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No-Hole L(2,1)-Colorings

by

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ABSTRACT

An $L(2,1)$-*coloring* of a graph G is a coloring of G's vertices with integers in $\{0, 1, \ldots, k\}$ so that adjacent vertices' colors differ by at least two and colors of distance-two vertices differ. We refer to an $L(2,1)$-coloring as a coloring. The *span* $\lambda(G)$ of G is the smallest k for which G has a coloring, a *span coloring* is a coloring whose greatest color is $\lambda(G)$, and the *hole index* $\rho(G)$ of G is the minimum number of colors in $\{0, 1, \ldots, \lambda(G)\}$ not used in a span coloring. We say that G is *full-colorable* if $\rho(G) = 0$. More generally, a coloring of G is a *no-hole coloring* if it uses all colors between 0 and its maximum color. Both colorings and no-hole colorings were motivated by channel assignment problems. We define the *no-hole span* $\mu(G)$ of G as ∞ if G has no no-hole coloring; otherwise $\mu(G)$ is the minimum k for which G has a no-hole coloring using colors in $\{0, 1, \ldots, k\}$. We prove that G is full-colorable if it has $\lambda(G)+1$ vertices. In addition, if G is not full-colorable and if it has at least $\lambda(G)+2$ vertices, then $\mu(G) \leq \lambda(G) + \rho(G)$. Moreover, for each $m \geq 1$ there is a graph with $\rho(G) = m$ and $\mu(G) = \lambda(G) + \rho(G)$.

1 INTRODUCTION

Let $\mathcal{G}$ be the family of simple graphs with nonempty, finite vertex sets. A *coloring* of $G = (V, E)$ in $\mathcal{G}$ is a map $f : V \to \mathbb{Z}$. It is an *ordinary coloring* if $f(x) \neq f(y)$ whenever $\{x, y\} \in E$, and a *no-hole coloring* if $f(V)$ is a set of consecutive integers. A *hole* in f is an integer i such that $\min f(V) < i < \max f(V)$ and $i \notin f(V)$.

Although every $G \in \mathcal{G}$ has an ordinary coloring with no holes, this need not be true for more demanding colorings. A case in point for $r \geq 1$ is an $(r+1)$-distant coloring which requires $|f(x) - f(y)| \geq r + 1$ whenever $\{x, y\} \in E$. Motivated by channel assignment problems, Roberts [9] and Sakai and Wang [11] study the question of what graphs have $(r+1)$-distant colorings that are no-hole colorings. The present paper does the same for $L(2,1)$-*colorings*, which are 2-distant colorings with the further restriction that $f(x) \neq f(y)$ whenever $\{x, z\}$, $\{y, z\} \in E$ for some $z \in V$. In other words, an $L(2,1)$-coloring is a coloring in which adjacent vertices' colors differ by at least two and colors of distance-two vertices differ.

We refer henceforth to an $L(2,1)$-coloring as a coloring, with $L(2,1)$ understood.

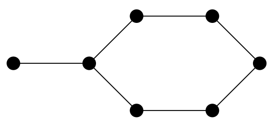
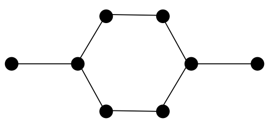
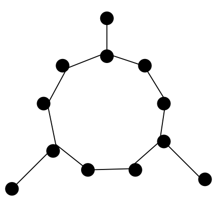
Colorings were first studied extensively by Yeh [13] and Griggs and Yeh [6] as a generalization of T-colorings formulated by Hale [7] for the channel assignment problem. Other studies of colorings include [1, 2, 3, 4, 5, 8, 10, 12]. Much of this work focuses on the *span* $\lambda(G)$ of $G \in \mathcal{G}$, defined as the minimum value of k for which G has a coloring with $f(V) \subseteq \{0, 1, \ldots, k\}$. We refer to a coloring with $\min f(V) = 0$ and $\max f(V) = \lambda(G)$ as a *span coloring*, and define the *hole index* $\rho(G)$ of G as the minimum, over all span colorings of G, of the number of holes in the coloring.

We say that G is *full-colorable* if $\rho(G) = 0$. The question of what graphs are full-colorable was the theme of Fishburn and Roberts [2]. We begin our present study of no-hole colorings by recalling selected results from that paper. In what follows, Δ denotes a graph's maximum vertex degree. We prove in [2] that:

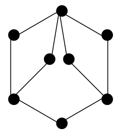
(1) all connected graphs with $\Delta = 2$ are full-colorable except for the 3-vertex path P_3 and cycles C_3, C_4 and C_6;

(2) all trees with $\Delta \geq 3$ other than stars are full-colorable;

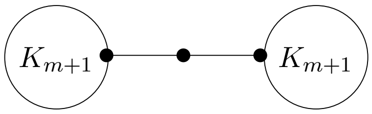
(3) every connected non-tree graph with $(\Delta, \lambda) = (3, 4)$ and $|V| \geq 5$ is full-colorable except for

 ,  and  ;

(4) the only connected non-tree graph with $(\Delta, \lambda) = (4, 5)$ and $6 \leq |V| \leq 8$ that is not full-colorable is the 8-vertex graph

 ;

(5) the hole index of a connected graph with span $2m \geq 4$ and $|V| \geq 2m+1$ can be as large as $m-1$. This value occurs for $|V| = 2m+3$ with the graph



in which the middle vertex has exactly one neighbor in each K_{m+1}.

The present paper introduces a new graph parameter for $L(2,1)$-colorings called the no-hole span. The *no-hole span* $\mu(G)$ of $G \in \mathcal{G}$ is ∞ if G has no no-hole coloring; otherwise $\mu(G)$ is the minimum k for which G has a no-hole coloring using colors in $\{0, 1, \ldots, k\}$. If G is not full-colorable then our definitions give $\rho(G) \geq 1$ and $\mu(G) \geq \lambda(G) + 1$. It is easily verified that each of the five graphs pictured above has $\mu(G) = \lambda(G) + 1$, so only one more color beyond those in the span color set $\{0, 1, \ldots, \lambda(G)\}$ is needed to achieve a no-hole coloring. Note in particular for (5) that a graph can have a large hole index and still have a no-hole coloring with $\mu(G) = \lambda(G) + 1$.

Let $n = |V|$. Because $|\{0, 1, \ldots, \lambda(G)\}| = \lambda(G)+1$, G is not full-colorable and $\mu(G) = \infty$ if $n \leq \lambda(G)$. This is true, for example, of every complete graph K_m with $m \geq 2$ and every star $K_{1,m}$ with $m \geq 1$.

The first of our three main results in the present paper concerns graphs with $\lambda(G) + 1$ vertices.

Theorem 1.1 *$G \in \mathcal{G}$ is full-colorable if $n = \lambda(G) + 1$.*

Theorem 1.1 identifies the only value of $n > 1$ in terms of λ that guarantees full-colorability. For example, $n = \lambda(G) + 2$ when G consists of two disjoint copies of K_3, and this graph is not full-colorable. An example for $n = \lambda(G) + 3$ is the first graph in (3) above.

Our second main result shows that a large hole index can entail a large no-hole span. It also implies that every graph with more than $\lambda(G) + 1$ vertices has a no-hole coloring. Consequently, G has a no-hole coloring if and only if $n \geq \lambda(G) + 1$.

Theorem 1.2 *If $G \in \mathcal{G}$ is not full-colorable and $n \geq \lambda(G) + 2$, then $\mu(G) \leq \lambda(G) + \rho(G)$. Moreover, for every $m \geq 1$ there is a $G \in \mathcal{G}$ with $\rho(G) = m$ and $\mu(G) = \lambda(G) + m$.*

One easy consequence of a result we will use to prove Theorem 1.2 (see Lemma 3.1, Section 3) is that $\mu(G)/\lambda(G) \leq 3/2$ when $\mu(G)$ is finite. A direct consequence of Theorem 1.2 is that if $n \geq \lambda(G) + 2$ then G has a no-hole coloring f with $f(V) = \{0, 1, \ldots, n-1\}$. For, by the theorem, G has a no-hole coloring. If two vertices have the same color, increase by 1 the color of one of them and all larger colors. Continuing in this way, we eventually arrive at a no-hole coloring in which each vertex gets a different color.

Our third main result improves on (5) by observing for each $m \geq 2$ that there is a connected graph with span $2m$, $n = 2m + 2$, and $\rho(G) = m$, which (by Lemma 3.1 below) is the maximum possible hole index for a graph with span $2m$.

Theorem 1.3 *For each $m \geq 2$ there is a connected graph G with $\lambda(G) = 2m$, $n = \lambda(G) + 2$, and $\rho(G) = m$.*

We prove Theorem 1.3 constructively in Section 5 after we complete the proofs of Theorems 1.1 and 1.2. In contrast to Theorem 1.2, the graphs for Theorem 1.3 have $\mu(G) = \lambda(G) + 1$.

We begin our proofs of Theorems 1.1 and 1.2 in the next section by constructing graphs that satisfy $\mu(G) = \lambda(G) + \rho(G)$ for every $\rho(G) \geq 2$. The graphs of (3) above have $\mu(G) = \lambda(G) + 1$ and $\rho(G) = 1$. Features of the construction in the next section will be evident in later proofs.

Section 3 proves a key lemma, Lemma 3.2, that analyzes hole structure, and the proofs of Theorems 1.1 and 1.2 are then completed in Section 4.

2 GRAPHS WITH LARGE NO-HOLE SPANS

In this section we construct a graph G_m for each $m \geq 2$ that has $\rho(G_m) = m$ and $\mu(G_m) = \lambda(G_m) + m$.

For each $m \geq 2$ let G_m be the graph with $n = 5(m + 1)$ that consists of disjoint copies A and B of K_{2m+2}, a K_{m+1} disjoint from $A \cup B$ that we denote by C, and assorted other edges between C and $A \cup B$ as follows. Denote the vertices by

$$\begin{array}{ll} a_0, a_2, a_4, \ldots, a_{4m+2} & \text{for } A \\ b_0, b_2, b_4, \ldots, b_{4m+2} & \text{for } B \\ c_1, c_5, c_9, \ldots, c_{4m+1} & \text{for } C\,. \end{array}$$

The other edges are between c_1 and each of $a_4, a_6, \ldots, a_{4m+2}$; between c_5 and each of a_0 and a_2; between c_5 and each of $b_8, b_{10}, \ldots, b_{4m+2}$; and between c_j and b_k if $j \geq 9$ and $|j - k| \geq 2$. Figure 1 pictures G_2 and G_3.

Figure 1 about here

Note that each $c_k \in C$ has exactly one edge to $\{a_j, b_j\}$ when $|j - k| \geq 2$, and has no edges to $\{a_{k-1}, a_{k+1}, b_{k-1}, b_{k+1}\}$. Each vertex in A has one edge to C and each vertex in B has exactly $m - 1$ edges to C. It follows that if $x \in A$ and $y \in B$, and if x and y do not have edges to the same vertex in C, then together they have edges to all but one of the $m + 1$ vertices in C.

By construction, the coloration of G_m that assigns color i to a_i, color j to b_j, and color k to c_k defines a coloring. Because $\lambda(K_{2m+2}) = 4m + 2$, $\lambda(G_m) = 4m + 2$ and every span coloring assigns precisely the colors in $\{0, 2, 4, \ldots, 4m+2\}$ to each of A and B. There are $4m+3$ integers between 0 and $4m+2$ inclusive. Hence any feasible assignment of colors between 0 and $4m+2$ to C has at least

$$4m + 3 - (2m + 2) - (m + 1) = m$$

holes. The coloration given shows that $\rho(G_m) = m$.

Suppose f is a no-hole coloring of G_m with $f(A \cup B \cup C) = \{0, 1, \ldots, \mu\}$. If color k is used more than once by f, then, since every vertex in C has distance at most 2 from all vertices of G_m, k is not used for C. Hence, k can be used only for one vertex in A and one in B, and the two cannot have edges to the same vertex in C. Suppose $1 \le k \le \mu - 1$ and $f(x) = f(y) = k$ for $x \in A$ and $y \in B$. Then x and y have edges to exactly m vertices in C, so none of those m has a color in $\{k-1, k+1\}$. Because no vertex in $A \cup B$ has a color in $\{k-1, k+1\}$, we are left with only one vertex whose color can be in $\{k-1, k+1\}$. This contradicts our no-hole supposition.

We conclude that only 0 and μ can be used for more than one vertex by f, and each can be assigned to no more than two vertices. If 0 and μ are in fact assigned by f to a total of four vertices, then $5m+1$ vertices remain uncolored and each must get a different color in $\{1, 2, \ldots, \mu - 1\}$. It is easily verified that such a no-hole coloring exists, so $\mu(G_m) = \mu = 5m + 2 = \lambda(G_m) + m$. A specific no-hole coloring of this type follows:

$$\begin{aligned}
C:&\quad f(c_1) = 5m+1, f(c_5) = 5(m-1)+1, f(c_9) = 5(m-2)+1, \ldots, f(c_{4m+1}) = 1\,;\\
A:&\quad f(a_0) = 5m+2, f(a_2) = 2, \text{ and } (f(a_4), f(a_6), \ldots, f(a_{4m+2})) = (0, 4, 7, 9, 12, 14, 17, 19, \ldots)\,.\\
B:&\quad (f(b_{4m+2}), f(b_{4m}), \ldots, f(b_0)) = (0, 3, 5, 8, 10, 13, 15, 18, \ldots)\,.
\end{aligned}$$

The f sequences shown for B and A alternate the colors from 3 through $5m$, skipping colors used for C.

3 KEY LEMMA

We now examine the hole structures of span colorings that omit as few colors as possible from $\{0, 1, \ldots, \lambda(G)\}$. Throughout this section and the next, $G = (V, E)$, $|V| = n$, $\lambda = \lambda(G)$, f is a span coloring of G with as few holes as possible,

$$V_j = \{v \in V : f(v) = j\}, \quad v_j = |V_j| \quad \text{for} \quad j = 0, 1, \ldots, \lambda$$

and

$$H = \{0 \le j \le \lambda : v_j = 0\} \quad \text{with} \quad |H| = \rho(G)\ .$$

Our first lemma limits the number of holes in H.

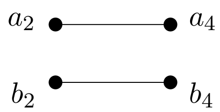
Lemma 3.1 *$0, \lambda \notin H$; if $h \in H$ then $h - 1 \notin H$ and $h + 1 \notin H$.*

Proof. By definition, 0 and λ cannot be holes in a span coloring. If $h, h+1 \in H$ with $1 \le h \le \lambda - 2$, we get a coloring and a contradiction by decreasing by 1 the color of every v with $f(v) \ge h+2$. ∎

Lemma 3.1 says that if h is a hole then V_{h-1} and V_{h+1} are nonempty. The following key lemma reveals a nice relationship between those two vertex sets.

Lemma 3.2 *Suppose $h \in H$ and $n \ge \lambda + 1$. Then $v_{h-1} = v_{h+1} \ge 2$ and there is a bijection $g : V_{h-1} \to V_{h+1}$ such that the set of edges of G within $V_{h-1} \cup V_{h+1}$ is $\{\{v, g(v)\} : v \in V_{h-1}\}$.*

REMARK. If G_m (see Figure 1) is colored by taking $f(a_i) = i$, $f(b_j) = j$, and $f(c_k) = k$, then $H = \{3, 7, \ldots\}$. For $h = 3$, each of V_2 and V_4 has one vertex in A and one in B with edge structure



Proof. Given $h \in H$, suppose $v_{h-1} > v_{h+1}$. Then because no two vertices in V_{h-1} can be adjacent to the same vertex, some $x \in V_{h-1}$ has no edge to V_{h+1}. A contradiction results by recoloring x with color h. A similar contradiction holds if $v_{h+1} > v_{h-1}$, so $v_{h-1} = v_{h+1}$.

Suppose $v_{h-1} = v_{h+1} \ge 2$. If a vertex in either set has no edge to the other set, recoloring the vertex with h contradicts the minimum-holes assumption. The distance-2 constraint then implies that every vertex in V_{h-1} and V_{h+1} has exactly one edge to the other set, and this establishes g.

Suppose $v_{h-1} = v_{h+1} = 1$ with $V_{h-1} = \{x\}$ and $V_{h+1} = \{y\}$. Because $n \ge \lambda + 1$, some color, say k, is used more than once by f. Assume without loss of generality that $k > h+1$. Every vertex in V_k has an edge to $\{x, y\}$, else we get a contradiction by recoloring such a vertex with h. Because no two vertices of the same color have a common neighbor, at most one vertex in V_k has an edge to x, and at most one has an edge to y. It follows that $|V_k| = 2$ and that V_k's two vertices, say z and w, are such that $\{x, z\}$ and $\{y, w\}$ are *not* edges. Now recolor the vertices as follows:

(i) all vertices with $f(v) \le h-1$ and $f(v) > k$ retain their original colors;

(ii) z gets color h;

(iii) every vertex with original color c such that $h+1 \le c < k$ gets color $h+k-c$, which implies that y gets color $k-1$;

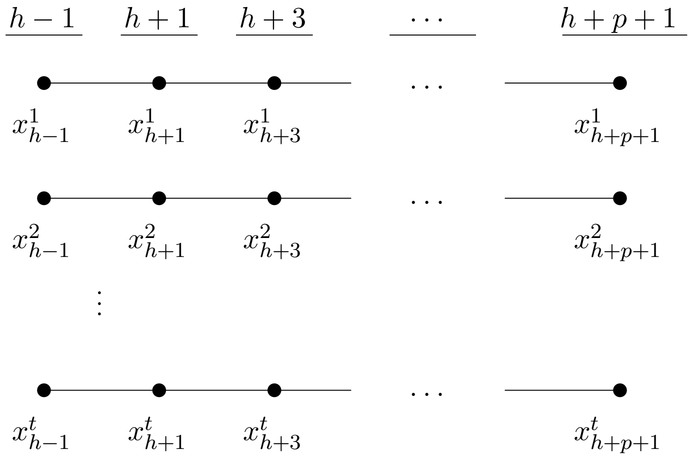
(iv) w retains color k.

This yields a span coloring with one fewer hole than f, a contradiction. We conclude that $v_{h-1} = v_{h+1} = 1$ is impossible, and the proof of Lemma 3.2 is complete. ∎

4 COMPLETIONS OF PROOFS OF THEOREMS 1.1 AND 1.2

To complete the proofs of Theorems 1.1 and 1.2, we continue with the notations of the preceding section and suppose that H is not empty. We show that this leads to a contradiction if $n = \lambda + 1$ and to $\mu(G) \leq \lambda + |H| = \lambda(G) + \rho(G)$ if G is not full-colorable and $n \geq \lambda + 2$.

Call distinct holes $h, h' \in H$ *adjacent* if $|h - h'| = 2$. A subset S of holes is a *string* if S is a nonempty maximal set of successively adjacent holes. (Each S for G_m is a singleton because $H = \{3, 7, 11, \ldots\}$.) If $S = \{h, h+2, \ldots, h+p\}$, it follows from Lemma 3.2 that the edges of G within $V_{h-1} \cup V_{h+1} \cup \cdots \cup V_{h+p+1}$ between successive sets in this union can be arrayed as



with $t \geq 2$. There can also be edges of G between vertices in nonadjacent columns, but there are no further edges between vertices in adjacent columns.

Let $S_1, S_2, \ldots, S_K$ with $K \geq 1$ be the strings of H ordered so that $h \in S_k$ is less than $h' \in S_{k+1}$. Also let $s_k = |S_k|$.

Suppose $n = \lambda + 1$. Then the total number of colors corresponding to holes and their adjacent colors ($h - 1$ and $h + 1$ for hole h) is

$$\sum_{k=1}^{K} (2s_k + 1) = 2|H| + K \ ,$$

and because $t \geq 2$, the total number of vertices assigned those colors is at least

$$\sum_{k=1}^{K} (2s_k + 2) = 2|H| + 2K \ .$$

This leaves $\lambda + 1 - (2|H| + K)$ colors for the span coloring f, and each of these is assigned to one or more vertices. Hence the number of vertices is at least

$$2|H| + 2K + [\lambda + 1 - (2|H| + K)] = \lambda + 1 + K \ .$$

This contradicts $n = \lambda + 1$, so the proof of Theorem 1.1 is complete.

Suppose $n \geq \lambda + 2$ and G is not full-colorable. Let S_K be as shown in the preceding array with $t \geq 2$. Recolor G by adding 1 to the colors of $x^1_{h-1}, x^1_{h+1}, \ldots, x^1_{h+p+1}$ and to all colors greater than $h + p + 1$. The other colors are unchanged. This yields a new coloring that increases the maximum color by 1, has s_K fewer holes than the original coloring, and has no hole to the right of S_{K-1}.

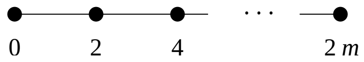
We now iterate the preceding routine for S_{K-1}, then $S_{K-2}, \ldots$, and finally S_1. Each iteration gives a coloring that increases the maximum color by 1 and removes all holes in the current S_k without creating any new holes. The final coloring has no holes, and the process gives

$$\mu(G) \leq \lambda + K \leq \lambda + |H| = \lambda(G) + \rho(G) \ .$$

This completes the proof of Theorem 1.2.

5 PROOF OF THEOREM 1.3: MAXIMUM HOLE INDICES

Lemmas 3.1 and 3.2 guide our construction of connected graphs that maximize ρ for a given λ. Suppose $\lambda(G) = 2m \geq 4$. Then $\rho(G) \leq m$ by Lemma 3.1, and if $\rho(G) = m$ then every span coloring of G must have hole set $\{1, 3, 5, \ldots, 2m-1\}$. It follows from Lemma 3.2 for $\rho(G) = m$ that every span coloring of G partitions G into $t \geq 2$ colored subgraphs

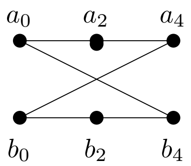


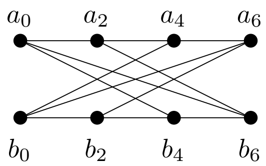
with no other edges between vertices whose colors differ by 2. Our ensuing construction takes $t = 2$.

Given $m \geq 2$, let F_m be the graph with $2m+2$ vertices $a_0, a_2, a_4, \ldots, a_{2m}, b_0, b_2, b_4, \ldots, b_{2m}$ and edge set

$$\{\{a_i, a_{i+2}\} : i = 0, 2, \ldots, 2m-2\} \cup \{\{b_i, b_{i+2}\} : i = 0, 2, \ldots, 2m-2\}$$
$$\cup \{\{a_i, b_j\} : |i - j| \geq 4\} \ .$$

Thus

$F_2 =$ 

and $F_3 =$ 

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Because $F_2 = C_6$ and every span coloring of C_6 uses only colors $0, 2$ and 4 ([2], Theorem 2.1), $\rho(F_2) = 2$. We focus on $m \geq 3$ henceforth. The bulk of the proof that $\lambda(F_m) = 2m$ and $\rho(F_m) = m$ is borne by the following lemma.

Lemma 5.1 *For every $m \geq 3$ and every coloring of F_m, at most two vertices have any given color, and if color k is used for two vertices then color k' is a hole when $|k' - k| = 1$.*

Proof. A pair of vertices $\{x, y\}$ is called a *dominating set* if every $z \neq x, y$ is adjacent to x or to y. Let $d(x, y)$ denote the distance between vertices x and y in graph F_m. It suffices to show that F_m satisfies the following property P: If $d(x, y) > 2$, then $\{x, y\}$ is a dominating set. For, if color k is used on vertices x and y, then $d(x, y) > 2$. By domination, colors k, $k-1$, and $k+1$ cannot be used on any other vertex.

It is easy to show that F_m satisfies property P. The pairs $\{x, y\}$ with $d(x, y) > 2$ for which one needs to prove domination are as follows: If $m \geq 6$, $\{a_i, b_i\}$, $i = 0, 2, \ldots, 2m$; if $m = 3$, $\{a_i, b_i\}$ plus $\{a_0, a_6\}$, $\{b_0, b_6\}$; if $m = 4$, $\{a_i, b_i\}$ plus $\{a_0, a_6\}$, $\{b_0, b_6\}$, $\{a_2, a_8\}$, $\{b_2, b_8\}$; and if $m = 5$, $\{a_i, b_i\}$ plus $\{a_2, a_8\}$, $\{b_2, b_8\}$. ■

For any coloring of F_m, let c_k be the number of vertices with color k and let $c = (c_0, c_1, c_2, \ldots, c_p)$ where $c_p > 0$ and $c_k = 0$ for all $k > p$. Note that $\Sigma c_k = 2m + 2$. By Lemma 5.1, $c_k \leq 2$, and if $c_k = 2$ then $c_j = 0$ when $|j - k| = 1$. The coloring f of F_m with $f(a_i) = f(b_i) = i$ for $i = 0, 2, \ldots, 2m$ has $c = (2, 0, 2, 0, \ldots, 0, 2)$ and shows that $\lambda(F_m) \leq 2m$. If c has 1's, then each maximal run of r 1's can be replaced by $2020 \cdots 201$ with r terms when r is odd, and by $2020 \cdots 2$ with $r - 1$ terms when r is even. This results in a new $c' = (c'_0, c'_1, c'_2, \ldots, c'_q)$ with $q \leq p$, no two consecutive terms positive, and $\Sigma c'_k = 2m + 2$. Clearly $q \geq 2m$, which shows that $\lambda(F_m) = 2m$. Moreover, $q = 2m$ if and only if $c'' = (2, 0, 2, 0, \ldots, 0, 2)$, which shows that every span coloring has $c_1 = c_3 = \cdots = c_{2m-1} = 0$, and hence that $\rho(F_m) = m$.

This completes the proof of Theorem 1.3.

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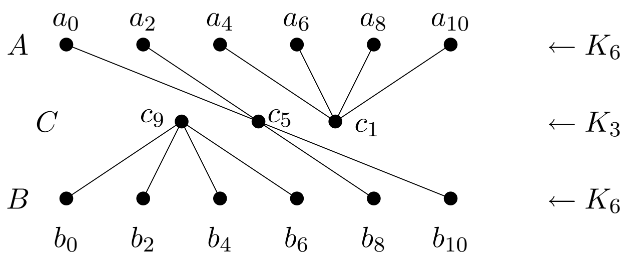
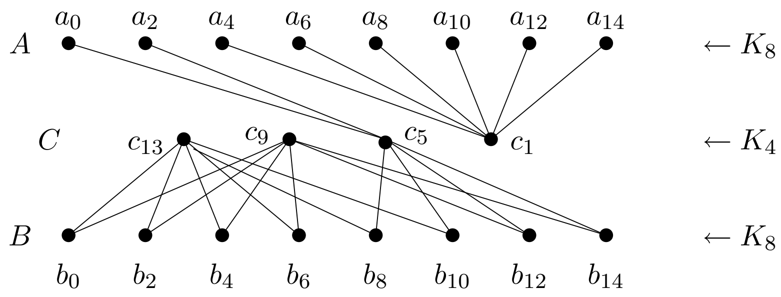


G_2



G_3

Figure 1. Graphs for $\mu(G_m) = \lambda(G_m) + m$. The edges between every two vertices within each row are *not* shown.