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Finding a maximum stable set in pentagraphs

by

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ABSTRACT

Penta is the configuration shown in Figure 1(a), where continuous lines represent edges and dotted lines represent non-edges. The vertex u in Figure 1(a) is called the *center* of Penta. A graph G is called a *pentagraph* if every induced subgraph H of G has a vertex v which is not a center of induced Penta in H.

The class of pentagraphs is a common generalization of chordal (triangulated) graphs and Mahadev graphs.

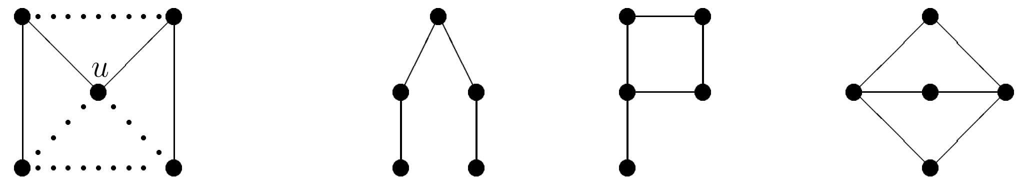
We construct a polynomial-time algorithm that either find a maximum stable set of G or concludes that G is not a pentagraph.

1 Introduction

Finding the stability number of a graph is one of the best known NP-complete problem. There are many clases where this problem can be solved in polynomial time. We propose a new class.

Definition 1. Penta *is the configuration shown in Figure 1(a), where continuous lines represent edges and dotted lines represent non-edges.*
The vertex u in Figure 1(a) is called the center *of Penta.*

It easy to see that Penta contains exactly three graphs, namely, P_5, Banner, and $K_{2,3}$ (Figure 1(b)) having one, two, and three centers, respectively.



(a) Penta (b) P_5 Banner $K_{2,3}$

Figure 1.

Definition 2. *A graph G is called a* pentagraph *if every induced subgraph H of G has a vertex v which is not a center of induced Penta in H.*

Example 1. Every chordal (triangulated) graph is a pentagraph. Indeed, a chordal graph contains a simplicial vertex (a vertex v such that either $N(v) = \emptyset$ or $N(v)$ induces a complete subgraph). Clearly, a simplicial vertex can nor be a center of induced Penta. Besides, each induced subgraph of a chordal graph is also a chordal graph.

Fact 1. *The stability number of a chordal graph can be found in polynomial time.*

It is also well known that the minimal forbidden induced subgraphs for chordal graphs are exactly the cycles C_n ($n \geq 4$). All of them, except C_4 and C_5 are also minimal forbidden induced subgraphs for pentagraphs.

Example 2. A *Mahadev graph* is a graph that does not contains an induced Penta, i.e., P_5, Banner, and $K_{2,3}$ are minimal forbidden induced subgraphs for this class. Clearly, every Mahadev graph is also a pentagraph.

Fact 2 (Mahadev [2]). *The stability number of a Mahadev graph can be found in polynomial time.*

2 Stability in pentagraphs

Here we construct a polynomial-time algorithm that produces a maximal stabile set of a pentagraph. The algorithm is robust, i.e., it either solves the problem, or concluded that graph is not a pentagraph.

Algorithm 1 (Pentagraph Stability).

Input: *a graph G.*
Output 1: *a maximal stable set S of G.*
Output 2: *G is not a pentagraph.*

Step 1. *Put $S = \emptyset$ and go to Step 2.*
Step 2. *If $V(G) = \emptyset$ then STOP (Output 1).*
Otherwise scan the vertices of G to find a vertex v that is not a center of induced Penta.
If there is no such a vertex then STOP (Output 2). Otherwise go to Step 3.
Step 3. *If v is a simplicial vertex then include v into S, delete $N[v]$ from G and go to Step 2. Otherwise go to Step 4.*
Step 4. *Delete v from S and go to Step 2.*

Theorem 1. *Pentagraph Stability either find a maximum stable set of G or concludes that G is not a pentagraph (in polynomial time).*

Proof. The algorithm produces Output 2 if it finds an induced subgraph that contains a vertex which is not a center of induced Penta. By the definition, G is not a pentagraph.

Suppose that the algorithm produces Output 1 (the set S). It is clear from Step 3 that S is stable. Let us show that S is a maximum stable set.

Claim 1. *If v (see Step 2 of the algorithm) is a simplicial vertex in G then there exists a maximum stable set S of G that contains v.*

Indeed, any maximum stable set has exactly one vertex (say, w) from $N[v]$. Them $(S\backslash\{w\}) \cup \{v\}$ is a required maximum stable set.

Claim 2. *If v is not a simplicial vertex in G and v is not a center of an induced Penta of G simplicial vertex in G then there exists a maximum stable set S of G that does not contain v.*

Proof. Since v is not a simplicial vertex, there are two non-adjacent vertices $w, x \in N[v]$. Let S be a maximum stable set of G. If $v \notin S$ then we are done.

Suppose that $v \in S$. Hence $w, x \notin S$. If $(S\backslash\{v\}) \cup \{w\}$ is a stable set then it is maximum and we are done. Otherwise there exists a vertex $y \in S\backslash\{v\}$ that is adjacent to w. Similarly, there exists a vertex $y' \in S\backslash\{v\}$ that is adjacent to x.

If $y \neq y'$ then v is the center of the Penta induced by $\{v, w, x, y, y'\}$, a contradiction. So $y = y'$.

Further, if $(S\backslash\{v, y\}) \cup \{w, x\}$ is a stable set then it is maximum and we are done. Therefore there exists a vertex $z \in S\backslash\{v, y\}$ which is adjacent to at least one of w, x. In that case we get the Penta induced by $\{v, w, x, y, z\}$ with the center v, a contradiction.

Claim 1 and Claim 2 imply the correctedness of the algorithm. Finally, it is obvious that Pentagraph Stability runs in polynomial time.

The following problem is of a theoretical interest.

Open Problem 1. *Find a characterization of pentagraphs in terms of forbidden induced subgraphs.*

We know the following minimal forbidden induced subgraphs for pentagraphs: $C_n (n \geq 6)$, and three graphs shown in Figure 2.

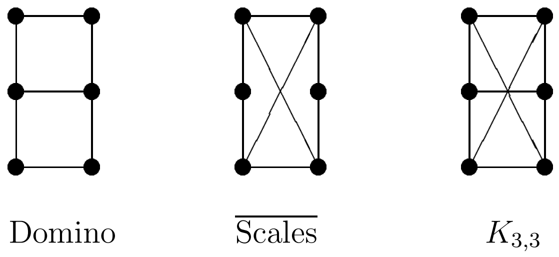


Figure 2. Some minimal forbidden induced subgraphs for pentagraphs

References

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[2] N. V. R. Mahadev, Vertex deletion and stability number, Technical Report 90/2 (Swiss Federal Institute of Technology, 1990).