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On the ratio of the stability number and the domination number

by

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ABSTRACT

Let $\alpha(G)$ and $\gamma(G)$ be the stability number and domination number of a graph G, respectively. Since $\alpha(G) \geq \gamma(G)$, $\alpha(G)/\gamma(G) \geq 1$. For each $r \geq 1$, we define a graph G to be an $\alpha/\gamma \leq r$-*perfect graph* if $\alpha(H)/\gamma(H) \leq r$ for each induced subgraph H of G.

We show that the class of all $\alpha/\gamma \leq r$-perfect graphs is determined by a unique forbidden induced subgraph.

It gives the possibility to approximate $\alpha(G)$ and $\gamma(G)$ in the corresponding classes.

Keywords: stability number, domination number, forbidden induced subgraph characterization

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is called *stable* if no two vertices in S are adjacent. The *stability number* of G, $\alpha(G)$, is the maximum cardinality of a stable set of G.

A set $D \subseteq V$ is called *dominating* if each vertex $u \in V \backslash D$ is adjacent to a vertex of D The *domination number*, $\gamma(G)$, is the minimal cardinality of a dominating set in G.

Accordingly, a stable set S with $|S| = \alpha(G)$ is a *maximum stable set*, and a dominating set D with $|D| = \gamma(G)$ is a *minimum dominating set*.

It is clear that $\alpha(G)/\gamma(G) \geq 1$ for every graph G.

Definition 1. *For each $r \geq 1$, we define a graph G to be an $\alpha/\gamma \leq r$-*perfect graph *if $\alpha(H)/\gamma(H) \leq r$ for each induced subgraph H of G.*

We denote by $\alpha/\gamma(r)$ the class of all $\alpha/\gamma \leq r$-perfect graphs.

The classes $\alpha/\gamma(r)$ are interesting because we can approximate $\alpha(G)$ and $\gamma(G)$ within them (for a fixed r). Indeed, let I be an arbitrary maximal (by inclusion) stable set of an $\alpha/\gamma \leq r$-perfect graph G. Then $\alpha/|I| \leq \alpha(G)/\gamma(G) \leq r$ and

$$|I| \geq \alpha(G)/r.$$

Since I is also a dominating set of G, $|I|/\gamma(G) \leq \alpha(G)/\gamma(G) \leq r$ and

$$|I| \leq r\gamma(G).$$

We show that there is exactly one minimal forbidden induced subgraph for every class $\alpha/\gamma \leq r$-perfect graphs ($r \geq 1$). Therefore, recognizing $\alpha/\gamma(r)$ can be done in polynomial time (when r is fixed).

Related problems of characterizing classes

$$\alpha i(k) = \{G : \alpha(H) - \gamma(H) \leq k \ \text{ for every induced subgraph } \ H \ \text{ of } \ G\}$$

and

$$\alpha\gamma(k) = \{G : \alpha(H) - \gamma(H) \leq k \ \text{ for every induced subgraph } \ H \ \text{ of } \ G\}$$

(where $\gamma(G)$ is the domination number) were solved in [1]. Also, the ratio of the irredundance number and the domination number was investigated in [2].

Theorem 1. *Let $r \in [p, p+1)$ for some $p \geq 1$. A graph G is an $\alpha/\gamma \leq r$-perfect graph if and only if it does not contain the star $K_{1,p+1}$ as an induced subgraph.*

Proof. Necessity is trivial, since $\alpha(K_{1,p+1})/\gamma(K_{1,p+1}) = p + 1 > r$.

Sufficiency. Let G be a graph which does not have an induced $K_{1,p+1}$. We may assume that G is *minimal*, i.e., each proper induced subgraph of G is $\alpha/\gamma \leq r$-perfect graphs.

Let us fix a maximum stable set S ($|S| = \alpha(G)$) and a minimum dominating set D ($|D| = \gamma(G)$).

Claim 1. $V(G) = S \cup D$.

Proof. Suppose that it is not true, i.e, there is a vertex $u \notin S \cup D$. We consider the subgraph $G - u$. Since $S \subseteq V(G - u)$, $\alpha(G - u) = |S| = \alpha(G)$. Clearly, D is a dominating set of $G - u$, hence $\gamma(G - u) \leq |D| = \gamma(G)$. Thus,

$$\alpha(G - u)/\gamma(G - u) \geq \alpha(G)/\gamma(G) > r,$$

a contradiction to the minimality of G. □

Claim 2. $S \cap D = \emptyset$.

Proof. Suppose that there is a vertex $v \in S \cap D$. By Claim 1, $V(G) = S \cup D$. Since S is a stable set, v can be adjacent to some vertices of D only.

It follows that $\alpha(G - v) \geq |S| - 1 = \alpha(G) - 1$. Let us consider a minimum dominating set D' of $G-v$. Since $D\backslash\{v\}$ is a dominating set of $G-v$, $|D'| \leq \gamma(G)-1$. On the other hand, $D' \cup \{v\}$ is a dominating set of G, so $|D' \cup \{v\}| \geq \gamma(G)$, or $|D'| \geq \gamma(G) - 1$. Thus, $\gamma(G - v) = |D'| = \gamma(G) - 1$.

It follows that

$$\alpha(G - v)/\gamma(G - v) \geq (\alpha(G) - 1)/(\gamma(G) - 1) \geq \alpha(G)/\gamma(G) > r$$

(since $\alpha(G) \geq \gamma(G)$, $(\alpha(G) - 1)/(\gamma(G) - 1) \geq \alpha(G)/\gamma(G)$), a contradiction to the minimality of G. □

Finally, we denote by m the number of edges of the form sd, where $s \in S$ and $d \in D$. Since D is a dominating set, each vertex of S is adjacent to a vertex of D. So

$$m \geq |S|. \tag{1}$$

On the other hand, G has no induced $K_{1,p+1}$. Therefore every vertex of D is adjacent to at most p vertices of S, and

$$m \leq p|D|. \tag{2}$$

From (1) and (2) we obtain that

$$\alpha(G) = |S| \leq m \leq p|D| = p\gamma(G),$$

or

$$\alpha(G)/\gamma(G) \leq p \leq r.$$

Thus, G is an $\alpha/\gamma \leq r$-perfect graph, a contradiction. □

A dominating set D of a graph G is called an *independent dominating set* is a vertex subset which is stable and dominating. The *independent domination number*, $i(G)$, of a graph G is the least cardinality of an independent dominating set in G.

A similar problem of characterizing $\alpha/i \leq r$-perfect graphs was solved in [3].

As in Definition 1, we can define $\gamma/i \leq r$-perfect graphs ($r \geq 1$).

Conjecture 1. *For every $r \geq 1$, the class of all $\gamma/i \leq r$-perfect graphs has finite forbidden induced subgraph characterization.*

The case $r = 1$ was considered in [4], where it was proved that there are exactly 17 minimal forbidden induced subgraphs for $\gamma/i \leq 1$-perfect graphs (*domination-perfect graphs*).

References

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