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Weighted well-covered graphs and complexity questions

by

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ABSTRACT

A weighted graph G is called *well-covered* if all its maximal stable sets have the same weight. Let S be a stable set of G (possibly, $S = \emptyset$). The subgraph $G - N[S]$ is called a *co-stable subgraph* of G. We denote by $\mathrm{CSub}(G)$ the set of all co-stable subgraphs of G (considered up to isomorphism). A class of weighted graphs $\mathcal{P}$ is called *co-hereditary* if it is closed under taking co-stable subgraphs, i.e., $G \in \mathcal{P}$ implies $\mathrm{CSub}(G) \subseteq \mathcal{P}$.

We note that the class $\mathcal{WW}$ of all weighted well-covered graphs is co-hereditary and characterize $\mathcal{WW}$ in terms of forbidden co-stable subgraphs.

Then we use a reduction from Satisfiability to show that the following decision problems are NP-complete.

Decision Problem 1 (Co-Stable Subgraph).
Instance: A graph G and a set $U \subseteq V(G)$ that induces a subgraph H.
Question: Is H a co-stable subgraph of G?

Decision Problem 2 (Co-Stable Subgraph H).
Instance: A graph G.
Question: Is H a co-stable subgraph of G?

Let $\Delta(G)$ be the maximum vertex degree of a graph G. We show that recognizing weighted well-covered graphs with bounded $\Delta(G)$ can be done in polynomial time.

1 Introduction

Let $G = (V, E)$ be a weighted graph with a real-valued weight function w defined on V. A set $S \subseteq V$ is *stable* if $e \nsubseteq S$ for every $e \in E$. A *maximal* stable set is not contained in any other stable set.

As usual, the *weight* of a subset $X \subseteq V$ is $w(X) = \sum_{x \in X} w(x)$.

Definition 1. *A weighted graph G is called* well-covered *if all its maximal stable sets have the same weight. This common weight is the* value *of G.*

The *neighborhood* $N(v)$ of a vertex v is the set of all vertices that are adjacent to v. The *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. For a set $X \subseteq V$, $N(X) = \bigcup_{x \in X} N(x)$ and $N[X] = X \cup N(x)$.

Definition 2. *Let S be a stable set of G (possibly, $S = \emptyset$). The subgraph $G - N[S]$ is called a* co-stable subgraph *of G.*

Since we consider graphs up to isomorphism, a graph which is isomorphic to a co-stable subgraph of G is also called a co-stable subgraph of G.

We denote by $\mathrm{CSub}(G)$ the set of all co-stable subgraphs of G (considered up to isomorphism). Note that the weight function of any subgraph H of G is a restriction of the weight function of G to $V(H)$.

Definition 3. *A class of weighted graphs $\mathcal{P}$ is called* co-hereditary *if it is closed under taking co-stable subgraphs, i.e., $G \in \mathcal{P}$ implies $\mathrm{CSub}(G) \subseteq \mathcal{P}$.*

We denote by $\mathcal{WW}$ the class of all weighted well-covered graphs.

Proposition 1. *The class $\mathcal{WW}$ is co-hereditary.*

Proof. Let G be a well-covered graph with a weight function w. We need to show that an arbitrary co-stable subgraph H of G is also well-covered. By the definition, $H = G - N[S]$ for a some stable set S.

Let A and B be maximal stable sets of H. It is easy to see that both $S \cup A$ and $S \cup B$ are maximal stable sets of G. Since G is a well-covered graph, $w(S \cup A) = w(S \cup B)$. Thus, $w(A) = w(B)$ and H is a well-covered graph.

2 A characterization of weighted well-covered graphs

By Proposition 1 we can characterize the class of all weighted well-covered graphs in terms of forbidden co-stable subgraphs.

Definition 4. *A graph F is a* minimal forbidden co-stable subgraph *for a co-hereditary class $\mathcal{P}$ if* $\mathrm{CSub}(F)\backslash\mathcal{P} = \{F\}$.

We denote by $Z(\mathcal{P})$ the set of all minimal co-stable subgraph for a co-hereditary class $\mathcal{P}$.

Proposition 2. *Let $\mathcal{P}$ be a co-hereditary class of weighted graphs. Then $G \in \mathcal{P}$ if and only if* $\mathrm{CSub}(G) \cap Z(\mathcal{P}) = \emptyset$.

All minimal forbidden co-stable subgraph for the class of all unweighted well-covered graphs were found by Zverovich [4]. Here we consider the weighted case.

Definition 5. *We define the set Z of weighted graphs as follows: $F \in Z$ if and only if there exists a partition*

$$V_1 \cup V_2 \cup \cdots \cup V_k = V(F) \tag{1}$$

such that

(Z1) $k \geq 2$;

(Z2) $V_i \neq \emptyset$ *and the induced subgraph $F(V_i)$ is well-covered for each* $i = 1, 2, \ldots, k$;

(Z3) *the values of $F(V_i)$ and $F(V_j)$ are distinct for all* $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$;

(Z4) *for all $1 \leq i \neq j \leq k$, each vertex of V_i is adjacent to each vertex of V_j.*

Theorem 1. $Z(\mathcal{WW}) = Z$.

Proof. Firstly, we show the inclusion $Z \subseteq Z(\mathcal{WW})$. Let $F \in Z$, i.e., there exists a partition (1) such that the conditions (Z1), (Z2), (Z3), and (Z4) hold.

By (Z1), (Z2), we can choose vertices $v_1 \in V_1$ and $v_2 \in V_2$. Let us extend $\{v_i\}$ to a maximal stable set S_i ($i = 1, 2$) of F. The condition (Z4) implies that $S_i \subseteq V_i$, $i = 1, 2$, i.e., S_i is a maximal stable set of $F(V_i)$. By (Z3), the values of $F(V_1)$ and $F(V_2)$ are distinct, so $w(S_1) \neq w(S_2)$ and $F \notin \mathcal{WW}$.

Let $H \in \mathrm{CSub(F)}\backslash\{F\}$. By the definition, $H = F - N[S]$ for some stable set $S \neq \emptyset$ of F. According to (Z4), $S \subseteq V_i$ for some $i \in \{1, 2, \ldots, k\}$ and $H = F(V_i) - N[S]$. Therefore H is a co-stable subgraph of $F(V_i)$. Then the condition (Z2) and Proposition 1 imply that H is well-covered.

Thus, $\mathrm{CSub(F)}\backslash\mathcal{WW} = \{F\}$, i.e., $F \in Z(\mathcal{WW})$.

Now we prove the reverse inclusion $Z(\mathcal{WW}) \subseteq Z$. Let F be in $Z(\mathcal{WW})$. By the definition of $Z(\mathcal{WW})$, $F_v = F - N[v]$ is a well-covered graph for every $v \in V(F)$.

Let S be an arbitrary maximal stable set of F that contains v. Then $S \backslash \{v\}$ is a maximal stable set of F_v. Since $F_v \in \mathcal{WW}$, all maximal stable sets containing v have the same weight ($w(v)$ plus the value of F_v). We will refer to this common weight as the *value of* v.

The binary relation "to have the same value" is an equivalence relation on $V(F)$. We denote by $V_1, V_2, \cdots, V_k$ the corresponding equivalence classes. The *value of* V_i is the common value of the vertices in V_i.

It remains to check the conditions (Z1), (Z2), (Z3), and(Z4).

(Z1) If $k = 1$ then all maximal stable sets of F have the same weight, i.e., F is a well-covered graph, a contradiction. Hence $k \geq 2$.

(Z4) Suppose that there are $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$, and non-adjacent vertices $v \in V_i$ and $w \in V_j$. Let S be a maximal stable set that contains $\{v, w\}$. Then the value of v is $w(S)$ and the value of w is $w(S)$, a contradiction to the definition of V_i and V_j. Thus, (Z4) holds.

(Z2) Let S be a maximal stable set of $F(V_i)$. Since $V_i \neq \emptyset$, $S \neq \emptyset$. By (Z4), S is also a maximal stable set of F. By the definition of V_i, $w(S)$ equals to the value of V_i Hence all maximal stable sets of $F(V_i)$ have the same weight, and F is a well-covered graph.

The condition (Z3) holds by the definition of V_i's.

3 NP-completeness of recognition co-stable subgraphs

For unweighted well-covered graphs the recognition problem is hard. In fact, Chvátal and Slater [2] and independently Sankaranarayana and Stewart [3] showed that it is co-NP-complete.

Here we prove some NP-complete results concerning co-stable subgraphs in unweighted graphs.

Decision Problem 1 (Co-Stable Subgraph).
Instance: A graph G and a set $U \subseteq V(G)$ that induces a subgraph H.
Question: Is H a co-stable subgraph of G?

Theorem 2. *Co-Stable Subgraph is an NP-complete problem.*

Proof. The problem is in NP. Indeed, if we guess a stable set S of G, we can check in polynomial time that $H = G - N[S]$.

Now we will use a reduction from Satisfiability Problem of CNF clauses (see Garey and Johnson [1]). Let $C = \{c_1, c_2, \dots, c_m\}$ be a set of clauses over a set a literals $L = \{x_1, \overline{x}_1, x_2, \overline{x}_2, \dots, x_n, \overline{x}_n\}$.

We construct a graph G as follows:

- $V(G) = U \cup C \cup L$;
- the sets U, C and L are pairwise disjoint;
- U induces H;
- C is a stable set;
- L induces nK_2, x_i being adjacent to $\overline{x}_i$ for all $i = 1, 2, \dots, n$;
- every vertex of U is adjacent to every vertex of C;
- a vertex $x \in L$ is adjacent to a vertex $c_j \in C$ if and only if the clause c_j contains the literal x;
- no edges between L and C.

Now we consider G and $H = G(U)$ as an instance to Co-Stable Subgraph.

Suppose that H is a co-stable subgraph of G. Then there exists a stable set $S \subseteq V(G)$ such that $H = G - N[S]$. Clearly, $S \subseteq L$. Evidently, every vertex of C is adjacent to some vertex of S (S *dominates* C). Since S is a stable set, S does not contain a pair of opposite literals (i.e., x_i and $\overline{x}_i$). Moreover, S contains exactly one literal from each pair $(x_i, \overline{x}_i)$ (since S is a maximal stable set).

Thus, we can define a truth assignment ϕ putting a literal x to be true if and only if $x \in S$. Clearly, ϕ is a satisfying assignment for C.

Conversely, if there is a truth assignment ϕ that satisfies C, then we define a stable set $S \subseteq L$: a literal x in L is in S if and only if $\phi(x) = 1$. Since exactly one of x, $\overline{x}$ is true, every vertex of L is in $N[S]$.

The assignment ϕ satisfies C, so S dominates C, and $C \subseteq N[S]$. We have $L \cup C \subseteq N[S]$. By the construction of G, $U \cap N[S] = \emptyset$. Thus, $N[S] = L \cup C$, or $H = G - N[S] \in \mathrm{CSub}(G)$ and the result follows.

Decision Problem 2 (Co-Stable Subgraph H).
Instance: A graph G.
Question: Is H isomorphic to a co-stable subgraph of G?

Theorem 3. *Co-Stable Subgraph H is an NP-complete problem for any graph H.*

Proof. Clearly, the problem is in NP.

Again, we will use a reduction from Satisfiability Problem. Let $C = \{c_1, c_2, \dots, c_m\}$ be a set of clauses over a set of literals $L = \{x_1, \overline{x}_1, x_2, \overline{x}_2, \dots, x_n, \overline{x}_n\}$.

We define a graph G^* as follows:

- $V(G^*) = U \cup C^* \cup L^*$;
- the sets U, C^* and L^* are pairwise disjoint;
- U induces H;
- $G^*(C^*)$ is a disjoin union of complete subgraphs $C^1, C^2, \dots, C^m$, each having exactly $k+1$ vertices, where $k = |V(H)|$;
- L^* consists of $2n$ pairwise disjoint sets $X_1, \overline{X}_1, X_2, \overline{X}_2, \dots, X_n, \overline{X}_n$ ($|X_i| = |\overline{X}_i| = k+1$ for every $i = 1, 2, \dots, n$); $G^*(L^*)$ is a union of complete graphs induced by $X_i \cup \overline{X}_i$, $i = 1, 2, \dots, n$;
- every vertex of U is adjacent to every vertex of C^*;
- vertices $x \in L^*$ and $c \in C^*$ are adjacent if and only if either $x \in X_i$, $c \in C^j$ and the clause c_j contains x_i, or $x \in \overline{X}_i$, $c \in C^j$ and the clause c_j contains $\overline{x}_i$;
- no edges between L and C^*.

In fact, we inflate every vertex of the set $C \cup L$ of the graph G (in the proof of Theorem 2) to a complete subgraph of order $k+1$.

As in the proof of Theorem 2, a set U induces a co-stable subgraph if and only if there exists a truth assignment for L that satisfies C. Therefore it is sufficient to prove the following statement: If $W \subseteq V(G)$ induces a co-stable subgraph of G isomorphic to H then $W = U$.

Suppose that W contains a vertex $w \notin U$. By the construction, w belongs to a complete subgraph K of order $k+1$ such that $N[a] = N[b]$ for all $a, b \in V(K)$ (K is one of C^j, $G(X_i)$, or $G(\overline{X}_i)$).

Since $|V(K)| > k = |V(H)| = |W|$, there is a vertex $z \in V(K) \backslash W$.

Further, W induces a co-stable subgraph, i.e., $W = V(G) \backslash N[S]$ for some stable set S of G. Since $w \in W$, $w \notin N[S]$. It follows that $S \cap N[w] = \emptyset$. But $N[w] = N[z]$, so $S \cap N[z] = \emptyset$. We obtain that $z \in V(G) \backslash N[S] = W$, a contradiction.

Thus, $W \subseteq U$. Since $|W| = |U|$, $W = U$.

4 A positive result

Let $\Delta(G)$ be the maximum vertex degree of a graph G. We show that recognizing weighted well-covered graphs with bounded $\Delta(G)$ can be done in polynomial time.

Theorem 4. *Let $\Delta_r = \{G : \Delta(G) \leq r\}$. For each fixed $r \geq 0$, there exists a polynomial-time algorithm that recognizes whether a weighted graph G in Δ_r is well-covered.*

Proof. Let G be an arbitrary graph in Δ_r. According to Theorem 1, it is sufficient to show that we can determine in polynomial time whether G contains a co-stable subgraph $F \in Z$ (Definition 5).

Claim 1. *Let $F \in Z$ be a co-stable subgraph of G. Then $|V(F)| \leq 2r$.*

Proof. By the Definition 5, F has a partition (1) satisfying (Z1), (Z2), (Z3), and (Z4). Clearly, $\Delta(F) \leq \Delta(G) \leq r$. Without loss of generality we may assume that $|V_1| \leq |V_2| \leq \cdots \leq |V_k|$.

Every vertex $v \in V_1$ is adjacent to all vertices of $V_2 \cup V_3 \cup \cdots \cup V_k$. Hence

$$r \geq \deg v \geq |V_2| + |V_3| + \cdots + |V_k|. \tag{2}$$

It follows from (2) that $|V_2| \leq r/(k-1)$ and $|V_1| \leq |V_2| \leq r/(k-1)$. Then

$$|V(H)| \leq \sum_{i=1}^{k} |V_i| \leq r/(k-1) + r = rk/(k-1) \leq 2r,$$

since $k/(k-1) \leq 2$ for all $k \geq 2$.

Claim 2. *We can find in polynomial time all subsets $X \subseteq V(G)$ that induce a subgraph $G(X) \in Z$.*

Proof. By Claim 1, we can restrict ourselves with subsets X of the cardinality at most $2r$. Since r is a constant, we can generate all such subsets in polynomial time. For every such X, we can determine in constant time whether $G(X) \in Z$ or not

According to Claim 2, we construct the list $\mathcal{L} = \{X \subseteq V(G) : G(X) \in Z\}$ in polynomial time.

Now we check whether a subset $X \in \mathcal{L}$ induces a co-stable subgraph. We put $\Gamma(X) = N[X] \backslash X$ and $R(X) = V(G) \backslash N[X]$.

Clearly, $G(X)$ is a co-stable subgraph if and only if there exists a stable set $S \subseteq R(X)$ that dominates $\Gamma(X)$. Each vertex of $\Gamma(X)$ is adjacent to at most $\Delta(G) - 1 \leq r - 1$ vertices of $R(X)$. Hence we have at most $N = (r-1)^{|\Gamma(X)|}$ variants for S. Since $|\Gamma(X)| \leq r|X| \leq 2r^2$, $N \leq (r-1)^{2r^2}$.

Thus, we have a constant number of variants for S. All of them can be generated in constant time. It remains to check for every $X \in \mathcal{L}$ whether there exists a variant with a stable set S or not.

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