2,3,4\}}$ and S_{n_6} we conclude that G can be covered by $C_{\geq 120}$.

■

Corollary 2 *The product of any at least $\lfloor \frac{93}{120}\Delta \rfloor + 7$ connected graphs of maximum degree at most Δ can be covered by $C_{\geq 120}$.*

Proof: The product of any $\lceil \frac{1}{8}\Delta \rceil$ (respectively $\lceil \frac{1}{9}\Delta \rceil$, $\lceil \frac{1}{15}\Delta \rceil$) connected graphs of maximum degree at most Δ can be covered by $S_{\leq 8}$ (respectively $S_{\leq 9}$, $S_{\leq 15}$) due to Theorem 5. Thus due to Lemma 12 the product of any $3\lceil \frac{1}{8}\Delta \rceil + 3\lceil \frac{1}{9}\Delta \rceil + \lceil \frac{1}{15}\Delta \rceil \leq \lfloor \frac{93}{120}\Delta \rfloor + 7$ such graphs can be covered by $C_{\geq 120}$. In case that the product consists of more graphs than necessary, we use Theorem 3 to get its covering. ■

Corollary 3 *For each $c > \ln \frac{25}{12}$ there exists c' such that the product of any at least $\lfloor c\Delta \rfloor + c'$ connected graphs of maximum degree at most Δ can be covered by $C_{\geq 120}$.*

Proof: Consider the product of $\sum_{i=2^{k+3}}^{2^{k+3}+2^k-1}\lceil\frac{1}{i}\Delta\rceil$ connected graphs of maximum degree at most Δ for $k \geq 0$. The product of $\lceil\frac{1}{i}\Delta\rceil$ such graphs can be covered by $S_{\leq i}$ due to Theorem 5. Thus there exists a covering of the considered product by $S_{\leq 2^{k+3}} \times \cdots \times S_{\leq 2^{k+3}+2^k-1}$. We can cover the considered product by $S_{\leq 2^{k+2}} \times \cdots \times S_{\leq 2^{k+2}+2^{k-1}-1}$ due to Lemma 3 applied to the pairs of the consecutive members of the product $S_{\leq 2^{k+3}} \times \cdots \times S_{\leq 2^{k+3}+2^k-1}$; we conclude by repeated application of Lemma 3 in this manner that the considered product can be covered by $S_{\leq 8}$. We obtain by the same reasoning that the product of $\sum_{i=2^{k+3}+2^k}^{2^{k+3}+2^{k+1}-1}\lceil\frac{1}{i}\Delta\rceil$, $\sum_{i=2^{k+4}-2^k}^{2^{k+4}-1}\lceil\frac{1}{i}\Delta\rceil$, connected graphs of maximum degree at most Δ for $k \geq 0$ can be covered by $S_{\leq 9}$, $S_{\leq 15}$. Thus we conclude in the same manner as in the proof of Corollary 2 that the following number of graphs in the product is enough:

$$3\sum_{i=2^{k+3}}^{2^{k+3}+2^k-1}\left\lceil\frac{1}{i}\Delta\right\rceil + 3\sum_{i=2^{k+3}+2^k}^{2^{k+3}+2^{k+1}-1}\left\lceil\frac{1}{i}\Delta\right\rceil + \sum_{i=2^{k+4}-2^k}^{2^{k+4}-1}\left\lceil\frac{1}{i}\Delta\right\rceil \leq$$

$$\leq \left\lfloor\left(3\sum_{i=2^{k+3}}^{2^{k+3}+2^{k+1}-1}\frac{1}{i} + \sum_{i=2^{k+4}-2^k}^{2^{k+4}-1}\frac{1}{i}\right)\Delta\right\rfloor + 3\cdot 2^{k+1} + 2^k$$

We examine the behaviour of the linear coefficient in the expression for large value of k:

$$\lim_{k\to\infty}\left(3\sum_{i=2^{k+3}}^{2^{k+3}+2^{k+1}-1}\frac{1}{i} + \sum_{i=2^{k+4}-2^k}^{2^{k+4}-1}\frac{1}{i}\right) = 3\ln\frac{5}{4} + \ln\frac{16}{15} = \ln\frac{25}{12}$$

We can choose for any $c > \ln\frac{25}{12}$ the value of k large enough to be the linear coefficient smaller than c; the value of c' is bounded by $3\cdot 2^{k+1} + 2^k$. ■

Lemma 13 *For any $\Delta > 32$, the product of a cycle of length at least 120 and at least $\lfloor\frac{1}{60}\Delta\rfloor + \lceil\log_2\Delta\rceil - 6$ graphs of maximum degree at most Δ can be covered by $C_{\geq\Delta}$.*

Proof: Let k be $\lceil\log_2\Delta\rceil - 6$; $k \geq 0$, since $\Delta > 32$. The theorem holds trivially due to Theorem 3 for $k = 0$ ($32 < \Delta \leq 64$). Otherwise, consider the product of $\sum_{i=1}^{k}\lceil\frac{1}{60\cdot 2^i}\Delta\rceil$ graphs of maximum degree at most Δ. For all $1 \leq i \leq k$ the product of $\lceil\frac{1}{60\cdot 2^i}\Delta\rceil$ graphs can be covered by $S_{\leq 60\cdot 2^i}$ due to Theorem 5. We conclude by repeated application of Theorem 1 to the product of the cycle and the stars of the coverings that the product of $\sum_{i=1}^{k}\lceil\frac{1}{60\cdot 2^i}\Delta\rceil$ graphs of maximum degree at most Δ and a cycle of length at least 120 can be covered by cycles of length at least $120\cdot 2^k \geq \Delta$. The lemma follows now from the simple upper estimate of the number of graphs needed:

$$\sum_{i=1}^{k}\left\lceil\frac{1}{60\cdot 2^i}\Delta\right\rceil \leq \left\lfloor\sum_{i=1}^{k}\frac{1}{60\cdot 2^i}\Delta\right\rfloor + k \leq$$

$$\leq \left\lfloor\sum_{i=1}^{\infty}\frac{1}{60\cdot 2^i}\Delta\right\rfloor + \lceil\log_2\Delta\rceil - 6 = \left\lfloor\frac{1}{60}\Delta\right\rfloor + \lceil\log_2\Delta\rceil - 6$$

■

The immediate corollaries of the previous lemma are the following:

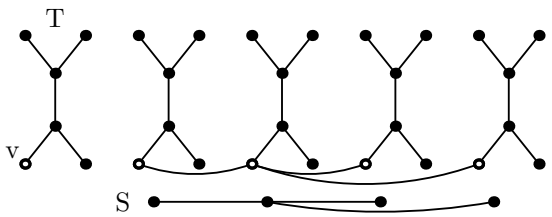


Figure 12: Construction of a small maximum degree spanning tree in a product of two trees

Corollary 4 *The product of any at least* $\lfloor \frac{19}{24}\Delta \rfloor + \lceil \log_2 \Delta \rceil + 1$ *connected graphs of maximum degree at most* Δ *can be covered by* $C_{\geq \Delta}$ *whenever* $\Delta > 32$.

Proof: Due to Corollary 2 the product of $\lfloor \frac{93}{120}\Delta \rfloor + 7$ of them can be covered by $C_{\geq 120}$; due to Lemma 13 the product of a cycle of length at least 120 and $\lfloor \frac{1}{60}\Delta \rfloor + \lceil \log_2 \Delta \rceil - 6$ connected graphs can be covered by $C_{\geq \Delta}$. The corollary immediately follows from these two facts and from Theorem 3 (the product may consist of more graphs than necessary). ■

If we use Corollary 3 instead of Corollary 2, we obtain the following:

Corollary 5 *For each* $c > \ln \frac{25}{12} + \frac{1}{60}$ *there exists* c' *such that the product of any at least* $\lfloor c\Delta \rfloor + c'$ *connected graphs of maximum degree at most* Δ *can be covered by* $C_{\geq 120}$ *for* $\Delta > 32$.

Lemma 14 *The product of any number of connected graphs of maximum degree at most* Δ *contains a spanning tree of maximum degree at most* $\Delta + 1$.

Proof: We prove that the product of a tree of maximum degree at most Δ and a tree of maximum degree at most $\Delta + 1$ contains a spanning tree of maximum degree at most $\Delta + 1$; the lemma easily follows from this statement by induction on the number of graphs in the product.

Let S be the tree of maximum degree at most Δ and let T be the tree of maximum degree at most $\Delta + 1$. Let v be any leaf of T. We describe a spanning tree of $S \times T$ of maximum degree at most $\Delta + 1$. The desired spanning tree consists of edges in copies of T (all the edges with the same S–vertex coordinate) and the edges of S which are in the copy of S with the T–vertex coordinate equal to v (see the Figure 12). The maximum degree of this tree is clearly at most $\Delta + 1$. ■

Theorem 8 *The product of any at least* $\lfloor \frac{19}{24}\Delta \rfloor + \lceil \log_2 \Delta \rceil + 3$ *connected graphs of maximum degree at most* Δ *is hamiltonian for* $\Delta > 32$.

Proof: We can suppose without loss of generality that one of the graphs in the product has maximum degree exactly Δ — call this graph H. Let G be any other of the graphs and let I be the product of the rest (at least $\lfloor \frac{19}{24}\Delta \rfloor + \lceil \log_2 \Delta \rceil + 1$) graphs. The graph I can be covered by $C_{\geq \Delta}$ due to Corollary 4 — let F_C be the edges of the cycles in this covering. I

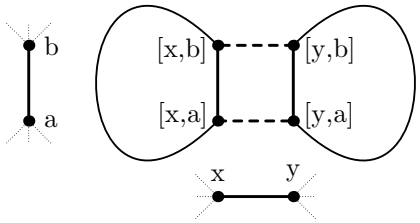


Figure 13: Concatenation procedure applied to two different cycles

contains a spanning tree of maximum degree at most $\Delta + 1$ due to Lemma 14; let us call this spanning tree T. Let F_T be the minimal set of edges of T such that the subgraph of I induced by the edges of $F_C \cup F_T$ is connected; clearly each vertex of I is adjacent to at most $\Delta + 1$ edges of F_T and the only cycles of the subgraph of I induced by the edges of $F_C \cup F_T$ are those of the covering of I. We can colour the edges of F_T by colours $\{1, \ldots, \Delta + 1\}$ such that no vertex of I is adjacent to two edges of the same colour; this is due to the fact that the subgraph of I induced by the edges of F_T is a forest.

The product of H and each cycle of the covering of I is hamiltonian due to Theorem 1; this gives us a covering of $H \times I$ by $C_{\geq 2\Delta}$; let E_C be the set of edges of cycles in this covering. Let $v_1, \ldots, v_{d+1}$ be any distinct vertices of H; it contains at least $\Delta + 1$ vertices since the maximum degree of H is exactly Δ. Let E_T be the set of all the edges $[v_i, x][v_i, y]$ such that $xy \in F_T$ and it is coloured by colour i. Clearly, the subgraph of $H \times I$ induced by edges of $E_T \cup E_C$ is connected and its only cycles are those of the covering of $H \times I$. Each vertex of $H \times I$ is adjacent to at most one edge of E_T due to the choice of the edges of E_T — there is no vertex in I adjacent to two edges of F_T coloured with the same colour.

We are now ready to prove that $G \times H \times I$ is hamiltonian. We assume that G is a tree; let u be any of its leaves and let v be its (only) neighbour. First, consider a product of a cycle of length at least 2Δ and G. We cover this product by the copies of the cycle corresponding to each vertex except for u and v; the product of the cycle and the edge uv we cover by a comb graph as shown in Figure 14. We introduce a concatenation procedure: Let $[x, a][x, b]$ and $[y, a][y, b]$ be edges of two distinct cycles in a covering and let xy be an edge, then we can concatenate these two cycles by replacing $[x, a][x, b]$ and $[y, a][y, b]$ with $[x, a][y, a]$ and $[x, b][y, b]$ (see Figure 13). We can apply the concatenation procedure to all the copies of one cycle, since there are at least Δ edges with the same G-coordinate in each such a copy (see Figure 14); actually the same idea was used in [1] to prove Theorem 1.

We now concatenate all the cycles covering $G \times H \times I$ together. Let xy be an edge of E_T. The edges $[u, x][v, x]$ and $[u, y][v, y]$ are in different cycles covering $G \times H \times I$ and we can apply the concatenation procedure to them — see Figure 14. Each vertex of $H \times I$ is adjacent to at most one edge of E_T and thus we can apply the concatenation procedure to all the edges of E_T simultaneously. Since the subgraph of $H \times I$ induced by $E_C \cup E_T$ is connected and its only cycles are those of the covering of $H \times I$, we obtain the hamiltonian cycle of $G \times H \times I$. ■

If we use Corollary 5 instead of Corollary 4 in the proof of Theorem 8, we obtain the fol-

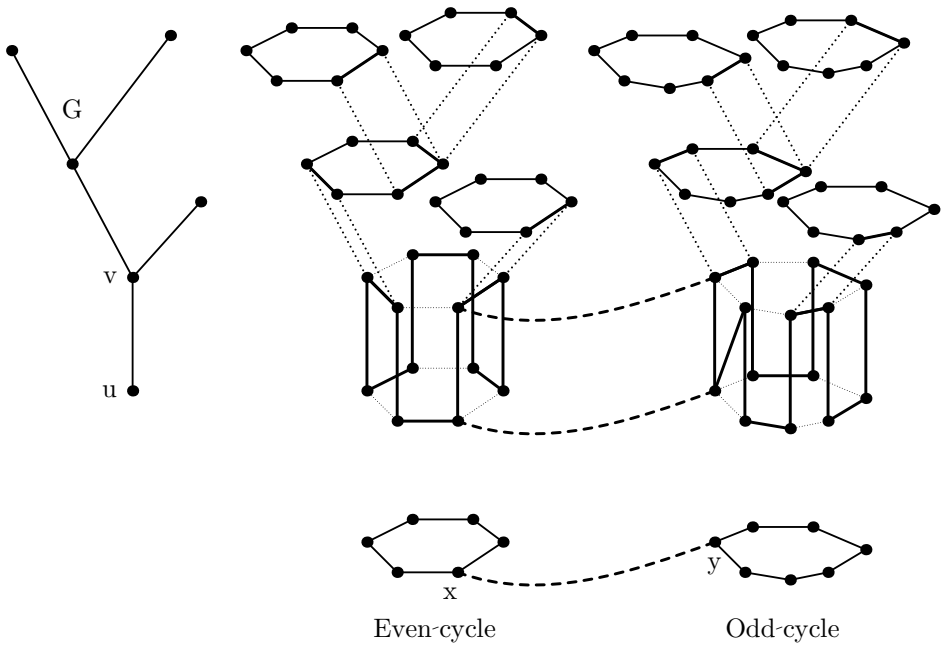


Figure 14: Covering the product of cycles and G by comb graphs and cycles and their concatenation

lowing theorem:

Theorem 9 *For each $c > \ln \frac{25}{12} + \frac{1}{60}$ there exists c' such that the product of any at least $\lfloor c\Delta \rfloor + \lceil \log_2 \Delta \rceil + c'$ connected graphs of maximum degree at most Δ is hamiltonian for $32 < \Delta$.*

If we use Corollary 1 instead of Corollary 2 in the proof of Theorem 8, we obtain the following theorem (the case $\Delta = 1$ not covered by Corollary 1 is trivial — in this case all the graphs of the product need to be P_2):

Theorem 10 *The product of any at least $\Delta + 2$ connected graphs of maximum degree at most Δ is hamiltonian.*

6 Conclusion

The question of precise determining of $h_{\max}(\Delta)$ remains open. We have proved the conjecture of Bermond et al. on $h_{\max}(\Delta)$, i.e. $h_{\max}(\Delta) \leq \Delta$ (Theorem 7). Theorem 8 and Theorem 9 get better results for large values of Δ — their linear (in Δ) upper bounds for $h_{\max}(\Delta)$ grow slower but their constant parts are large. It seems to be an easier (but not less attractive) question to describe the linear behaviour of $h_{\max}(\Delta)$ for large Δ. Theorem 2 states that:

$$\liminf_{\Delta \to \infty} \frac{h_{\max}(\Delta)}{\Delta} \geq \ln 2 \approx 0.6931$$

On the other hand Theorem 9 states that:

$$\limsup_{\Delta\to\infty} \frac{h_{\max}(\Delta)}{\Delta} \leq \ln \frac{25}{12} + \frac{1}{60} \approx 0.7506$$

We conjecture that:

$$\lim_{\Delta\to\infty} \frac{h_{\max}(\Delta)}{\Delta} = \ln 2$$

The products of stars are in the just suggested question the worst case: Let $h^{\mathrm{S}}_{\max}(\Delta)$ be the minimal k such that all the graphs $S^k_{\leq\Delta}$ are hamiltonian, then it follows from the proof of Theorem 8 (note that the hamiltonian cycle of any graph of $S^{h^{\mathrm{S}}_{\max}(\Delta)}_{\leq\Delta}$ has the length at least $2^{\ln 2\Delta} \geq \Delta$, since $h^{\mathrm{S}}_{\max}(\Delta) \geq h(S_\Delta) \geq \Delta \ln 2$) that:

$$h_{\max}(\Delta) \leq h^{\mathrm{S}}_{\max}(\Delta) + 2$$

Thus it holds that:

$$\limsup_{\Delta\to\infty} \frac{h_{\max}(\Delta)}{\Delta} \leq \limsup_{\Delta\to\infty} \frac{h^{\mathrm{S}}_{\max}(\Delta)}{\Delta}$$

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