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Distribution of Distances in Star Graphs

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ABSTRACT

The distribution of distances in the star graph ST_n $(1 < n \in \mathcal{Z})$ is determined from any fixed vertex, in particular to the vertices of each one of the n so-called efficient dominating sets, or 1-perfect codes.

1 Introduction

The star graph ST_n ($1 < n \in \mathcal{Z}$) is the Cayley graph of the symmetric group S_n with respect to the set of transpositions $\Sigma_n = \{(1\ i),\ i = 2, \dots, n\}$, (see [1, 2]). Define the *weight* of a vertex u of ST_n as its distance to the identity permutation vertex $12\dots n$. We determine the weight distribution of certain subsets C of ST_n, that is: how many vertices of C are at distance ω from $12\dots n$, for $0 \le k \le D(ST_n)$, where $D(G)$ stands for the diameter of a graph G. This is done for $C = ST_n$ in Theorem 5.3.

The present work is motivated by the study of efficient dominating sets or E-sets, (see [4, 3]), (also called 1-perfect codes, see [5, 6]) of star and related graphs. The E-sets of ST_n form a partition of its vertex set. In Section 6, we determine their weight distributions, (Theorem 6.1 and its subsequent remark).

In obtaining the cited results we use a rooted directed tree $\Lambda_n = \Lambda(ST_n)$. In Section 7, we extend Λ_n to a quotient graph Γ_n of ST_n whose vertices represent the different cycle structures of the permutations associated to the vertices of ST_n. Moreover, the Γ_n form a nested sequence that converges to a universal graph Γ_∞ associated to the infinite star graph ST_∞.

2 Definition and Examples of $\Lambda_n = \Lambda(ST_n)$

Let $n > 1$. Let $\Sigma = \sigma_1\sigma_2\dots\sigma_n$ be a permutation of $12\dots n$. A cycle of Σ is said to be *proper* if it is not of the form (σ_i), where σ_i ($i \in \{1, \dots, n\}$) is fixed by Σ. The *cycle structure* of $\Sigma = \sigma_1\sigma_2\dots\sigma_n$ is the set of proper cycles produced by the transformation that sends $1, 2, \dots n$ respectively onto $\sigma_1, \sigma_2, \dots, \sigma_n$. More generally, two such nontrivial permutations, say Σ^1 and Σ^2, are said to have a common *1-invariant cycle structure* (or 1-ics) if there is a permutation of $12\dots n$ with 1 fixed taking the cycle structure of Σ^1 onto that of Σ^2. In that case, we say that Σ^2 has the *1-invariant cycle structure* (or 1-ics) of Σ^1. Each vertex u of Λ_n is represented the form

$$\boxed{\begin{array}{c} w(u), c(u) \\ \Sigma(u) \end{array}}$$

where **(a)** $\Sigma(u)$ represents a permutation $\sigma_1\sigma_2\dots\sigma_n$ of $12\dots n$ such that if $i \in \{2, \dots, n\}$ is the smallest index with $\sigma_j = j$, for $i \le j \le n$ then $\sigma_j \ne j$ for $1 < j < i$; $\Sigma(u)$ is shown shortened to $\sigma_1 \dots \sigma_{i-1}$ if no confusion arises; **(b)** $w(u)$ is the weight of $\Sigma(u)$; **(c)** $c(u)$ is the cardinality of the set $S(u)$ of permutations with the 1-ics $\Pi(u)$ of $\Sigma(u)$.

We define the length $\lambda(\Sigma(u))$ of $\Sigma(u)$ to be the number of elements in the notation of $\Sigma(u)$ given in (a) above (but identify $\Sigma(u)$ with $\sigma_1\sigma_2\dots\sigma_n$ if no confusion arises). Given an edge e of Λ_n, let u_e and u^e be the source and the target of e, respectively. There are two types of edges e in Λ_n: **(1)** horizontal edges e with $\lambda(\Sigma(u^e)) = 1 + \lambda(\Sigma(u_e))$, traced from left to right and labeled with the multiplicative operator $\times m_e$, where $c(u^e) = c(u_e) \times m_e$, whenever $\sigma_1(u^e) \ne 1$; **(2)** vertical edges f traced from top to bottom and labeled with the divisive operator $\div d_f$, where $c(u^f) = c(u_f) \div d_f$, whenever $\sigma_1(u^f) = 1$ and there is not an

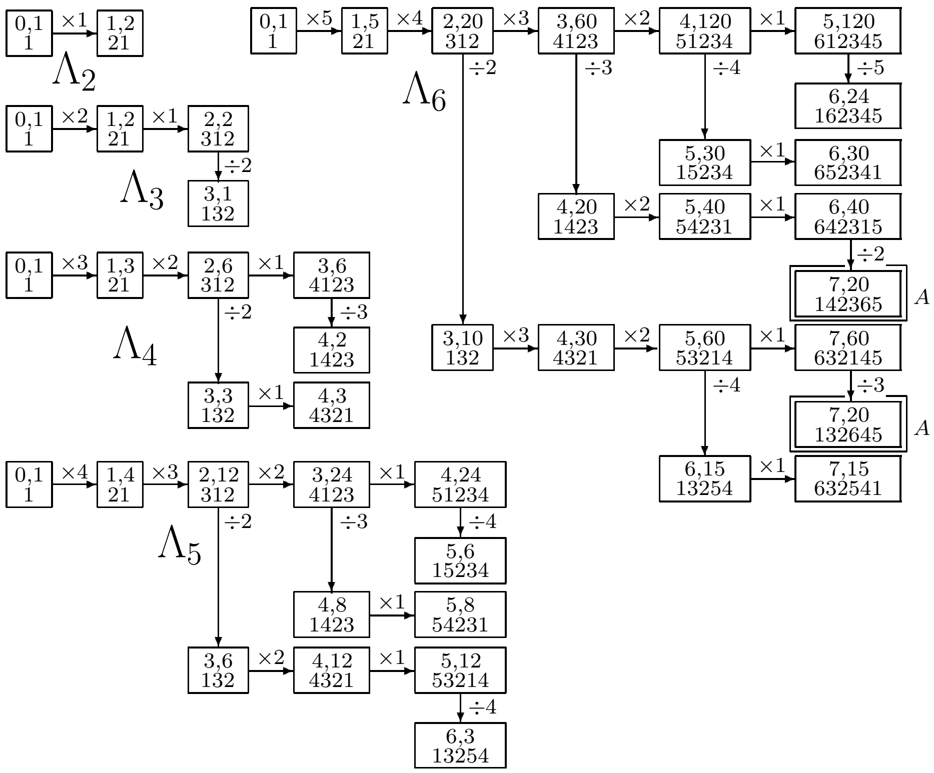

Figure 1: Representations of Λ_n, for $n = 2, 3, 4, 5, 6$.

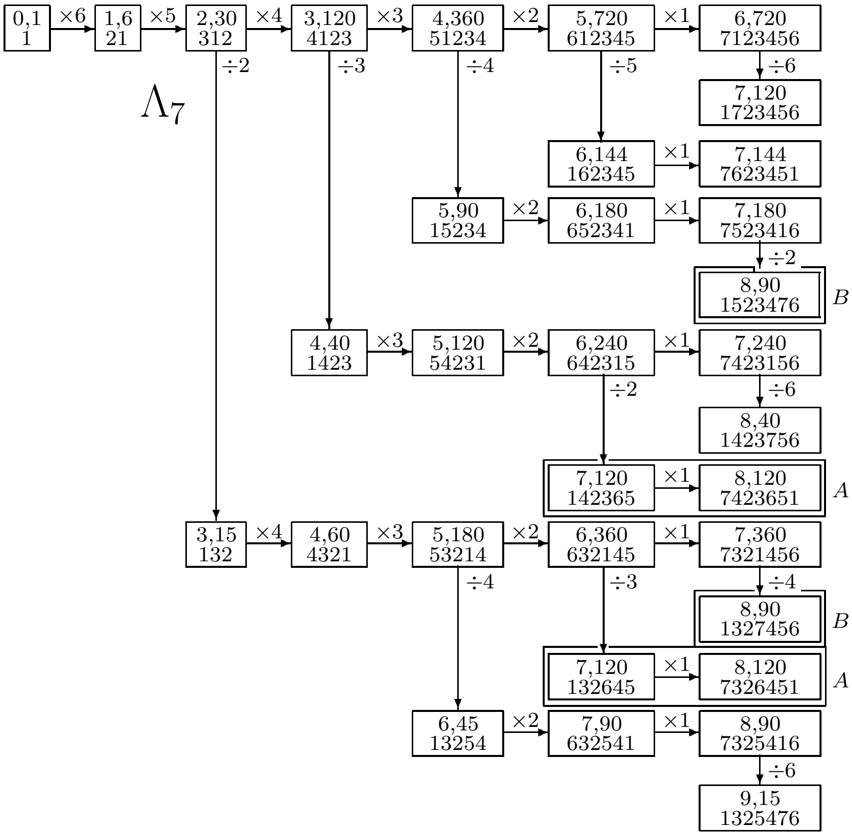

Figure 2: Representation of Λ_7.

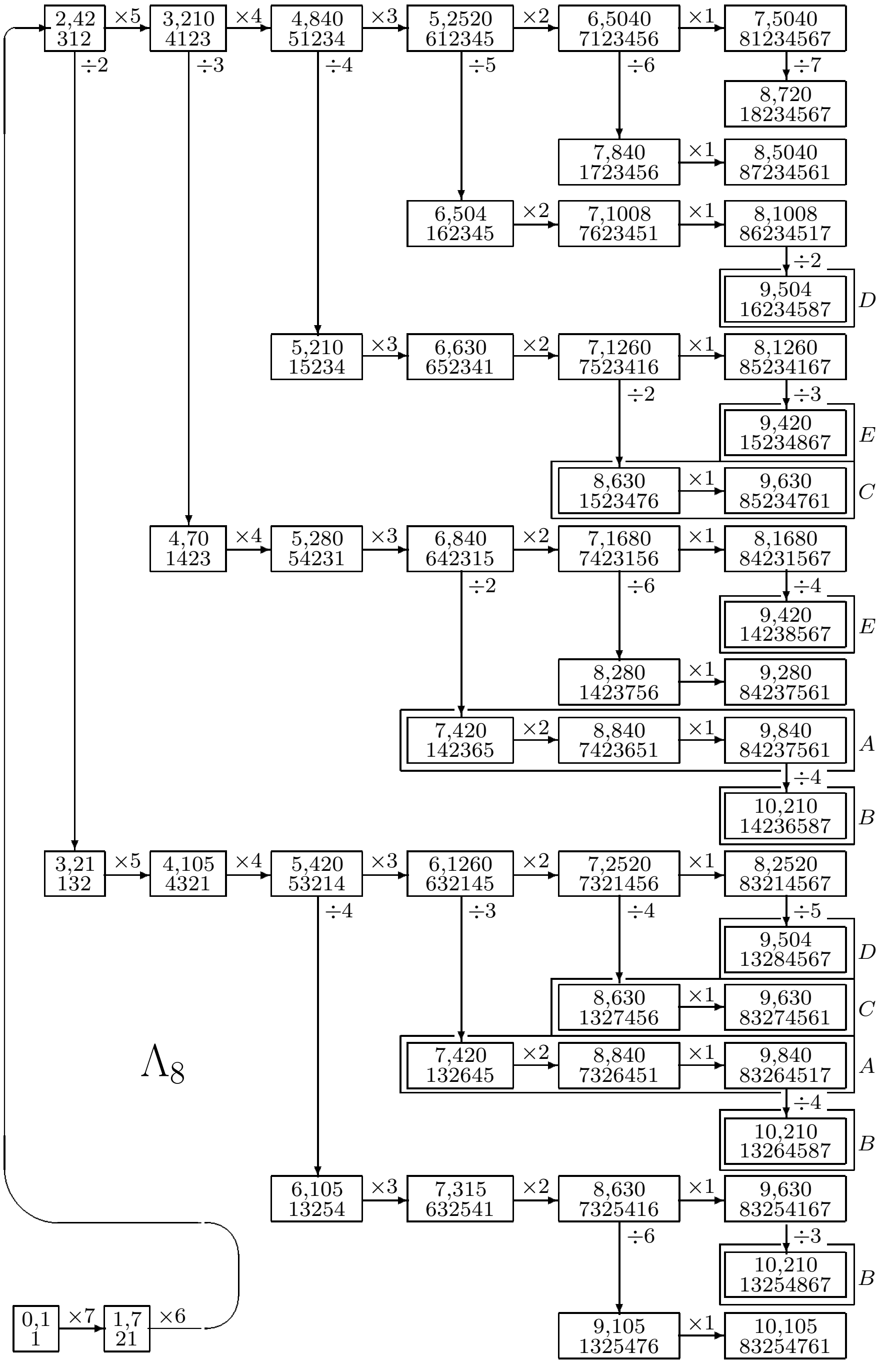

Figure 3: Representation of Λ_8.

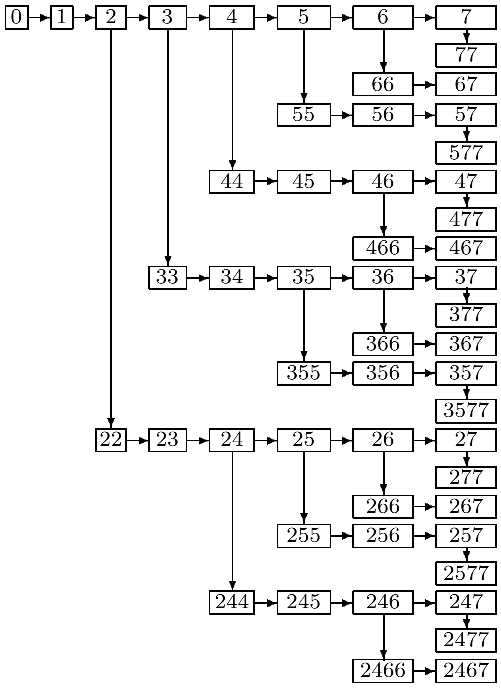

Figure 4: Index string representation of Λ_8.

horizontal edge e as in item (1) above with $u_e = u^f$ and $u^e = u_f$, (to avoid a 'vertical' edge inverse to the first horizontal one in each Λ_n).

The only additional requirement now in the definition of Λ_n is that it has exactly one universal source vertex $u_0 = u_0^n$ (so it is the only vertex of Λ_n which is not a target of any edge of Λ_n), namely the one in the upper-left corner of the graphical representation of Λ_n, with $w(u_0) = 0$, $c(u_0) = 1$ and $\Sigma(u_0) = 1$.

Given a maximal horizontal directed path (or mhdp) P of Λ_n, the *depth* of P is the number of vertical edges of Λ_n preceding P from u_0.

Examples. Figures 1,2,3 contain the representations of Λ_n, for $n = 2, \ldots, 8$, respectively, (with last case in Figure 3 starting at lower left corner for lack of space), where pairs of encased mhdp's U_I, V_I, improper (i.e. consisting of an only vertex) or proper, and labeled with a common capital letter $I = A, B, \ldots$ on their right, have corresponding vertices u_j^I, v_j^I representing each a complete set of permutations with a common 1-ics, and thus having a common cardinality number $c(u_j^I) = c(v_j^I)$, so that, to determine the weight distribution of ST_n, the Pruning Algorithm of section 3 below leaves only one of these encased mpdh's with a common capital letter I. ■

Remark. A vertex labeling accounting for all of Λ_{n+1} is apparent via the following inductive definition of properties (j), for $j = 0, 1 \ldots \lfloor n/2 \rfloor$:

(0) There is an mhdp $u_0 u_1 \ldots u_n$ of depth 0 in Λ_{n+1}.

(j) For each $u_{i_0 i_1 \ldots i_{j-2} i_{j-1}}$ as in property (j-1) with $1 < i_{j-1} - i_{j-2}$ there is a vertical edge

$u_{i_0i_1\ldots i_{j-1}}u_{i_0i_1\ldots i_{j-1}i_{j-1}}$ and an mhdp

$u_{i_0i_1\ldots i_{j-1}i_{j-1}}\ldots u_{i_0i_1\ldots i_{j-1}n}$ of depth j in Λ_{n+1}.

As an accompanying example, see Figure 4 representing the disposition of the index strings $i_0i_1\ldots i_j$ of vertices $u_{i_0i_1\ldots i_j}$ according to Figure 3, for $n=8$. ∎

3 Redefinition and Pruning of $\Lambda_n = \Lambda(ST_n)$

If a vertex u of Λ_n is the source vertex of an horizontal, (a vertical), edge e labeled $\times m_e$, $(\div d_e)$, then we write $m_u = m_e$, $(d_u = d_e)$; if u is not a source, then we write $m_u = 0$, $(d_u = 0)$. The following redefinition allows to consider Λ_n as a labeled subdigraph of Λ_{n+1} for any n so the limit Λ_∞ of the nested sequence $\{\Lambda_n; n > 0\}$ of rooted directed trees makes sense: replace the label $\times m_u = \times m_e$ of the horizontal edge e of Λ_n departing from each source vertex u of Λ_n by $\bullet\ell_u$, where $\ell_u = n - m_u$ and $\bullet$ is the operation given by $c(u)\bullet\ell_u = c(u)\times(n-m_u)$. Let $\Lambda_n^\bullet$ be the resulting labeled digraph; redefine $\Lambda_n = \Lambda_n^\bullet$. The new labels for the horizontal edges allow now the claimed containment of labeled subdigraphs. Incidentally, the cardinalities of the set of vertices of the resulting Λ_∞ that are sources of edges labeled $\bullet_i$ form a Fibonacci sequence according to the increasing values of $i = 0, 1, \ldots$. This is apparent from the number of vertices in the successive columns from left to right, in the representations of the Λ_n's as in Figures 1,2,3, for increasing values of $n = 2, 3, \ldots$.

Let ρ_n be the relation defined on the vertex set of ST_n by $u\rho_n v$ if and only if u and v represent permutations with a common 1-ics. The following second redefinition of Λ_n allows to have its vertex set in bijective correspondence with the family of equivalence classes of ST_n under ρ_n, which in turn allows to use Λ_n to compute the weight distribution of ST_n: perform the Pruning Algorithm of Λ_n given below, leaving a maximal subdigraph Λ'_n of Λ_n in which there are not pairs of mhdp's $v_0v_1\ldots v_s$ and $v'_0v'_1\ldots v'_s$ of the same length s with corresponding vertices v_i and v'_i having common 1-ics $\Pi(v_i) = \Pi(v'_i)$, for $i = 0, 1, \ldots, s$; redefine $\Lambda_n = \Lambda'_n$; we have still Λ_n as a subdigraph of Λ_{n+1} for every n, so a Λ_∞ persists.

Pruning Algorithm. The vertices $u_{i_0i_1\ldots i_j}$ of Λ_n are treated first in the increasing order of their lengths $j+1$ and then, for each fixed length $j+1$, in the lexicographical order of their subindex strings $i_0i_1\ldots i_j$, namely:

$$u_0, u_1, \ldots, u_{n-1}, u_{2,2}, u_{2,3}, \ldots, u_{2,n-1}, u_{3,3}, \ldots,$$

$$u_{n-1,n-1}, u_{2,4,4}, \ldots, u_{2,4,6,6}, \cdots$$

Each such a vertex $u = u_{i_0i_1\ldots i_j}$ is considered as composed by the following fields: **(1)** The notation $u = u_i^w = u_{i_0i_1\ldots u_j}^{w(u)}$ of vertex u, where $w(u) = w(\Sigma(u))$. **(2)** The notation $\Sigma(u)$ of the corresponding permutation of $\{1,\ldots n\}$ associated to u. **(3)** The 1-ics $\Pi(u)$ of $\Sigma(u)$. **(4)** The number $\ell_u = n - m_u$; **(5-7)** Either a blank in each of the three cases (5),(6),(7), if u is the first or second vertex of an mhdp, or: **(5)** the notation $\Sigma[u]$ of the permutation obtained from $\Sigma(u)$ by permuting σ_1 and $\sigma_k = 1$, $(k \neq 1)$; **(6)** the 1-ics $\Pi[u]$ of $\Sigma[u]$ possibly written differently from its form in (3) above; **(7)** a tuple $C(u) = s_1, \ldots, s_h$ composed by the orders

s_j of the cycles composing $\Pi[u]$. **(8)** The number d_u expressed as a product $b_u \times a_u$, where **(8a)** $a_u = i_j - i_{j-1}$ under the convention $i_{-1} = 0$ and **(8b)** $b_u \neq 0$ if and only if the value of item (7) above is not a blank and the resulting tuple $C(u)$ was not present in previously treated vertices of Λ_n.

Finally, the Pruning Algorithm consists in determining the mentioned fields of the vertices of Λ_n in the prescribed order, allowing a partial reconstruction of Λ_n in the form of Λ'_n which accepts and copies all the vertices and edges of Λ_n in Λ'_n but for the following case: if $b_u = 0$ and there is a vertical edge e whose source is u then e is not copied from Λ_n into Λ'_n, which is interpreted as the pruning of e and its descendant vertices and edges. This is done to avoid repetitions of mhdp's, as in the encased mpdh's having a common capital-letter label at their right in Figures 1,2 and 3; it is here that the desired specific action of our Pruning Algorithm takes place. ■

Example. The table $\mathcal{P}_n = \mathcal{P}_9$ below shows the running of the algorithm for $n = 9$, where commas are deleted in subindices of the u_i^w (item 1) and in the $C(u)$ (item 7) and where the $\Pi(u)$ and $\Pi[u]$ are shown as subindices of their corresponding $\Sigma(u)$ and $\Sigma[u]$ with each ')(' replaced by '.'.

u_i^w	$\Sigma(u)_{\Pi(u)}$	ℓ_u	$\Sigma[u]_{\Pi[u]}$	$C(u)$	$b_u a_u$
u_0^0	1	I		1	00
u_1^1	$21_{(12)}$	2			01
u_2^2	$312_{(132)}$	3	$132_{(32)}$	2	12
u_3^3	$4123_{(1432)}$	4	$1423_{(432)}$	3	13
u_4^4	$51234_{(15432)}$	5	$15234_{(5432)}$	4	14

u_i^w	$\Sigma(u)_{\Pi(u)}$	ℓ_u	$\Sigma[u]_{\Pi[u]}$	$C(u)$	$b_u a_u$
u_5^5	$612345_{(165432)}$	6	$162345_{(65432)}$	5	15
u_6^6	$7123456_{(1765432)}$	7	$1723456_{(765432)}$	6	16
u_7^7	$81234567_{(18765432)}$	8	$18234567_{(8765432)}$	7	17
u_8^8	$912345678_{(198765432)}$	9	$192345678_{(98765432)}$	8	18
u_{22}^3	$132_{(23)}$	3			00
u_{23}^4	$4321_{(14.32)}$	4			01
u_{24}^5	$53214_{(154.32)}$	5	$13254_{(54.32)}$	22	22
u_{25}^6	$632145_{(1654.32)}$	6	$132645_{(654.32)}$	23	13
u_{26}^7	$7321456_{(17654.32)}$	7	$1327456_{(7654.32)}$	24	14
u_{27}^8	$83214567_{(187654.32)}$	8	$13284567_{(87654.32)}$	25	15
u_{28}^9	$932145678_{(1987654.32)}$	9	$132945678_{(987654.32)}$	26	16
u_{33}^4	$1423_{(432)}$	4			00
u_{34}^5	$54231_{(15.432)}$	5			01
u_{35}^6	$642315_{(165.432)}$	6	$142365_{(65.432)}$	32	02
u_{36}^7	$7423156_{(1765.432)}$	7	$1423756_{(765.432)}$	33	23
u_{37}^8	$84231567_{(18765.432)}$	8	$14238567_{(8765.432)}$	34	14
u_{38}^9	$942315678_{(198765.432)}$	9	$142395678_{(98765.432)}$	35	15
u_{44}^5	$15234_{(5432)}$	5			00
u_{45}^6	$652341_{(16.5432)}$	6			01
u_{46}^7	$7523416_{(176.5432)}$	7	$1523476_{(76.5432)}$	42	02
u_{47}^8	$85234167_{(1876.5432)}$	8	$15234867_{(876.5432)}$	43	03
u_{48}^9	$952341678_{(19876.5432)}$	9	$152349678_{(9876.5432)}$	44	24
u_{55}^6	$162345_{(65432)}$	6			00
u_{56}^7	$7623451_{(17.65432)}$	7			01
u_{57}^8	$86234517_{(187.65432)}$	8	$16234587_{(87.65432)}$	52	02
u_{58}^9	$962345178_{(1987.65432)}$	9	$162345978_{(987.65432)}$	53	03
u_{66}^7	$1723456_{(765432)}$	7			00
u_{67}^8	$87234561_{(18.765432)}$	8			01
u_{68}^9	$972345618_{(198.765432)}$	9	$172345698_{(98.765432)}$	62	02
u_{77}^8	$18234567_{(8765432)}$	8			00
u_{78}^9	$982345671_{(19.8765432)}$	9			01
u_{88}^9	$192345678_{(98765432)}$	9			00
u_{244}^6	$13254_{(54.32)}$	5			00
u_{245}^7	$632541_{(16.54.32)}$	6			01
u_{246}^8	$7325416_{(176.54.32)}$	7	$1325476_{(23.45.67)}$	222	32
u_{247}^9	$83254167_{(1876.54.32)}$	8	$13254867_{(23.45.687)}$	223	13
u_{248}^{10}	$932541678_{(19876.54.32)}$	9	$132549678_{(23.45.6987)}$	224	14

u_i^w	$\Sigma(u)_{\Pi(u)}$	ℓ_u	$\Sigma[u]_{\Pi[u]}$	$C(u)$	$b_u a_u$
u_{255}^7	$132645_{(654.32)}$	6			00
u_{256}^8	$7326451_{(17.654.32)}$	7			01
u_{257}^9	$83264517_{(187.654.32)}$	8	$13264587_{(23.465.78)}$	232	02
u_{258}^{10}	$932645178_{(1987.654.32)}$	9	$132645978_{(23.465.798)}$	233	13
u_{266}^8	$1327456_{(7654.32)}$	7			00
u_{267}^9	$83274561_{(18.7654.32)}$	8			01
u_{268}^{10}	$932745618_{(198.7654.32)}$	9	$132745698_{(23.4765.89)}$	242	02
u_{277}^9	$13284567_{(87654.32)}$	8			00
u_{278}^{10}	$932845671_{(19.87654.32)}$	9			01
u_{288}^{10}	$132945678_{(987654.32)}$	9			00
u_{366}^8	$1423756_{(765.432)}$	7			00
u_{367}^9	$84237561_{(18.765.432)}$	8			01
u_{368}^{10}	$942375618_{(198.765.432)}$	9	$142375698_{(243.576.89)}$	332	02
u_{377}^9	$14238567_{(8765.432)}$	8			00
u_{378}^{10}	$942385671_{(19.8765.432)}$	9			01
u_{388}^{10}	$142395678_{(98765.432)}$	9			00
u_{488}^{10}	$152349678_{(9876.5432)}$	9			00
u_{2466}^9	$1325476_{(76.54.32)}$	7			00
u_{2467}^{10}	$83254761_{(18.76.54.32)}$	8			01
u_{2468}^{11}	$932547618_{(198.76.54.32)}$	9	$132547698_{(23.45.67.89)}$	2222	42
u_{2477}^{10}	$13254867_{(876.54.32)}$	8			00
u_{2478}^{11}	$932548671_{(19.876.54.32)}$	9			01
u_{2488}^{11}	$132549678_{(9876.54.32)}$	9			00
u_{2588}^{11}	$132645978_{(987.654.32)}$	9			00
u_{24688}^c	$132547698_{(98.76.54.32)}$	9			00

This table shows and generalizes to the patterns expressed in the following theorem. For $u = u_{i_0 i_1 \dots i_j}$ in Λ_n, let $\ell_u = \ell_{i_0 i_1 \dots i_j}$, etc.

Theorem 3.1 *Let $i_{-1} = 0$ and let $t_k = i_k - i_{k-1}$, for $k = 0, 1, \dots, j-1$. Then:*

1. *The 1-ics $C(u)$ in the penultimate field of the line associated to a vertex $u = u_{i_0 i_1 \dots i_j}$ in $\mathcal{P}_n$ is of the form $t_0, t_1, \dots, t_j$, where the order of the t_k is irrelevant.*
2. *The vertices $u_{i_0 i_1 \dots i_j}$ of Λ_n (left after applying the Pruning Algorithm) have subindex strings $i_0 i_1 \dots i_j$ completely determined by the following conditions:*

   (a) $0 \le i_0 \le n-1$; (b) *if $j > 0$ then $2 \le i_0$*;
   (c) $t_k \le t_{k+1}$, *for $k = 0, \dots, j-2$*; (d) $i_{j-1} \le i_j$.

3. *The weight $w(u)$ of a vertex $u = u_{i_0 i_1 \dots i_j}$ of Λ_n is $w(u) = w(u_{i_0 i_1 \dots i_j}) = i_j + j$.*

4. *The number ℓ_u associated to a vertex $u_{i_0 i_1 \dots i_j}$ of Λ_n is $\ell_u = \ell_{i_0 i_1 \dots i_j} = i_j + 1$. Thus, the corresponding multiplicative factor m_u is $m_u = m_{i_0 i_1 \dots i_j} = n - i_j - 1$.*

5. *The number $d_u = b_u.a_u$ has $a_u = t_j$. Moreover, $b_u > 0$ if and only if either $j = 0$ and $i_0 > 1$ or $j > 0$ and $2 \leq i_0 \leq t_1 \leq t_2 \leq \dots \leq t_j$. Furthermore, if $b_u > 0$ then $b_u = 1$ unless $i_0 = t_1 = t_2 = \dots = t_j$, in which case $b_u = j + 1$.*

To compute the weight distribution of ST_n, a table $\mathcal{T}_n$ constructed out of the resulting pruned version of Λ_n and satisfying the following additional conditions will be used: **(a)** the subindex strings $i_0 i_1 \dots i_j$ of the vertices $u_{i_0 i_1 \dots i_j}$ of Λ_n are distributed on columns according to their weight; **(b)** each row contains the subindex strings of the vertices of an mhdp P of Λ_n, given from left to right according to the orientation of P; **(c)** each mhdp is presented in its row in lexicographical order; **(d)** the rows of each complete set of common-depth mhdp's are presented contiguously by the decreasing order of their path lengths, thus forming upper triangular matrices (because of (a)); **(e)** the upper triangular matrices in (d) are presented downward by the increasing order of their depths.

Example. $\mathcal{T}_{11}$ is as follows with $a = 10$ and $b = 11$, where vertices with $j = 2$, $i_0 = 3$ and previous to 366 do not appear as they were pruned, etc.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	2	3	4	5	6	7	8	9	a					
			22	23	24	25	26	27	28	29	$2a$				
				33	34	35	36	37	38	39	$3a$				
					44	45	46	47	48	49	$4a$				
						55	56	57	58	59	$5a$				
							66	67	68	69	$6a$				
								77	78	79	$7a$				
									88	89	$8a$				
										99	$9a$				
											aa				

6	7	8	9	10	11	12	13	14	15
244	245	246	247	248	249	$24a$			
	255	256	257	258	259	$25a$			
		266	267	268	269	$26a$			
			277	278	279	$27a$			
				288	289	$28a$			
					299	$29a$			
						$2aa$			
		366	367	368	369	$36a$			
			377	378	379	$37a$			
				388	389	$38a$			
					399	$39a$			
						$3aa$			
				488	489	$48a$			
					499	$49a$			
						$4aa$			
						$5aa$			
			2466	2467	2468	2469	$246a$		
				2477	2478	2479	$247a$		
					2488	2489	$248a$		
						2499	$249a$		
							$24aa$		
					2588	2589	$258a$		
						2599	$259a$		
							$25aa$		
							$26aa$		
						3699	$369a$		
							$36aa$		
						24688	24689	$2468a$	
							24699	$2469a$	
								$246aa$	
								$247aa$	
									$2468aa$

Each vertex $u = u_{i_0 i_1 \ldots i_j} = i_0 i_1 \ldots i_j$ of Λ_n reachable from $u_0 = 0$ by a unique path P has associated cardinality $c(u) = M/(A.B)$, where: **(a)** M, (A), is the product of the numbers $m_{i_0 i_1 \ldots i_j} = n - i_j - 1$, $(a_{i_0 i_1 \ldots i_j} = t_j = i_i - i_{j-1})$ of all sources $i_0 i_1 \ldots i_j$ of horizontal, (vertical), edges in P; **(b)** B is the product of all the numbers $b_{i_0 i_1 \ldots i_j}$ of sources $i_0 i_1 \ldots i_j$ of vertical edges in P with $i_0 = t_1 = \ldots = t_j$.

We need a quick procedure to compute the path from u_0 to any given vertex u of Λ_n, best done by going backwards from $i_0 i_1 \ldots i_j$ to 0 by means of table $\mathcal{T}_n$ and consisting of the following steps: **(1)** set $u = i_0 i_1 \ldots i_j$; **(2)** if u is not the first vertex of an mhdp, then

go backwards through the vertices of the mhdp containing $i_0i_1\dots i_j$; **(3)** once arrived to the first vertex v of an mhdp, or in the case that $u=v$ is such a first vertex, consider its vertical predecessor, that is the source z of the vertical edge in Λ_n with target v, (which is in the previous column to that containing v); **(4)** set $u=z$ and repeat item 2; **(5)** the procedure continues until vertex 0 is reached.
Example. Let $i_0i_1\dots i_j = 2468aa$ be the vertex of Λ_{11}, whose weight is 15. This is the first vertex of its (improper) mhdp. Its vertical predecessor in column 14 is $2468a$. This is preceded horizontally by 24689 and this by 24688, in respective columns 13 and 12. The vertical predecessor of 24688 is 2468, in column 11, preceded horizontally by 2467 and this by 2466, in respective columns 10 and 9. The vertical predecessor of 2466 is 246, in column 8, preceded horizontally by 245 and this by 244, in respective columns 7 and 6. The vertical predecessor of 244 is 24, in column 5, preceded horizontally by 23 and this by 22, in respective columns 4 and 3. The vertical predecessor of 22 is 2, in column 2, preceded horizontally by 1 and this by $0=u_0$, in respective columns 1 and 0. Thus we get the path (with commas replaced by superindices m_u, for horizontal edge sources u, and subindices d_u, for vertical edge sources u, respectively):

$$0^{10}1^{9}2_{2}22^{8}23^{7}24_{4}244^{6}245^{5}246_{6}2466^{4}2467^{3}2468_{8}24688^{2}24689^{1}2468a_{10}$$

$$2468aa.$$

We arrive at $c(2468aa) = 9\times 7\times 5\times 3$. ∎

This generalizes to the following result.

Theorem 3.2 *If $n = 2k+1$ then the paths realizing the diameter $D(ST_n)$ and starting at $12\dots n$ end up at exactly $(n-2)(n-4)\dots(n-2k)\dots 3$ vertices u of the form $\Sigma(u) = \sigma_1\sigma_2\dots\sigma_n$ with $\sigma_1 = 1$ and $\Pi(u)$ expressible as a product of $k+1$ independent transpositions.*

4 ...via Counting in Λ_∞

A string $i_0i_1\dots i_j$ is said to be admissible if $u_{i_0i_1\dots i_j}$ is a vertex of Λ_∞. Given a positive integer $d \le D(ST_n)$, we want first to find an expression for the cardinality of the set V_d of vertices of Λ_∞ having d as their weight in ST_{d+1}. Toward this task, we start exemplifying some sequences of admissible strings for lower values of d, where subindex strings $i_0i_1\dots u_j$ of vertices $u_{i_0i_1\dots u_j}$ are expressed in a suitable order, avoiding commas and using the following additional shorthand rule for a dot notation for certain subsequences: Let $i_0i_1\dots i_{k-1}i_k.i_{k+1}\dots i_{j-1}i_j$ be the sequence composed by all admissible strings $i_0i_1\dots i_{k-1}\iota_k.\iota_{k+1}\dots\iota_{j-1}i_j$ in Λ_n with $\iota_\ell \ge i_\ell$ for $k\le \ell < j$.
Examples. Some subsequences of admissible strings in Λ_∞ are:

$$\begin{array}{lllllll}
2.2 & = & \{22\}; & 2.i_1 & = & \{2i_1, 3i_1, \dots, i_1i_1\}, & \text{for } i_1 > 2;\\
24.4 & = & \{244\}; & 24.i_2 & = & \{24i_2, 25i_2, \dots, 2i_2i_2\}, & \text{for } i_2 > 4;\\
36.6 & = & \{366\}; & 36.i_2 & = & \{36i_2, 37i_2, \dots, 3i_2i_2\}, & \text{for } i_2 > 6;\\
246.6 & = & \{2466\}; & 246.i_3 & = & \{246i_3, 247i_3, \dots, 24i_3i_3\}, & \text{for } i_3 > 6;\\
369.9 & = & \{3699\}; & 369.i_3 & = & \{369i_3, 36ai_3, \dots, 36i_3i_3\}, & \text{for } i_3 > 9.
\end{array}$$

Let V_ω be the subset of vertices in Λ_∞ having fixed weight ω. For $\omega = 0, 1, \ldots, 15 = f$ we can express V_ω as follows, where hexadecimal notation is used:

$$\begin{array}{llll} V_0 = \{0\} & V_1 = \{1\} & V_2 = \{2\} & \\ V_3 = \{3, \ \ 2.2\} & V_4 = \{4, \ \ 2.3\} & V_5 = \{5, \ \ 2.4\} & V_6 = \{6, \ \ 2.5, \ \ 24.4\} \end{array}$$

$$\begin{array}{lllllll}
V_7 = \{7, & 2.6, & 24.5\} & & & & \\
V_8 = \{8, & 2.7, & 24.6, & 36.6\} & & & \\
V_9 = \{9, & 2.8, & 24.7, & 36.7, & & & \\
 & & 246.6\} & & & & \\
V_a = \{a, & 2.9, & 24.8, & 36.8, & 48.8, & & \\
 & & 246.7, & 257.7\} & & & \\
V_b = \{b, & 2.a, & 24.9, & 36.9, & 48.9, & & \\
 & & 246.8, & 257.8, & 268.8\} & & \\
V_c = \{c, & 2.b, & 24.a, & 36.a, & 48.a, & 5a.a, & \\
 & & 246.9, & 257.9, & 268.9, & 279.9, & \\
 & & 369.9 & & & & \\
 & & 2468.8\} & & & & \\
V_d = \{d, & 2.c, & 24.b, & 36.b, & 48.b, & 5a.b, & \\
 & & 246.a, & 257.a, & 268.a, & 279.a, & 28a.a, \\
 & & 369.a, & 37a.a, & & & \\
 & & 2468.9, & 2579.9\} & & & \\
V_e = \{e, & 2.d, & 24.c, & 36.c, & 48.c, & 5a.c, & 6c.c, \\
 & & 246.b, & 257.b, & 268.b, & 279.b, & 28a.b, \ 29b.b, \\
 & & 369.b, & 37a.b, & 38b.b, & & \\
 & & 2468.a, & 2579.a, & 268a.a\} & &
\end{array}$$

$$\begin{array}{lllllll}
V_f = \{f, & 2.e, & 24.d, & 36.d, & 48.d, & 5a.d, & 6c.d, \\
 & & 246.c, & 257.c, & 268.c, & 279.c, & 28a.c, 29b.c, \ 2ac.c, \\
 & & 369.c, & 37a.c, & 38b.c, & 39c.c & \\
 & & 2468.b, & 2579.b, & 268a.b, & 279b.b, & \\
 & & 2468a.a\} & & & &
\end{array}$$

Some of these subsets can be presented more succintly as:

$$\begin{array}{llllll}
V_6 = \{6, & 2.5, & 2.44\} & & & V_7 = \{7, \ \ 2.6, \ \ 2.45\} \\
V_8 = \{8, & 2.7, & 2.46\} & & & V_9 = \{9, \ \ 2.8, \ \ 2.47, \ \ 2.466\} \\
V_a = \{a, & 2.9, & 2.48, & 2.467\} & & V_b = \{b, \ \ 2.a, \ \ 2.49, \ \ 2.468\} \\
V_d = \{d, & 2.c, & 2.4b, & 2.46a, & 2.4689\} & \\
V_e = \{e, & 2.d, & 2.4c, & 2.46b, & 2.468a\} & \\
V_f = \{f, & 2.e, & 2.4d, & 2.46c, & 2.468b, & 2.468aa\}
\end{array}$$

What strings of length λ does V_ω have? The examples of V_w above show patterns taking to the following conclusions. Let $V_\omega^{i_0}$ be the subset of strings of V_ω starting at i_0.

1. String length $\lambda = 1$ happens in V_ω and only for the string ω, where $\omega \geq 0$.
2. String length $\lambda = 2$ happens in V_ω only in:

$$V_\omega^2 \text{ and only for the members of } 2.(\omega - 1), \text{ where } \omega \geq 3;$$

3. String length $\lambda = 3$ happens in V_ω only in:

$$\begin{array}{l} V_\omega^2 \text{ and only for the members of } 24.(\omega - 2), \text{ where } \omega \geq 6; \\ V_\omega^3 \text{ and only for the members of } 36.(\omega - 2), \text{ where } \omega \geq 8; \\ \ldots \\ V_\omega^k, \ (k \geq 2) \text{ and only for the members of } k(2k).(\omega - 2), \\ \text{ where } \omega \geq 2(k+1). \end{array}$$

4. String length $\lambda = 4$ happens in V_ω only in:

$$\begin{array}{l} V_\omega^2 \text{ and only for the members of } 246.(\omega - 3) \text{ where } \omega \geq 9; \\ V_\omega^3 \text{ and only for the members of } 369.(\omega - 3) \text{ where } \omega \geq 12; \\ \ldots \\ V_\omega^k \ (k \geq 2), \text{ and only for the members of } k(2k)(3k).(\omega - 3) \\ \text{ where } \omega \geq 3(k+1). \end{array}$$

The following result is obtained.

Theorem 4.1 *String length $\lambda = 1$ happens in V_ω and only for the strings of V_ω starting at i_0. Moreover, any fixed string length $\lambda > 1$ happens in V_ω only in the subsets V_ω^k, $(k \geq 2)$ and only for the members of $k(2k)(3k)\ldots((\lambda-1)k).(\omega - \lambda + 1)$, where $\omega \geq (\lambda - 1)(k+1)$.*

Let W_ω^k be the subset of V_ω consisting of the strings of length $\lambda = k$. Then $|W_\omega^1| = 1$ and $|W_\omega^k| = 0$ whenever $\omega < 3k$, for $k \geq 2$. Moreover, if

$$\begin{array}{lll} S_j^0 = & 1, & \text{for every } j \geq 1; \\ S_j^h = & \sum_{k=1}^{j} S_k^{h-1} & \text{for every } j \geq 1, \quad \text{where } h > 0, \end{array}$$

then $|W_\omega^k| = T_\omega^k$, for every weight ω in $ST_{\omega+1}$ and every string length k, where

$$\begin{array}{lll} T_1^k & = & S_1^k; \\ T_\omega^k & = & S_\omega^k \ + \ S_{\omega-(k+1)}^k \ + \ S_{\omega-2(k+1)}^k \ + \ \ldots \ + \ S_{\omega - \lfloor \frac{\omega}{k+1} \rfloor (k+1)}^k. \end{array}$$

Theorem 4.2 *For $0 < \omega \in \mathcal{Z}$, the number of vertices of $ST_{\omega+1}$ having weight ω is given by a finite sum $|V_\omega| = T_\omega^1 + T_\omega^2 + \ldots + T_\omega^k + \ldots$.*

It is easy to check the following expression for the diameter $D(n) = D(ST_n)$ of ST_n.

Proposition 4.1 *The diameter of ST_n is $D(n) = \lfloor \frac{n-1}{2} \rfloor + n - 1$.*

Let $V_\omega(n)$ be the set of vertices of Λ_n having weight ω. Let $W^k_\omega(n)$ be the subset of admissible strings corresponding to vertices of $V_\omega(n)$ whose length λ is equal to k. Then, from the tables $\mathcal{T}_n$ we get:

$$\begin{array}{lcccll}
|W^k_\omega(n)| & = & |W^k_\omega| & , & (0 \le k \le n-1); & \\
|W^n_\omega(n)| & = & |W^n_\omega| & - & |W^0_\omega| & = |W^n_\omega| - 1; \\
|W^{n+1}_\omega(n)| & = & |W^{n+1}_\omega| & - & (|W^0_\omega| + |W^1_\omega|); & \\
|W^{n+2}_\omega(n)| & = & |W^{n+2}_\omega| & - & (|W^0_\omega| + |W^1_\omega| + |W^2_\omega|); & \\
\ldots & & \ldots; & & & \\
|W^{D(n)}_\omega(n)| & = & |W^{D(n)}_\omega| & - & (|W^0_\omega| + |W^1_\omega| + \ldots & + |W^{D(n)-n+1}_\omega|);
\end{array}$$

where the last superindex reduces to $D(n) - n + 1 = \lfloor \frac{n-1}{2} \rfloor$. The main result of the section follows.

Theorem 4.3 *The cardinality of the set of vertices of ST_n having weight ω is*

$$|V_\omega(n)| = |W^0_\omega(n)| + |W^1_\omega(n)| + \ldots + |W^{D(n)}_\omega(n)|.$$

Proof. This arises naturally from the patterns in the tables $\mathcal{T}_n$. ∎

5 Weight Distributions of E-Sets in Star Graphs

Given a graph G, an isolated vertex subset of G is said to be an E-set of G if every vertex of $G \backslash C$ is neighbor of exactly one vertex of C. It was proved in [3] that if $1 \le i \le n$, then, the vertex subset C_i of ST_n corresponding to the permutations $\sigma_1\sigma_2\ldots\sigma_n$ having $\sigma_1 = i$ fixed form an E-set. Moreover, this is the only way to get an E-set in ST_n. Clearly, the E-sets of ST_n form a partition of its vertex set.

Having established in Section 5 the distribution of weights of vertices of ST_n, there is still interest on how such a distribution restricts to each C_i. Two cases are left to consider here: $i = 1$ and $i \neq 1$.

Proposition 5.1 *Those vertices u of Λ_n having $\Sigma(u) = \sigma_1\sigma_2\ldots\sigma_n$ with $\sigma_1 = 1$ represent in fact all the vertices of ST_n having $\sigma_1 = 1$. Such vertices of Λ_n have associated admissible strings $i_0i_1\ldots i_{j-1}i_j$ with $i_{j-1} = i_j$.*

Proof. This is clear from the developments above. ∎

Let $V^i_\omega(n)$ be the set of vertices of C_i having weight ω in ST_n, for $1 \le i \le n$.

Theorem 5.1 *The weight distribution of the subsets C_i of ST_{n+1}, for $2 \le i \le n+1$, is:*

$$\begin{array}{ll}
|V^i_0(n+1)| = 0; & \\
|V^i_\omega(n+1)| = |V_{\omega-1}(n)|, & \textit{for } \omega = 1, 2, \ldots, 2\lfloor \frac{D(n+1)}{2} \rfloor; \\
|V_{D(n+1)}(n+1)| = 0, & \textit{for } n \textit{ even,} \\
 & \textit{(only remaining case not covered above).}
\end{array}$$

Proof. For each $i \in \{2, \dots, n+1\}$, the permutations $\sigma_1\sigma_2 \dots \sigma_{n+1}$ with $\sigma_i = i$ induce a copy H_i of ST_n in ST_{n+1} containing the identity permutation $12\dots(n+1)$. Each vertex h of H_i has a unique neighbor h^i in $ST_{n+1} \backslash H_i$. Then the collection of all h^i is C_i, for each $i \in \{2, \dots n+1\}$ fixed. ■

Remark. According to Theorem 6.1, the n vertex subsets C_i in ST_{n+1} with $1 < i \leq n+1$ have equivalent weight distributions. Thus, by multiplying the quantities obtained in Theorem 6.1 by n and substracting the results correspondingly from those obtained in Theorem 5.3 for ST_{n+1}, the restriction of Theorem 5.3 to C_1 can now easily be obtained, because, in addition, if n is odd then $|V_{D(n)}| = (n-2)(n-4)\dots \times 5 \times 3$, by Theorem 4.1. ■

6 Threading Λ_n into a Quotient of ST_n

We now modify the Pruning Algorithm into a Threading Algorithm to produce a quotient Γ_n of ST_n whose vertices are the vertices of Λ_n (remaining after applying the algorithm) and whose edge set contains the edge set of Λ_n. This Threading Algorithm consists in running the Pruning Algorithm (on the previously defined Λ_n), checking whether the last field $b_u \times a_u$ of each line of the table $\mathcal{P}_n$ that is being generated has $b_u = 0$ and $a_u \geq 2$. If this is the case, then a *thread*, or new edge, is added to Λ_n from u to a vertex $\psi(u)$ determined as follows. It happens that the penultimate field $C(u)$ was present in a previous line of $\mathcal{P}_n$ corresponding to the source vertex $\phi(u)$ of a vertical edge $e(u)$ of Λ_n having target vertex $\psi(u)$. Then $\psi(u)$ is the target vertex of $e(u)$.

Example. Starting from $\mathcal{P}_9$, the threads appearing by means of the Threading Algorithm are departing from the vertices u with subindex strings 35, 46, 47, 57, 58, 68, 257, 268, 368, whose values $C(u)$ are respectively 32, 42, 43, 52, 53, 62, 232, 242, 332 and whose fields $b_u \times a_u = 0 \times a_u$ have $a_u = 2, 2, 3, 2, 3, 2, 2, 2, 2$, respectively. But the vertices $\phi(u)$ with respective subindex strings 25, 26, 37, 27, 38, 28, 257, 268, 368, have the same corresponding values $C(u)$, presented in $\mathcal{P}_9$ in nondecreasing order: 23, 24, 34, 25, 35, 26, 223, 224, 233, so the corresponding 1-ics's are the same in both cases. We obtain the desired quotient of ST_9 by adding a thread from each one of the eight mentioned vertices respectively into the vertices $\psi(u)$ whose subindex strings are 255, 266, 377, 277, 388, 288, 2477, 2488, 2588, which are the targets of the respective edges $e(u)$ (that departed from the vertices $\phi(u)$ mentioned above). ■

Theorem 6.1 *Any pair $(u, \phi(u))$ appearing during the running of the Threading Algorithm has the vertices u and $\phi(u)$ with $C(u) = C(\phi(u))$, the order of the elements in the two sides of this equality being irrelevant. Thus, in the running of the Threading Algorithm, each consideration of a vertex u of Λ_n with $C(u)$ equal to the $C(v)$ of a previously considered vertex $v = \phi(u)$ determines a thread from u onto the corresponding $\psi(u)$.*

Proof. The statement follows from the previous discussion and Theorem 3.1, item 1. ■

Remark. The Threading Algorithm insured by Theorem 7.1 produces a quotient graph Γ_n of ST_n whose vertices represent the 1-ics's of the permutations on n elements, that is each vertex of Γ_n represents all the permutations on n elements having a specific 1-ics, and there is a bijective correspondence between the vertices of Λ_n and the 1-ics's of permutations on n elements. Thus Γ_n may be referred to as the 1-ics quotient graph of ST_n. Each edge of ST_n projects into a specific edge of Γ_n. We still consider that the edges of Γ_n are 'horizontal' and 'vertical', as in the case of Λ_n, where threads of Γ_n are 'vertical'. Moreover, the vertices and edges of Γ_n may be considered as preserving the labels they inherit from Λ_n, including the threads, which preserve the labels of the edges removed by the Pruning Algorithm. As said above, the labels of horizontal edges are of the form $\bullet\ell_u$, so we still have that the quotient graphs Γ_n form a nested sequence of labeled digraphs and that their limit labeled digraph Γ_∞ is well defined and constitutes a universal graph for this situation. This corresponds to the infinite star graph ST_∞ that can be defined as the Cayley graph of the symmetric group S_∞ with respect to the set of transpositions $\Sigma_\infty = \{(1\ i),\ i = 2, \ldots n, \ldots\}$. ■

Theorem 6.2 *Γ_n can be interpreted as a quotient graph of ST_n via the quotient map $\Phi_n : ST_n \to \Lambda_n$ given by $\Phi_n^{-1}(u) = \rho$-equivalence class of $\Sigma(u)$, for each vertex u of Λ_n. Then:*

1. *the value $c(u)$ of each vertex u of Γ_n is the cardinality of $\Phi_n^{-1}(u)$ and*
2. *the inverse image Φ_n^{-1} of an horizontal, (vertical), edge e of Λ_n is formed by $c(u^e)$, $(c(u_e))$, edges subdivided into $c(u^e)/m_e$, $(c(u_e)/d_e)$, subsets of m_e, (d_e), edges incident each to a common corresponding vertex in $\Phi_n^{-1}(u_e)$, $(\Phi_n^{-1}(u^e))$.*

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