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Graph Bipartition with Hereditary Properties

by

I. E. Zverovich [1]
RUTCOR–Rutgers Center for Operations Research
Rutgers, The State University of New Jersey
640 Bartholomew Rd.
Piscataway, NJ 08854-8003, U.S.A.
igor@rutcor.rutgers.edu

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ABSTRACT

A hereditary class P is called *finitely generated* if the set of all minimal forbidden induced subgraphs for P is finite. For a pair of hereditary classes P and Q, we define a hereditary class $P * Q$ of all graphs G which have a partition $A \cup B = V(G)$ such that $G(A) \in P$ and $G(B) \in Q$ ($G(X)$ is the subgraph of G induced by $X \subseteq V(G)$). We investigate the problem of recognizing finitely generated classes of the form $P*Q$.

We use the following model. Let H^0 and H^1 be hypergraphs with the same vertex set V. The ordered pair $H = (H^0, H^1)$ is called a *bihypergraph*. A bihypergraph $H = (H^0, H^1)$ is called *bipartite* if there is an ordered partition $V^0 \cup V^1 = V(H)$ such that V^i is stable in H^i, $i = 0, 1$. If the maximum cardinality of hyperedges in H is at most r and every k-subset of $V(H)$ contains at least one hyperedge then $H \in C(k, r)$.

We prove that there exists a finite number of minimal non-bipartite bihypergraphs in $C(k, r)$ (when k and r are fixed).

Let P and Q be hereditary classes of graphs. Suppose that the stability number $\alpha(H)$ is bounded above for all $H \in P$, and the clique number $\omega(H)$ is bounded above for all $H \in Q$. An ordered partition $A \cup B = V(G)$ is called a *ramseian* $P * Q$*-partition* if $G(A) \in P$ and $G(B) \in Q$. Let Ramsey($P * Q$) be the set of all graphs which have a ramseian $P * Q$-partition.

Generalizing a result of Gyárfás [4], we prove that if both P and Q are finitely generated then Ramsey($P*Q$) is also finitely generated. In particular, every class of (α, β)-polar graphs (which are generalizations of split graphs) has a finite forbidden induced subgraph characterization.

1 r-Bounded k-complete bipartite bihypergraphs

Let V be a finite set (possibly, empty) and E be a set of subsets of V (possibly, $\emptyset \in E$). The pair $H = (V, E)$ is called a *hypergraph* with the *vertex set* $V = V(H)$ and the *hyperedge set* $E = E(H)$. The number of vertices $n(H) = |V(H)|$ is the *order* of H and the maximum cardinality among hyperedges is the *rank* $r(H)$. A hypergraph H is called *r-bounded* if $r(H) \leq r$. A set $S \subseteq V(H)$ is *stable* in H if S contains no hyperedges of H.

Definition 1. *A set $T \subseteq V(H)$ which intersects every hyperedge of H is called a* transversal. *A* minimal *transversal does not contain any other transversal. A transversal T is called* t-bounded *if $|T| \leq t$.*

We denote by $MT_t(H)$ the set of all minimal t-bounded transversals in H.

Note that $\emptyset \in MT_t(H)$ if and only if $E(H) = \emptyset$. Also, H has no transversals if and only if $\emptyset \in E(H)$.

Definition 2. *Let H^0 and H^1 be hypergraphs with the same vertex set V. The ordered pair $H = (H^0, H^1)$ is called a* bihypergraph *with the set of* 0-edges $E(H^0)$ *and the set of* 1-edges $E(H^1)$. *Every hyperedge of either H^0 or H^1 we consider as an* hyperedge *of H.*

The *order* of H is $n(H) = |V|$. The *rank* of H is $r(H) = \max\{r(H^0), r(H^1)\}$.

Definition 3.

- *If $r(H) \leq r$ then H is* r-bounded.
- *For a fixed $k \geq 1$ a bihypergraph H is called* k-complete *if every k-subset of $V(H)$ contains at least one hyperedge.*

We denote by $C(k, r)$ the class of all k-complete r-bounded bihypergraphs.

Definition 4. *A bihypergraph $H = (H^0, H^1)$ is called* bipartite *if there exists an ordered partition $V^0 \cup V^1 = V(H)$ (*bipartition*) such that the set V^i is stable in H^i, $i = 0, 1$.*

We denote by Bip *the set of all bipartite bihypergraphs.*

When $H^0 = H^1$ the problem of recognizing the bipartiteness of H coincides with the NP-complete problem [3] of recognizing the bipartiteness of H^0. As follows from Theorem 1 below, there is a polynomial algorithm for recognizing bipartite bihypergraphs in the class $C(k, r)$.

Let G be a hypergraph and $X \subseteq V(G)$. By $G(X)$ we denote the *induced subhypergraph* which has the vertex set X and the hyperedge set $\{e \in E(G) : e \subseteq X\}$. Note that if $X = \emptyset$ then either $E(G(X)) = \emptyset$ (when $\emptyset \notin E(G)$) or $E(G(X)) = \{\emptyset\}$ (when $\emptyset \in E(G)$). Given a bihypergraph $H = (H^0, H^1)$ and X a subset of $V(H)$, the *subhypergraph induced by* X is $H(X) = (H^0(X), H^1(X))$. Let us denote by $\mathrm{ISub}(H)$ the set of all induced subhypergraphs of H.

Definition 5. *A class P of bihypergraphs is* hereditary *if* $\mathrm{ISub}(H) \subseteq P$ *for every* $H \in P$. *Let Z be a set of bihypergraphs. We put*

$$\mathrm{FIS}(Z) = \{H : \mathrm{ISub}(H) \cap Z = \emptyset\}.$$

Clearly, a class P is hereditary if and only if $P = \mathrm{FIS}(Z)$ for a set Z.

Definition 6. *All bihypergraphs in Z are* forbidden induced subhypergraphs *for P. A forbidden induced subhypergraph G for P is a* minimal *if* $\mathrm{ISub(G)}\backslash P = \{G\}$.

Lemma 1. *There is a function $f(r,t)$ with the property that every r-bounded hypergraph H contains a subset $U \subseteq V(H)$ such that $|U| \leq f(r,t)$ and $MT_t(H) = MT_t(H(U))$.*

Proof. Let H is an r-bounded hypergraph. If $\emptyset \in E(H)$ then we put $U = \emptyset$. It is clear that both H and $H(U)$ do not have any transversals ($H(U)$ has $\emptyset$ as an hyperedge). Let $\emptyset \notin E(H)$. If $E(H) = \emptyset$ then we put $U = \emptyset$. Clearly, $H(U)$ has no hyperedges. In that case we have $MT_t(H) = MT_t(H(U)) = \{\emptyset\}$.

Let now $\emptyset \notin E(H)$ and $E(H) \neq \emptyset$. We define sets $S_1, S_2, \dots, S_t$ (S_i will consist of some i-subsets of $V(H)$), $F \subseteq E(H)$, and $T = MT_t(H)$ by the following algorithm.

Algorithm 1.

Step 1. *Choose $e \in E(H)$ and put $S_1 = \{\{x\} : x \in e\}, F = \{e\}$, and $T = \emptyset$.*

Step i $(2 \leq i \leq t+1)$. *For every $X \in S_{i-1}$ we produce as follows:*

a) *if X is a transversal in H, then add to T all minimal transversals of H which are contained in X;*

b) *if X is not a transversal in H, then choose $e \in E(H)$ which does not intersect X, add e to F, and, when $|X| = i-1 < t$, add $X \cup \{x\}$ to S_i for every $x \in e$.*

Now we show that this algorithm gives $T = MT_t(H)$.

The inclusion $T \subseteq MT_t(H)$ is obvious. Let $X \in MT_t(H)$. We denote by Y a subset of X which is contained in some S_l and has the maximum cardinality among all such subsets of X. Since $Y \in S_l$, $|Y| = l$. If $Y \neq X$ then Y is not a transversal in H (by the minimality of X) and $l = |Y| < |X| \leq t$. So on the step $l+1$ the algorithm performs b) and choose $e \in E(H)$ which does not intersect Y. Since X is a transversal in H, there is $x \in X \cap e$. So the set S_{l+1} contains $Y \cup \{x\}$, a contradiction with the choice of Y. Hence $Y = X$. Since $X \in MT_t(H) \cup S_l$, in Step $(l+1)$ the algorithm performs a) and add X to T. Thus, $MT_t(H) \subseteq T$.

We put

$$U = \bigcup_{e \in F} e.$$

Claim 1. *Every t-bounded transversal X in $H(U)$ is a transversal in H.*

Proof. Let m be the largest integer such that X contains some subset $Y \in S_m$. The existence of Y follows from Step 1. By the definition of S_m, $|Y| = m$. Clearly, $m \le |X| \le t$.

If Y is not a transversal in H, then on the step $m+1$ a hyperedge e which does not intersect Y is included in F. Since X is a transversal of $H(U)$, there exists $x \in X \cap e$. According to b) (Step $(m+1)$) $Y \cup \{x\}$ is included to S_m, a contradiction to the choice of Y. □

Now we prove that $MT_t(H) = MT_t(H(U))$.

Let $X \in MT_t(H)$. Since $MT_t(H) = T$ and, by the algorithm, every vertex $x \in X$ belongs to some hyperedge $e \in F$, then $x \in U$ and $X \subseteq U$. We show that $X \in MT_t(H(U))$. Obviously, X is a t-bounded transversal in $H(U)$. We prove the minimality of X. If X contains a transversal $Y \in MT_t(H(U))$ then, by Claim 1, Y is a transversal in H. By the minimality of X in H, $Y = X$. The minimality of X in $H(U)$ is proved. Thus, $MT_t(H) \subseteq MT_t(H(U))$.

To prove the reverse inclusion we consider now a minimal t-bounded transversal X in $H(U)$. By Claim 1, X is a transversal in H. The minimality of X in $H(U)$ implies the minimality of X in H. We obtain $X \in MT_t(H)$. Thus, $MT_t(H) = MT_t(H(U))$.

Let us estimate the cardinality of the set U.

After step 1, we have $s_1 = |S_1| \le r$ and $f_1 = |F| = 1$. After step i, we denote $s_i = |S_i|$ and $f_i = |F|$. By the algorithm, $s_{i+1} \le rs_i$, so $s_{i+1} \le r^{i+1}$ and

$$f_{i+1} \le f_i + s_i \le f_1 + (s_1 + s_2 + \cdots + s_i) \le 1 + r + r^2 + \cdots + r^i.$$

We obtain $|U| \le rf_{t+1} \le r + r^2 + \cdots + r^{t+1}$. □

Theorem 1. *Let $k \ge 1$, $r \ge 1$ and $\Psi = \mathrm{FIS}(Z_\Psi) \subseteq C(k,r)$ be a hereditary class of bihypergraphs. Then there is a finite set $Z(\Psi,k,r) \subseteq \Psi$ of bihypergraphs such that*

$$\Psi \cap \mathrm{Bip} = \mathrm{FIS}(Z_\Psi \cup Z(\Psi,k,r)).$$

Proof. Since both Ψ and Bip are hereditary classes, $\Psi \cap \mathrm{Bip}$ is also a hereditary class. Let $G = (G^0, G^1)$ be a minimal forbidden induced subhypergraph for $\Psi \cap \mathrm{Bip}$.

If $G \notin \Psi = \mathrm{FIS}(Z_\Psi)$ then $G \in Z_\Psi$ (by the minimality of G).

Let $G \in \Psi$. Since $G \notin \Psi \cap \mathrm{Bip}$, $G \notin \mathrm{Bip}$. By the minimality, $\mathrm{ISub}(G) \backslash \{G\} \subseteq \Psi \cap \mathrm{Bip}$.

Hence

$$G \in Z(\Psi,k,r) \quad \text{and} \quad \Psi \cap \mathrm{Bip} = \mathrm{FIS}(Z_\Psi \cup Z(\Psi,k,r)),$$

where

$$Z(\Psi,k,r) = \{G \in \Psi : \mathrm{ISub}(G) \backslash \mathrm{Bip} = \{G\}\}$$

is the set of all minimal non-bipartite bihypergraphs in Ψ.

To show that $Z(\Psi,k,r)$ is finite it is sufficient to prove that $n(G)$ is bounded above by a function of k and r.

Since $G \in Z(\Psi,k,r) \subseteq \Psi$ and $\Psi \subseteq C(k,r)$, $G \in C(k,r)$.

Let us fix a vertex $u \in V(G)$. By the minimality, $G-u \in \mathrm{Bip}$, i.e., there exists a bipartition of $V(G-u)$ into parts V^0 and V^1. (The bihypergraph $G-u$ is produced from G by deleting u with all incident hyperedges).

Claim 2. *Let $G' \in \mathrm{ISub}(G)$ and $W^0 \cup W^1 = V(G')$ is a bipartition of G' (W^i is stable in G^i, $i = 0,1$). Then both $A^0 = V^0 \cap W^1$ and $A^1 = V^1 \cap W^0$ contains at most $k-1$ vertices.*

Proof. Since $A^0 \subseteq V^0$, $E(G^0(A^0)) = \emptyset$. Since $A^0 \subseteq W^1$, $E(G^1(A^0)) = \emptyset$. Hence A^0 does not contain hyperedges of G. It follows from $G \in C(k,r)$ that $|A^0| < k$. Similarly, $|A^1| < k$. □

Firstly we give an informal description of the construction below. We consider two possible variants for adding u to the partition $V^0 \cup V^1$:

$$((V^0 \cup A^1 \cup \{u\})\backslash A^0) \cup ((V^1 \cup A^0)\backslash A^1) \tag{1}$$

and

$$((V^0 \cup A^1)\backslash A^0) \cup ((V^1 \cup A^0 \cup \{u\})\backslash A^1), \tag{2}$$

where $A^0 \subseteq V^0$ and $A^1 \subseteq V^1$.

It follows from $G \notin \mathrm{Bip}$ that neither (1) nor (2) is a bipartition of G. We will construct a subhypergraph $G' \in \mathrm{ISub}(G)$ of bounded order such that $u \in V(G')$ and neither (1) nor (2) induces a bipartition of G' for all $A^0 \subseteq V^0$ and $A^1 \subseteq V^1$. By Claim 2, we may assume that $|A^0| < k$ and $|A^1| < k$.

Firstly we consider the case (1), i.e. we try to add u to V^0. Since G is not bipartite, adding u to V^0 produces at least one hyperedge of G^0 in a hypergraph $F^0 = G^0(V^0 \cup \{u\})$. Then some minimal transversal $X \subseteq V^0$ distinct from $\{u\}$ in F^0 must be transferred to V^1. If $|X| < k$ then we include X into A^0 (the set A^0 depends on X). By Claim 2, we may omit the case when $|X| \geq k$ (otherwise $|A^0| \geq |X| \geq k$). The set $(V^0\backslash X) \cup \{u\}$ is stable in G^0.

Further, let $F^1 = G^1(V^1 \cup X)$. If F^1 has no transversal $Y \subseteq V^1$ for every transversal X ($|X| < k$), then u cannot be added to V^0. Otherwise we include a transversal Y into A^1 (the set A^1 depends on X and Y). By Claim 2, we may assume that $|Y| < k$ (otherwise $|A^1| \geq |Y| \geq k$). The set $(V^1 \cup X)\backslash Y$ is stable in G^1. Then we add A^1 to $(V^0 \cup \{u\})\backslash A^0$ and so on.

Now we begin a formal description. To consider all possible variants for A^0 and A^1 we define the rooted tree T^0 recursively. The vertices of T^0 are some non-empty subsets of $V(G)$.

- The root of T^0 is $R_0 = \{u\}$. We will consider the root as a unique vertex of the *level* 0.
- Every vertex of the *level* i in T^0 is on the distance i from the root.
- When $i \geq 1$ is even (respectively, odd) every vertex of the level i is a subset of V^1 (respectively, V^0).
- Suppose that for a fixed $i \geq 0$ all vertices of the level i are already defined. We will define the level $i+1$. For every path

  $$P = (R_0 = \{u\}, R_1, R_2, \dots , R_i)$$

  in T^0 (where each R_j is a vertex of level j) we put

  $$A^0 = \cup\{R_i : i \geq 1 \ \text{ is } \ \text{odd}\} \subseteq V^0$$

  and

  $$A^1 = \cup\{R_i : i \geq 2 \ \text{ is } \ \text{even}\} \subseteq V^1.$$

  When $i = 0$ (respectively, $i = 1$) we put $A^0 = A^1 = \emptyset$ (respectively, $A^1 = \emptyset$).

  The following definitions of a hypergraph $F = F_P$ and an integer t depend on P and the parity of i.

  Definition 7. *If $i \geq 0$ is even then $V(F) = V^0 \backslash A^0$ and*

  $$E(F) = \{e\backslash(A^1 \cup \{u\}) : e \in E(G^0), e \subseteq (V^0\backslash A^0) \cup (A^1 \cup \{u\})\}.$$

  Put $t = k - |A^0| - 1$.

  If $i \geq 1$ is odd then $V(F) = V^1\backslash A^1$ and

  $$E(F) = \{e\backslash A^0 : e \in E(G^1), e \subseteq (V^1\backslash A^1) \cup A^0\}.$$

  Put $t = k - |A^1| - 1$.

- Since G is r-bounded, F is also r-bounded. By Lemma 1, there is a function $f(r, t)$ and a subset $U_P \subseteq V(F)$ such that $|U_P| \leq f(r, t)$ and $MT_t(F) = MT_t(F(U_P))$. In particular, the number of minimal t-bounded transversals in F is bounded above by a function $\phi(k, r)$ (because $t \leq k$).
- If $MT_t(F) = \emptyset$ (i.e., $\emptyset \in E(F)$) then R_i is defined as a pendant vertex of T^0. Otherwise all elements of $MT_t(F) = MT_t(F(U_P))$ are included to the level $i + 1$ as vertices of T^0 and all of them are adjacent to R_i.
- The construction of T^0 is finished when we obtain an empty level.

Note that every new vertex $R_{i+1} \in MT_t(F)$ in T^0 satisfies $R_{i+1} \cap A^0 = R_{i+1} \cap A^1 = \emptyset$, so $R_0, R_1, \dots$ are pairwise disjoint (for a fixed path P).

We show that $R_{i+1} \neq \emptyset$ for every vertex R_{i+1} of T^0. Suppose that R_{i+1} is obtained from the path P above. As G is not bipartite, (1) is not a bipartition of $V(G)$. Hence $E(F) \neq \emptyset$. If $\emptyset \in E(F)$, then $MT_t(F) = \emptyset$ and R_{i+1} is not defined as a vertex of T^0. Otherwise every minimal t-bounded transversal in F is not empty (regardless of the particular value of t). In particular, $R_{i+1} \neq \emptyset$.

Show that the level $2k$ of the tree T^0 does not contain any vertices. Indeed, consider a path $P = (R_0 = \{u\}, R_1, R_2, \dots, R_{2k-1})$ in T^0. Since $R_j \neq \emptyset$ for all $j = 0, 1, \dots, 2k-1$, we have $|A^0| \geq k$. Since $A^0 \subseteq V^0$, V^0 is stable in G^0 and G is k-complete, there exists a hyperedge $e \in E(G^1)$ such that $e \subseteq A^0$. It follows that the hypergraph $F = F_P$ has $\emptyset = e \backslash A^0$ as a hyperedge. Hence, F has no transversals and T^0 has no vertices on the level $2k$.

Further, every vertex in T^0 of the level i is adjacent to at most $\phi(k, r)$ vertices of the level $i+1$. Since the number of all levels is at most $2k$ (including the level 0), the order of T^0 is bounded above by a function of k and r. Hence the number of all paths in T^0 connecting the root with a vertex of T^0 is bounded above by a function of k and r. Finally, the union S^0 of all sets U_P constructed for every path $P = (R_0, R_1, \dots)$ in T^0 has cardinality which is bounded above by a function $g(k, r)$.

Claim 3. *The bihypergraph $G(\{u\} \cup S^0)$ has no bipartition of the form $(W^0 \cup \{u\}) \cup W^1$, where $W^0 \cup \{u\}$ is stable in G^0 and W^1 is stable in G^1.*

Proof. Suppose that $G(\{u\} \cup S^0)$ has a bipartition $(W^0 \cup \{u\}) \cup W^1$. We denote

- $B^0 = V^0 \cap W^1$,
- $B^1 = V^1 \cap W^0$,
- $C^0 = V^0 \cap W^0$,
- $C^1 = V^1 \cap W^1$.

We have $W^0 = B^1 \cup C^0$ and $W^1 = B^0 \cup C^1$. By Claim 2, $|B^0| < k$ and $|B^1| < k$.

We construct a sequence of pairwise disjoint sets

$$R_0, R_1, \dots, R_{2k-1}$$

of S^0. Put $R_0 = \{u\}$, $P_0 = (R_0)$ and $A^0 = A^1 = \emptyset$. We will use the notation $F = F_P$ and U_P defined above.

The set $R_0 \cup V^0$ is not stable in G^0 (since G is not bipartite). Hence the hypergraph $F_0 = F_{P_0}$ has an hyperedge. By the construction of T^0, S^0 contains the set $U_0 = U_{P_0}$ such that $U_0 \subseteq V(F_0) = V^0$ and $MT_{k-1}(F_0) = MT_{k-1}(F_0(U_0))$. It follows from $E(F_0) \neq \emptyset$ that $MT_{k-1}(F_0) \neq \{\emptyset\}$. So a minimal $(k-1)$-bounded non-empty transversal R_1 of $F_0(U_0)$ is contained in B^0 (otherwise $W^0 \cup \{u\}$ is not stable in G^0).

Suppose that a sequence $P_i = (R_0, R_1, \dots, R_i)$, $i \geq 1$, of pairwise disjoint non-empty sets R_j is already defined and

- $A^0 = \cup\{R_j : j \geq 1 \text{ is odd}\} \subseteq B^0$;
- $A^1 = \cup\{R_j : j \geq 2 \text{ is even}\} \subseteq B^1$;
- for each $j = 1, 2, \ldots, i$, the set $R_j \subseteq U_j = U_{P_j}$ is a minimal transversal of $F_{j-1} = F(P_{j-1})$;
- when $i \geq 1$ is odd, the set $(V^0 \cup A^1 \cup \{u\})\backslash A^0$ is stable in G^0;
- when $i \geq 2$ is even, the set $(V^1 \cup A^0)\backslash A^1$ is stable in G^1.

If i is odd then $(V^1 \cup A^0)\backslash A^1$ is not stable in G^1 (since G is not bipartite). So $F_i = F_{P_i}$ contains a hyperedge. By the construction of T^0, $MT_t(F_i) = MT_t(F_i(U_i))$, where $t = k - |A^1| - 1$ (since $|B^1| < k$) and $U_i = U_{P_i} \subseteq V(F_i)$. It follows from $E(F_i) \neq \emptyset$ that there is a minimal non-empty t-bounded transversal $R_{i+1} \subseteq U_i$ in B^1 (otherwise W^1 is not stable in G^1).

When i is even, we define $R_{i+1} \subseteq B^0$ in a similar way.

The existence of P_{2k-1} leads to a contradiction: B^0 contains k disjoint non-empty sets $R_1, R_3, \ldots, R_{2j-1}, \ldots, R_{2k-1}$ and $|B^0| < k$. □

In a symmetric way we define a rooted tree T^1 changing V^0 (respectively, V^1) by V^1 (respectively, V^0) and we find the corresponding set S^1 such that $|S^1| \leq g(k, r)$ and $G(\{u\} \cup S^1)$ has no bipartition of the form $W^0 \cup (W^1 \cup \{u\})$ (an analogue of Claim 3).

It follows that $G' = G(\{u\} \cup S^0 \cup S^1) \notin \text{Bip}$. By the minimality of G, we have $G = G'$ and $n(G) \leq 1 + 2g(k, r)$. □

Corollary 1. *For fixed k and r the set of all minimal non-bipartite bihypergraphs in $C(k, r)$ is finite.*

2 Ramseian partitions of graphs

The set of all induced subgraphs of a graph G we denote by $\mathrm{ISub}(G)$. A class P of graphs is called *hereditary* if $\mathrm{ISub}(G) \subseteq P$ for every $G \in P$. Let Z be a set of graphs. We put

$$\mathrm{FIS}(Z) = \{G : \mathrm{ISub}(G) \cap Z = \emptyset\}.$$

If $P = \mathrm{FIS}(Z)$ then Z is a set of *forbidden induced subgraphs* for P. A graph H is called *minimal forbidden induced subgraph* for a hereditary class P if $H \notin P$ and $\mathrm{ISub}(H) \backslash \{H\} \subseteq P$.

Definition 8. *We denote by Z_P the set of all minimal forbidden induced subgraphs for a hereditary class P. It is well-known that $P = \mathrm{FIS}(Z_P)$. A hereditary class P is called* finitely generated *if the set Z_P is finite.*

The *stability number* $\alpha(G)$ of a graph G is the maximum cardinality of stable sets in G. The *clique number* $\omega(G)$ of a graph G is the maximum order of complete subgraphs in G. A class P of graphs is called *α-bounded* (respectively, *ω-bounded*) if there is a constant c such that $\alpha(G) \le c$ (respectively, $\omega(G) \le c$) for every $G \in P$.

Definition 9. *Let P and Q be classes of graphs. An ordered partition $A \cup B = V(G)$ is called a* (P, Q)-partition *of a graph G if $G(A) \in P$ and $G(B) \in Q$.*

We denote by $P * Q$ the class of all graphs having a (P, Q)-partition.

Definition 10. *Let P be an α-bounded hereditary class and Q be a ω-bounded hereditary class. Then a (P, Q)-partition will be called* Ramseian.

We denote by $\mathrm{Ramsey}(P * Q)$ *the set of all graphs which have a Ramseian (P, Q)-partition.*

Theorem 2. *Let P be α-bounded hereditary class and Q is ω-bounded hereditary class. If both P and Q are finitely generated, then the class* $\mathrm{Ramsey}(P * Q)$ *is also finitely generated.*

Proof. Denote by Z_P (respectively, Z_Q) the set of all minimal forbidden induced subgraphs for P (respectively, for Q). Let G be an arbitrary graph. We define a bihypergraph $\psi(G) = H = (H^0, H^1)$ in the following way: $V(H) = V(G) = V$,

$$E(H^0) = \{X \subseteq V : G(X) \in Z_P\}$$

and

$$E(H^1) = \{X \subseteq V : G(X) \in Z_Q\}.$$

Let $\Psi = \{\psi(G) : G \text{ is a graph}\}$. Clearly, Ψ is a hereditary class of bihypergraphs; $\Psi = \mathrm{FIS}(Z_\Psi)$. Since P and Q are both finitely generated, there is

$$r = \max\{|V(G)| : G \in Z_P \cup Z_Q\}.$$

So every bihypergraph $H \in \Psi$ is r-bounded.

Let $\alpha(G) < n$ for every $G \in P$ and $\omega(G) < m$ for every $G \in Q$. We denote by k the Ramsey number $r(m,n)$. Then every graph of order k contains either a complete subgraph K_m or an stable set of the cardinality n. Hence either $G(X) \notin P$ or $G(X) \notin Q$ for every graph G and every k-subset $X \subseteq V(G)$, i.e., X contains a hyperedge of $\psi(G)$. So Ψ consists of k-complete bihypergraphs only. Thus, $\Psi \subseteq C(k,r)$.

By Theorem 1, there exists a finite set $Z(\Psi,k,r) \subseteq \Psi$ such that

$$\Psi \cap \mathrm{Bip} = \mathrm{FIS}(Z_\Psi \cup Z(\Psi,k,r)).$$

We denote by Z_{PQ} the set of all minimal forbidden induced subgraphs for Ramsey$(P * Q)$.

Claim 4. *If $G \in Z_{PQ}$ then $\psi(G) \in Z(\Psi,k,r)$.*

Proof. Since $G \notin \mathrm{Ramsey}(P*Q)$, $\psi(G) \notin \mathrm{Bip}$.

For every $F \in \mathrm{ISub}(\psi(G))\backslash\{\psi(G)\}$ we have $F = \psi(H)$, where $H = G(V(F))$. By minimality of G, $H \in \mathrm{Ramsey}(P*Q)$. So $F = \psi(H) \in \mathrm{Bip}$.

Since $F = \psi(H) \in \Psi \cap \mathrm{Bip}$ and $\psi(G) \notin \Psi \cap \mathrm{Bip}$, then $\psi(G)$ is a minimal forbidden induced subhypergraph for$\Psi\cap$Bip. It follows that $\psi(G) \in Z_\Psi \cup Z(\Psi,k,r)$. We have $\Psi = \mathrm{FIS}(Z_\Psi)$ and $Z_\Psi \cap \Psi = \emptyset$. Since $\psi(G) \in \Psi$, $\psi(G) \notin Z_\Psi$. Thus, $\psi(G) \in Z(\Psi,k,r)$. ☐

The set $\psi^{-1}(H) = \{G : \psi(G) = H\}$ is finite for every bihypergraph H. Then the set

$$\psi^{-1}(Z(\Psi,k,r)) = \bigcup_{H \in Z(\Psi,k,r)} \psi^{-1}(H)$$

is finite. By Claim 4, $Z_{PQ} \subseteq \psi^{-1}(Z(\Psi,k,r))$. Thus, the set Z_{PQ} is finite, i.e. Ramsey$(P*Q)$ is a finitely generated class. ☐

As an example we consider the classes of (α,β)-*polar* graphs.

The complete graph, the edgeless graph, the path, and the cycle of order n is denoted by K_n, O_n, P_n and C_n respectively. The union $G_1 \cup G_2 \cup \cdots \cup G_k$ of graphs $G_1, G_2, \ldots, G_k$ we consider as disjoint (i.e., $V(G_i) \cap V(G_j) = \emptyset$ for $1 \le i \ne j \le k$). The complementary graph for G is denoted by $\overline{G}$.

Definition 11 (Tyshkevich and Chernyak [5]). *$G \in (\alpha,\beta)$ if and only if there exists a partition $A \cup B = V(G)$ such that*

- $G(A) \in \mathrm{FIS}(K_1 \cup K_2; O_{\alpha+1})$ *(the complete multipartite graphs having the cardinality of parts at most α) and*
- $G(B) \in \mathrm{FIS}(P_3; K_{\beta+1})$ *(disjoint union of complete graphs of orders at most β).*

In particular, the class of all *split* graphs coincides with the class of $(1,1)$-polar graphs and it can be characterized by three forbidden induced subgraphs, namely $2K_2, C_4$ and C_5 (Földes and Hammer [1]). Recently Gagarin and Metelsky [2] proved that the class of all (1, 2)-polar graphs has a finite FIS-characterization (18 minimal forbidden induced subgraphs). Theorem 2 implies the existence of a finite FIS-characterization for every class of (α, β)-polar graphs (α and β are finite). Note that (α, β)-polar graphs are defined also when α is infinite and/or β is infinite [5].

Since each class of (α, β)-polar graphs is defined as a class of type $\mathrm{Ramsey}(P*Q)$, we obtain

Corollary 2. *For every (finite) $\alpha \geq 1$ and $\beta \geq 1$ the class of all (α, β)-polar graphs has a finite forbidden induced subgraph characterization.*

Problem 1. *Let $P = \mathrm{FIS}(O_2)$ (complete graphs) and $Q = \mathrm{FIS}(K_3)$ (triangle-free graphs). Find a forbidden induced subgraph characterization for the class $\mathrm{Ramsey}(P*Q)$,*

V. Zverovich used his software Graph Navigator to find 350 minimal forbidden induced subgraphs for this class. The maximum order among them is 10.

3 Families of hereditary classes of graphs

Conjecture 1. *Let P and Q be hereditary classes (distinct from the class of all graphs). The class of all graphs having (P, Q)-partition is finitely generated if and only if*

either **(i)** *P is α-bounded finitely generated class and Q is ω-bounded finitely generated class (or vice versa);*

or **(ii)** *P is a finite class and Q is a finitely generated class (or vice versa).*

We obtain results supporting this conjecture.

Definition 12. *We define eight families of hereditary classes $J(\alpha, \omega)$, where $J \in \{F, I\}$ and $\alpha, \omega \in \{c, \infty\}$ in the following way:*

- *if $J = F$ (respectively, $J = I$) then every class in $J(\alpha, \omega)$ is (respectively, is not) finitely generated;*
- *if $\alpha = c$ (respectively, $\alpha = \infty$) then every class in $J(\alpha, \omega)$ is (respectively, is not) α-bounded;*
- *if $\omega = c$ (respectively, $\omega = \infty$) then every class in $J(\alpha, \omega)$ is (respectively, is not) ω-bounded.*

[It is supposed that every family $J(\alpha, \omega)$ contains all hereditary classes satisfying these conditions.]

Thus, we partite all hereditary classes into eight families:

- $F(c, c)$;
- $F(c, \infty)$;
- $F(\infty, c)$;
- $F(\infty, \infty)$;
- $I(c, c)$;
- $I(c, \infty)$;
- $I(\infty, c)$; and
- $I(\infty, \infty)$.

Proposition 1. *$I(c,c)=\emptyset$.*

Proof. By Ramsey's theorem, every class $P \in I(c,c)$ is finite. Let us denote $N = \max\{|V(G)| : G \in P\}$. We put $Z = Z_1 \cup Z_2$, where

$$Z_1 = \{H : |V(H)| \leq N \text{ and } H \notin P\}$$

and Z_2 is the set of all graphs of the order $N+1$. Clearly, the set Z is finite and $Z_P \subseteq P$. It contradicts to the fact that $P \in I(c,c)$ is not finitely generated. $\square$

We will use the following hereditary classes:

- $O = \mathrm{FIS}(K_2)$ is the class of all edgeless graphs;
- $K = \mathrm{FIS}(O_2)$ is the class of all complete graphs;
- $O \cup K = \mathrm{FIS}(P_3, \overline{P}_3)$ is the union of O and K;
- $M = \mathrm{FIS}(\overline{P}_3)$ is the class of all complete multipartite graphs;
- $M_\alpha = \mathrm{FIS}(\overline{P}_3, O_{\alpha+1})$ is the class of all complete multipartite graphs with cardinalities of parts at most α ($\alpha \geq 1$);
- $D = \mathrm{FIS}(P_3)$ is the class of all graphs which are disjoint union of complete graphs;
- $D_\beta = \mathrm{FIS}(P_3, K_{\beta+1})$ is the class of all graphs which are disjoint union of complete graphs of order at most β ($\beta \geq 1$).

We reformulate Theorem 1 as

Theorem 3. *For every hereditary classes $P \in F(c,\omega)$ and $Q \in F(\alpha, c)$, where $\alpha, \omega \in \{c, \infty\}$, the class $P * Q$ is finitely generated.*

This result leads to the following problem.

Problem 2. *Recognize pairs $\{J(\alpha,\omega), J'(\alpha',\omega')\}$ such that for every hereditary classes $P \in J(\alpha,\omega)$ and $Q \in J'(\alpha',\omega')$ the class $P * Q$ is finitely generated.*

A pair $\{J(\alpha,\omega), J'(\alpha',\omega')\}$ is considered as unordered since $P * Q = Q * P$. Theorem 3 gives a positive solution for pairs

$$\{F(c,c), F(c,c)\}, \{F(c,c), F(\infty,c)\}, \{F(c,c), F(c,\infty)\} \text{ and } \{F(c,\infty), F(\infty,c)\}.$$

We will show that the positive solution of Problem 2 there exists also for pair $\{F(c,c), F(\infty,\infty)\}$ only.

Now we consider the pair $\{F(c,c), F(\alpha,\omega)\}$.

Definition 13.

- *A hypergraph H is called* hereditary *if $\emptyset \in E(H)$ and $2^e \subseteq E(H)$ for each edge $e \in E(H)$.*
- *A hypergraph H is called* r-bounded *($r \geq 0$) if $|e| \leq r$ for each $e \in E(H)$.*

We denote by $H(r, s)$, where $r \geq 0$ and $s \geq 0$, the set of all bihypergraphs $H = (H^0, H^1)$ in which H^0 is a hereditary r-bounded hypergraph, while H^1 is a s-bounded hypergraph and $\emptyset \notin E(H^1)$.

Definition 14. *A 0-edge e is called 0-*transversal *of a bihypergraph if $e \cap f \neq \emptyset$ for each 1-edge f. A bihypergraph $H \in H(r, s)$ is called 0-*minimal *if there are no 0-transversals in it, but every induced subbihypergraph $H(X) \neq H$ has 0-transversal.*

Theorem 4. *For any fixed $r \geq 0$ and $s \geq 0$, the set of all 0-minimal bihypergraphs in $H(r, s)$ is finite.*

Proof. Let $H = (H^0, H^1) \in H(r, s)$ and H be a 0-minimal bihypergraph.

If $E^1 = E(H^1) = \emptyset$ then $\emptyset \in E^0 = E(H^0)$ is a 0-transversal in H, a contradiction to the definition of 0-minimal bihypergraph. Hence $E^1 \neq \emptyset$.

We define a sequence $B_0, B_1, \ldots, B_r$ of induced subbihypergraphs in H as follows.

Step 0. Choose an arbitrary 1-edge e in H and put $V_0 = e$ and $B_0 = H(V_0)$.

Step i + 1 ($0 \leq i < r$). Let bihypergraph B_i is already defined. For each 0-edge f in H, we assign a 1-edge f^* which does not intersect f. The existence of f^* follows from the fact that none of 0-edges is a 0-transversal. We put $W_i = \cup f^*$, where the union is taken over all 0-transversals f of the bihypergraph B_i. Further, $V_{i+1} = V_i \cup W_i$ and $B_{i+1} = H(V_{i+1})$.

Let $n_i = |V_i| = |V(B_i)|$. By the construction, $n_{i+1} < n_i + s2^{n_i}$ ($i = 0, 1, \ldots, r-1$) since the number of 0-transversal in B_i is less than 2^{n_i} and $|f^*| \leq s$ for each 1-edge f^*. It follows from $n_0 \leq r$ that n_r is bounded above by a function $f(r, s)$.

We show that the bihypergraph B_r has no 0-transversals. Let e be an arbitrary 0-transversal in B_r. Denote by $e_i = e \cap V_i$ for $i = 0, 1, \ldots, r$. By hereditariness, $e_i \in E^0$ and so e_i is a 0-edge of B_i.

We use induction on i to prove that $|e_i| \geq i + 1$, $i = 0, 1, \ldots, r$. The basis of induction: since $V_0 = V(B_0)$ is an 1-edge and e is 0-transversal, $e \cap V_0 \neq \emptyset$, i.e., $|e_0| \geq 1$.

Now suppose that $|e_i| \geq i + 1$, where $i \in \{0, 1, \ldots, r-1\}$ and show that

$$|e_{i+1}| \geq i + 2.$$

Since e is a 0-transversal in B_r and B_i is a subbihypergraph in B_r, e_i is a 0-transversal in B_i. By the definition, B_{i+1} contains a 1-edge e_i^* which does not intersect e_i. Since e_i is a 0-transversal in B_r, $e \cap e_i^* \neq \emptyset$. Therefore $|e_{i+1}| \geq |e_i| + |e \cap e_i| \geq i + 2$.

In particular, $|e_r| = |e| \geq r + 1$, a contradiction to the condition that H^0 is r-bounded.

This contradiction shows that H_r has no transversals. By 0-minimality, H is a subbihypergraph in H_r. It follows that $|V(H)| \leq f(r, s)$. □

Corollary 3. *If a hereditary class P is a finitely generated and a hereditary class Q is a finitely generated, than the class $P * Q$ is also finitely generated.*

Proof. Let $Z_Q = \{G_1, G_2, \ldots, G_k\}$. We assign a bihypergraph $H = (H^0, H^1)$ to every graph G: H has the vertex set $V = V(G)$, the set of 0-edges

$$E^0 = \{X \subseteq V : X \neq \emptyset \text{ and } G(X) \in P\} \cup \{\emptyset\}$$

and the set of 1-edges

$$E^1 = \{X \subseteq V : X \neq \emptyset \text{ and } G(X) \in Z_Q\}.$$

Let $r = \max\{|V(G)| : G \in P\}$. Since P is hereditary and finite, H^0 is hereditary and r-bounded.

Since Z_Q is a finite set, H^1 is s-bounded, where $s = \max\{|V(G)| : G \in Z_Q\}$. Thus, $H \in H(r, s)$ for any graph G.

If G is a forbidden induced subgraph for $P * Q$ then H has no 0-transversals. If G is a minimal forbidden induced subgraph for $P * Q$ then H is 0-minimal.

By Theorem 4, the number of 0-minimal bihypergraphs in $H(r, s)$ is finite. Besides, there are finitely many minimal forbidden induced subgraphs for $P * Q$ with the same assigned bihypergrah. □

Corollary 4. *If $P \in F(c, c)$ and $Q \in F(\alpha, \omega)$, where $\alpha, \omega \in \{c, \infty\}$, then the class $P * Q$ is finitely generated.*

Proof. By Ramsey's theorem, a class $P \in F(c, c)$ is finite and the result follows from Corollary 3 □

4 Construction of infinite series of minimal forbidden induced subgraphs

We will show that Problem 2 for a pair $\{J(\alpha,\omega), J'(\alpha',\omega')\}$ has a negative solution except cases which are covered by Theorem 3 and Corollary 4. In other words, we show that there exist hereditary classes $P \in J(\alpha,\omega)$ and $Q \in J'(\alpha',\omega')$ such that the class $P * Q$ is not finitely generated.

The number of cases may be reduced by excluding the *complementary* class $\overline{P} = \{\overline{G} : G \in P\}$. A class $P * Q$ is finitely generated if and only if the class $\overline{P * Q} = \overline{P} * \overline{Q}$ is finitely generated. Clearly that if $P \in J(\alpha,\omega)$ then $\overline{P} \in J(\omega,\alpha)$. So we will consider only one of two pairs $\{J(\alpha,\omega), J'(\alpha',\omega')\}$ and $\{J(\omega,\alpha), J'(\omega',\alpha')\}$.

Case 1: $\{F(\infty,c), F(\infty,c)\}$
(the complementary case is $\{F(c,\infty), F(c,\infty)\}$)

We choose the class $O = \mathrm{FIS}(K_2) \in F(\infty,c)$. By König's theorem, the class $O * O$ of all bipartite graphs is $B = \mathrm{FIS}(C_{2n+1} : n \geq 1)$ and it is not finitely generated.

Case 2: $\{F(c,\infty), F(\infty,\infty)\}$
(the complementary case is $\{F(\infty,c), F(\infty,\infty)\}$)

We choose classes $K = \mathrm{FIS}(O_2) \in F(c,\infty)$ and $D = \mathrm{FIS}(P_3) \in F(\infty,\infty)$.

Proposition 2. *The class $K * D$ of all $(1,\infty)$-polar graphs is not finitely generated.*

Proof. For any $n \geq 2$, we prove that the graph $G = \overline{C}_{2n+1}$ is a minimal forbidden induced subgraph for $K * D$.

Suppose that $G \in K * D$, i.e., there exists a partition $A \cup B = V(G)$ such that $G(A) \in K$ and $G(B) \in D$. Let $(v_0, v_1, \ldots, v_{2n})$ be the cyclic ordering of vertices of $\overline{G} = C_{2n+1}$. We will refer to vertices v_i and v_{i+1} $(i = 0, 1, \ldots, 2n-1)$ as well as vertices v_{2n} and v_0 as *neighbors*.

Since A induces a complete subgraph and neighbors are non-adjacent, A contains at most one vertex of each pair of neighbors. It follows that there are neighbors in B. By the symmetry, we may assume that $v_0, v_1 \in B$. Since $\{v_0, v_1, v_3\}$ induces P_3, $v_3 \notin B$. So $v_3 \in A$. Then $v_2, v_4 \in B$. We see that $\{v_1, v_2, v_4\} \subseteq B$ induces P_3, a contradiction to $G(B) \in D = \mathrm{FIS}(P_3)$. Thus, $G \notin K * D$.

Now we prove the minimality of G. Each induced subgraph $H \neq G$ is the complement of a bipartite graph. Hence there is a partition $A \cup B = V(H)$ such that both A and B induce complete subgraphs. It follows from $K \subseteq D$ that $H \in K * D$. □

Case 3: $\{F(\infty,\infty), F(\infty,\infty)\}$

We consider the class $O \cup K = \mathrm{FIS}(P_3, \overline{P}_3) \in F(\infty,\infty)$.

Proposition 3. *The class $(O \cup K) * (O \cup K)$ is not finitely generated.*

Proof. We show that $G = C_{2n+1}$, $n \geq 2$, is a minimal forbidden induced subgraph for $(O \cup K) * (O \cup K)$.

Suppose that $G \in (O \cup K) * (O \cup K)$, i.e., there exists a partition $A \cup B = V(G)$ such that $G(A), G(B) \in (O \cup K) * (O \cup K)$.

Let $(v_0, v_1, \dots, v_{2n})$ be the cyclic ordering of vertices in G. Since the number of all vertices is odd, there are two neighbors in the same class. By the symmetry, we may assume that $v_0, v_1 \in A$. Then A induces a complete subgraph. Hence $v_2, v_3, v_4 \in B$. The set $\{v_2, v_3, v_4\}$ induces P_3, a contradiction to $G(B) \in \mathrm{FIS}(P_3, \overline{P}_3)$. $G \notin (O \cup K) * (O \cup K)$.

Now we prove minimality of G. Any induced subgraph $H \neq G$ is bipartite. So there is a partition $A \cup B = V(H)$ such that both A and B induce edgeless subgraphs. Since $O \subseteq O \cup K$, $H \in (O \cup K) * (O \cup K)$. $\square$

Case 4: $\{F(c, c), I(\alpha, \omega)\}$, where $\alpha, \omega \in \{c, \infty\}$

We consider the following classes:

- $P = \mathrm{FIS}(K_1) = \emptyset \in F(c, c)$ and
- $Q_1 = \mathrm{FIS}(C_n : n \geq 3) \in I(\infty, c)$ (for $\alpha = \infty$, $\omega = c$);
- $Q_2 = \mathrm{FIS}(\overline{C}_n : n \geq 3) \in I(c, \infty)$ (for $\alpha = c$, $\omega = \infty$);
- $Q_3 = \mathrm{FIS}(C_n : n \geq 4) \in I(\infty, \infty)$ (for $\alpha = \omega = \infty$)

(by Proposition 1, $I(c, c) = \emptyset$).

Clearly, for each of Q_i above, the class $P * Q_i = Q_i$ is not finitely generated.

Case 5: $\{F(\infty, c), I(\infty, c)\}$
(the complementary case is $\{F(c, \infty), I(c, \infty)\}$)

We consider the class $O = \mathrm{FIS}(K_2) \in F(\infty, c)$ of all edgeless graphs and $B = \mathrm{FIS}(C_{2n+1} : n \geq 1) \in I(\infty, c)$ of all bipartite graphs.

Proposition 4. *The class $O * B$ of all 3-colorable graphs is not finitely generated.*

Proof. For any $n \geq 1$, we construct a graph G_n as follows. Let C_{2n+1} be a cycle with the vertex set $(v_0, v_1, \dots, v_{2n})$ (in the cyclic order). We add new vertices u_{2i-1} $(i = 1, 2, \dots, n)$ along with edges $v_{2i-2}u_{2i-1}$, $v_{2i-1}u_{2i-1}$ and $v_{2i}u_{2i-1}$.

It is not hard to check that G_n is a minimal forbidden induced subgraph for $O * B$. $\square$

Case 6: $\{F(c, \infty), I(\infty, c)\}$

noindent (the complementary case is $\{F(\infty, c), I(c, \infty)\}$)

We choose the classes $K = \mathrm{FIS}(O_2) \in F(c, \infty)$ of all complete graphs and $AC = \mathrm{FIS}(C_n : n \geq 3) \in I(\infty, c)$ of all acyclic graphs.

Proposition 5. *The class $K * AC$ is not finitely generated.*

Proof. We show that, for all $m \geq 3$ and $n \geq 3$, the graph $G = C_m \cup C_n$ is a minimal forbidden induced subgraph for the class $K * AC$.

Suppose that $G \in K * AC$, i.e., there exists a partition $A \cup B = V(G)$ such that $G(A) \in K$ and $G(B) \in AC$. Since A induces a complete graph then either $A \cap V(C_m) = \emptyset$ or $A \cap V(C_n) = \emptyset$. In any case, $G(B)$ contains a cycle, a contradiction. Thus, $G \notin K * AC$.

Let us prove the minimality of G. Each induced subgraph $H \neq G$ contains at most one cycle. Therefore there is a vertex $u \in V(H)$ such that $H - u$ has no cycles. We put $A = \{u\}$ and $B = V(H) \backslash A$. □

Case 7: $\{F(c, \infty), I(\infty, \infty)\}$
the complementary case is $\{F(\infty, c), I(\infty, \infty)\}$)

We consider the classes $K = \mathrm{FIS}(O_2) \in F(c, \infty)$ of all complete graphs and $T = \mathrm{FIS}(C_n : n \geq 4) \in I(\infty, \infty)$ of all triangulated graphs.

The class $K * T$ is not finitely generated because there exists an infinite series $\{C_m \cup C_n : m, n \geq 4\}$ of minimal forbidden induced subgraphs. The proof is essentially the same as in Proposition 10.

Case 8: $\{F(\infty, \infty), I(\infty, c)\}$
(the complementary case is $\{F(\infty, \infty), I(c, \infty)\}$)

We consider the classes $O \cup K = \mathrm{FIS}(P_3, \overline{P}_3) \in F(\infty, \infty)$ of all edgeless and complete graphs and $AC = \mathrm{FIS}(C_n : n \geq 3) \in I(\infty, c)$.

Proposition 6. *The class $(O \cup K) * A$ is not finitely generated.*

Proof. We show that $G = K_4 \cup C_n$ $(n \geq 3)$ is a minimal forbidden induced subgraph for $(O \cup K) * AC$.

Suppose that $G \in (O \cup K) * AC$, i.e., there is a partition $A \cup B = V(G)$ such that $G(A) \in O \cup K$ and $G(B) \in AC$. Since $G(B)$ has no cycles, at least two vertices of K_4 belong to A. Then A induces a complete subgraph. So C_n is a cycle of $G(B)$, a contradiction to $G(B) \in AC$.

Now we prove the minimality of G. Any induced subgraph $H \neq G$ either does not contain C_n (in that case we put $A = V(H) \cap V(K_4)$ and $B = V(H) \cap A$), or does not contain K_4 as a component (in that case we put $A = \{u, v\}$, where $u \in V(C_n) \cap V(H)$, $v \in V(K_4) \cap V(H)$ and $B = V(H) \backslash A$). □

Case 9: $\{F(\infty, \infty), I(\infty, \infty)\}$

We choose the classes $O \cup K = \mathrm{FIS}(P_3, \overline{P}_3) \in F(\infty, \infty)$ of all edgeless and complete graphs and $T = \mathrm{FIS}(C_n : n \geq 4) \in I(\infty, \infty)$ of all triangulated graphs.

Proposition 7. *The class $(O \cup K) * T$ is not finitely generated.*

Proof. Define a graph G which consists of a cycle $C = C_n$ with vertex set $(v_1, v_2, \dots, v_n)$ (vertices are ordered cyclically) and disjoint cycles $H_i = C_4$, $i = 1, 2, \dots, n$, with the condition that every vertex v_i is adjacent to every vertex of $V(H_i)$.

We show that, for all $n \geq 4$, G is a minimal forbidden induced subgraph for the class $(O \cup K) * T$.

Suppose that $G \in (O \cup K) * T$, i.e., there exists a partition $A \cup B = V(G)$ such that $G(A) \in O \cup K$ and $G(B) \in T$. Since $G(B)$ has no C_4's, at least one vertex of each $H_i = C_4$ belongs to A. So $G(A)$ is an edgeless graph. Then C is contained in $G(B)$, a contradiction to $G(B) \in T$. Thus, $G \notin (O \cup K) * T$.

Now we prove the minimality of G. It is sufficient to show that $G - v \in (O \cup K) * T$ for every vertex $v \in V(G)$.

If $v \in V(H_i)$ then we include into A a vertex of each subgraph H_j, $j = 1, 2, \dots, i-1, i+1, \dots, n$, and v_i. Clearly, $G(A) \in O \subset O \cup K$. The set $B = V(G - v) \backslash A$ induces a triangulated graph. So $G - v \in (O \cup K) * T$.

If $v \in V(C)$ then we include into a vertex of each subgraph H_j, $j = 1, 2, \dots, n$. It is clear that A induces an edgeless graph while $B = V(G - v) \backslash A$ induces a triangulated graph. □

Case 10: $\{I(\infty, c), I(\infty, c)\}$
(the complementary case is $\{I(c, \infty), I(c, \infty)\}$)

We consider the class $B = \mathrm{FIS}(C_{2n+1} : n \geq 1) \in I(\infty, c)$ of all bipartite graphs.

Proposition 8. *The class $B * B$ is not finitely generated.*

Proof. For any $n \geq 1$, we construct a graph G_n in the following way. Let C_{2n+1} be a cycle with the vertex set $(v_0, v_1, \dots, v_{2n})$ (vertices are ordered cyclically). Add vertices u_{2i-1} and w_{2i-1} $(i = 1, 2, \dots, n)$ along with the edges $v_{2i-2}u_{2i-1}$, $v_{2i-2}w_{2i-1}$, $v_{2i-1}u_{2i-1}$, $v_{2i-1}w_{2i-1}$, $v_{2i}u_{2i-1}$, $v_{2i}w_{2i-1}$ and $u_{2i-1}w_{2i-1}$.

It is not hard to check that G_n is a minimal forbidden induced subgraph for $B * B$. □

Case 11: $\{I(c, \infty), I(\infty, c)\}$

We consider the classes $\overline{AC} = \mathrm{FIS}(\overline{C}_n : n \geq 3) \in I(c, \infty)$ (complements to acyclic graphs) and $AC = \mathrm{FIS}(C_n : n \geq 3) \in I(\infty, c)$ (acyclic graphs).

Proposition 9. *The class $\overline{AC} * AC$ is not finitely generated.*

Proof. We show that the graph $G = C_l \cup C_m \cup C_n$ $(l, m, n \geq 3)$ is a minimal forbidden induced subgraph for the class $\overline{AC} * AC$.

Suppose that $G \in \overline{AC} * AC$, i.e., there is a partition $A \cup B = V(G)$ such that $G(A) \in \overline{AC}$ and $G(B) \in AC$. Since $G(B)$ has no cycles, at least one vertex of each of cycles C_l, C_m and C_n belongs to A. These three vertices induce $\overline{C}_3$, a contradiction to $G(A) \in \overline{AC}$. Thus, $G \notin \overline{AC} * AC$.

Now we prove the minimality of G. Every induced subgraph $H \neq G$ contains at most two cycles. Let us include into A a vertex of every cycle of H. Clearly, $G(A) \in \overline{AC}$ and the set $B = V(H)\backslash A$ induces an acyclic graph. □

Case 12: $\{I(\infty, \omega), I(\infty, \infty)\}$, $\omega \in \{c, \infty\}$
(the complementary case is $\{I(\omega, \infty), I(\infty, \infty)\}$)

We consider classes $AC = \mathrm{FIS}(C_n : n \geq 3) \in I(\infty, c)$ of all acyclic graphs and $T = \mathrm{FIS}(C_n : n \geq 4) \in I(\infty, \infty)$ of all triangulated graphs.

Proposition 10. *The classes $AC * T$ and $T * T$ are not finitely generated.*

Proof. We define a graph G of order $6n + 3$ with the vertex set $V_0 \cup V_1 \cup \cdots \cup V_{2n}$, where $|V_i| = 3$ for all $i = 0, 1, \ldots, 2n$. Each vertex of V_i is adjacent to each vertex of V_{i+1} ($i = 0, 1, \ldots, 2n$) and each vertex of V_{2n} is adjacent to each vertex of V_0. We show that, for every $n \geq 1$, G is a minimal forbidden induced subgraph for $AC * T$ and $T * T$.

Suppose that $G \in T * T$, i.e., there exists a partition $A \cup B = V(G)$ such that $G(A) \in T$ and $G(B) \in T$. If $|V_i \cap A| \geq 2$ then V_i is called *A-set*. Otherwise V_i is a *B-set*. Since the number of sets V_i is odd, there are either two neighboring A-sets or two neighboring B-sets. By the symmetry, we may assume that $|V_0 \cap A| \geq 2$ and $|V_1 \cap A| \geq 2$. We choose vertices $u_i, v_i \in V_i \cap A$, $i = 0, 1$. Clearly, $\{u_0, u_1, v_0, v_1\}$ induces C_4, a contradiction to $G(A) \in T$. Thus, $G \notin T * T$. Since $AC * T \subset T * T$, $G \notin AC * T$.

Now we prove the minimality of G. It is sufficient to show that $G - v \in AC * T$ for each vertex $v \in V(G)$. Since $AC * T \subset T * T$, it will be follow that $G - v \in T * T$. By the symmetry, we may assume that $v \in V_0$. Let $V_0 \backslash \{v\} = \{x, y\}$. We put

$$A = \{x\} \cup (\bigcup_{i=1}^{n} V_{2i-1})$$

and

$$B = \{y\} \cup (\bigcup_{i=1}^{n} V_{2i}).$$

Clearly, $G(A) \in AC$ and $G(B) \in AC \subset T$. □

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