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PACKING TREES IN GRAPHS

by

Alexander Kelmans [1]
RUTCOR, Rutgers University,
New Brunswick, New Jersey 08903
and
Department of Mathematics, University of Puerto Rico
San Juan, PR 00931,
Email: kelmans@rutcor.rutgers.edu

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ABSTRACT

Let $\mathcal{G}_r^s$ denote the set of graphs with each vertex of degree at least r and at most s, $v(G)$ the number of vertices, and $\tau_k(G)$ the maximum number of disjoint k–edge trees in G. In this paper we show that

$(a1)$ if $G \in \mathcal{G}_2^s$ and $s \geq 4$ then $\tau_2(G) \geq v(G)/(s+1)$,

$(a2)$ if $G \in \mathcal{G}_2^3$ and G has no 5–vertex components then $\tau_2(G) \geq v(G)/4$,

$(a3)$ if $G \in \mathcal{G}_1^s$ and G has no k–vertex component where $k \geq 2$ and $s \geq 3$ then $\tau_k(G) \geq (v(G)-k)/(sk-k+1)$, and

$(a3)$ the above bounds are attained for infinitely many connected graphs.

Our proofs provide polynomial time algorithms for finding the corresponding packings in a graph.

Keywords: packing, 2–edge and k–edge paths, k–edge trees, polynomial time approximation algorithms.

1 Introduction

We consider simple undirected graphs. All notions on graphs that are not defined here can be found in [1]. Given a graph G and a set $\mathcal{F}$ of subgraphs of G, an *$\mathcal{F}$–packing* of G is a subgraph of G whose components are members of $\mathcal{F}$. The *$\mathcal{F}$–packing problem* is that of finding an $\mathcal{F}$–packing having the maximum number of vertices. Various $\mathcal{F}$–packing problems have extensively been studied by many authors for different families $\mathcal{F}$ (e.g. [2, 3, 4, 5, 6, 9, 11, 12, 13, 14]).

It is not surprising that $\mathcal{F}$–packing turns out to be NP–hard for most of the families $\mathcal{F}$. Surprisingly the problem can be solved in polynomial time for some non-trivial families. For example, Edmonds (see [14]) showed that a classical matching problem can be solved in polynomial time. It is also known that the problem of packing stars of at least one and at most k edges is polynomially solvable even if the stars we pack are required to be induced [6, 8]. On the other hand, the problem of finding in G the maximum number of disjoint subgraphs, isomorphic to a given connected graph H of at least three vertices, is NP–hard [4].

In this paper we consider, in particular, a special case of H–packing problem when $H = \Lambda$, a path of two edges. Although the Λ–packing problem is NP–hard, i.e. possibly intractable in general, this problem turns out to be tractable for some natural classes of graphs. Here are some examples of such results.

Let $v(G)$ denote the number of vertices of a graph G and $\tau_k(G)$ denote the maximum number of disjoint k–edge trees in G. We also put $\tau_2(G) = \lambda(G)$.

A graph is called *claw–free* if it contains no induced subgraph isomorphic to $K_{1,3}$ (which is called a *claw*).

A block of a connected graph is called an *end–block* if it has at most one vertex in common with any other block of the graph.

Obviously $\lambda(G) \leq \lfloor v(G)/3 \rfloor$.

1.1 [5] *Suppose that G is a connected claw–free graph having at most two end–blocks (in particular, a 2–connected claw–free graph). Then $\lambda(G) = \lfloor v(G)/3 \rfloor$.*

Let G^{Δ} be the graph obtained from a cubic graph by "replacing" every vertex by a triangle. From **1.1** we have, in particular, for cubic graphs:

1.2 *Suppose that G is a connected cubic graph having at most two end–blocks (in particular, a 2–connected graph). Then $\lambda(G^{\Delta}) = \lfloor v(G^{\Delta})/3 \rfloor$.*

Let $eb(G)$ denote the number of end–blocks of G.

1.3 [5] *Suppose that G is a simple connected claw–free graph and $eb(G) \geq 2$. Then $\lambda(G) \geq \lfloor (v(G) - eb(G) + 2)/3 \rfloor$, and this lower bound is sharp.*

1.4 [9] *Let T be a tree on t vertices. Let G_n be a d_n–regular graph on n vertices. Suppose that $d_n \to \infty$ when $n \to \infty$. Then G_n contains at least $(1 - o(1))n/t$ (and, obviously, at most n/t) disjoint trees isomorphic to T.*

In particular, $\lambda(G_n) = (1 - o(1))n/3$.

1.5 [10] *Let G be a cubic graph. Then $\lambda(G) \geq v(G)/4$.*

Let $\mathcal{G}_r^s$ denote the set of graphs with each vertex of degree at least r and at most s. Our main question on Λ–packings is:

How many disjoint 2–edge paths must an n–vertex graph G from $\mathcal{G}_2^s$ have?

Our main results here is:

1.6 *Let $G \in \mathcal{G}_2^3$ and G has no 5–vertex components. Then $\lambda(G) \geq v(G)/4$.*

Obviously **1.5** follows from **1.6** because if G is a cubic graph then $G \in \mathcal{G}_2^3$ and G has no 5–vertex components.

1.7 *Let $G \in \mathcal{G}_2^s$, $s \geq 4$. Then $\lambda(G) \geq v(G)/(s+1)$.*

We give constructions (see Section 4) that allow us to prove the following:

1.8 *There are infinitely many connected graphs for which the bounds in* **1.6** *and* **1.7** *are attained. Moreover, for every integer $k \geq 1$ there are infinitely many subdivisions of k–connected graphs for which the bounds in* **1.6** *and* **1.7** *are attained.*

We also consider a special case of the $\mathcal{F}$–packing problem when $\mathcal{F}$ is the set of all connected subgraphs of G having k edges. Let $\mathcal{T}_k$ denote the set of all trees having k edges. Obviously this problem is equivalent to the $\mathcal{T}_k$–packing problem, the problem of finding in a graph the maximum number of disjoint trees of k edges. Notice that Theorem **1.4** provides a similar asymptotic result for the $\mathcal{T}_k$–packing problem. Our question on $\mathcal{T}_k$–packings is:

How many disjoint k–edge trees must an n–vertex graph G from $\mathcal{G}_1^s$ have?

Our main result here is:

1.9 *Let s and k be integers, $s \geq 3$, $k \geq 2$, $G \in \mathcal{G}_1^s$, and G has no k–vertex component. Then $\tau_k(G) \geq (v(G)-k)/(sk-k+1)$.*

One of the constructions in Section 4 allows to prove:

1.10 *There are infinitely many connected graphs for which the bound in* **1.9** *is attained.*

Our proofs provide polynomial time algorithms for finding in a cubic graph G the corresponding packings. Therefore these algorithms are polynomial approximation algorithms for the corresponding problems.

2 Main notions, notation and simple observations

Let G be a simple graph. We use the following notation and notions:

$V = V(G)$ and $E = E(G)$ are the sets of vertices and edges of a graph G, respectively,

$v(G) = |V(G)|$ and $e(G) = |E(G)|$,

if $x \in V(G)$, then $d(x, G) = d(x)$ is the degree of x in G,

$L(G)$ is the set of vertices of degree one in G,

xPy is a path with the end–vertices x and y,

an $\mathcal{F}$–*packing* of G is a subgraph of G whose components are members of $\mathcal{F}$,

a Λ–*packing* is a subgraph of G whose components are 2–edge paths in G,

$\lambda(G)$ is the maximum number of vertex disjoint 2–paths in G,

$\tau_k(G)$ is the maximum number of disjoint k–edge trees in G where $k \geq 1$, and so $\lambda(G) = \tau_2(G)$,

A *block* of G is a maximal 2–connected subgraph of G. A block of a connected graph is called an *end–block* if it has at most one vertex in common with any other block of the graph.

A *path–thread* in a graph G is a maximal path P in G such that each vertex of degree two in P is also of degree two in G. A *cycle–thread* in G is a cycle C in G such that each vertex of C, except for possibly one, is of degree two in G. A *thread* is either a path–thread or a cycle–thread.

Let, as above, $\mathcal{G}_r^s$ denote the set of graphs with each vertex of degree at least r and at most s.

If G is a connected graph in $\mathcal{G}_2^3$, then an end–block of G is called a *leaf*. A B–*leaf* is a leaf isomorphic to a graph B. If k is an integer then a k–*leaf* is a leaf having k vertices.

Let $\mathcal{F}_2^3$ denote the set of graphs such that

$(a1)$ $d(x, G) \in \{2, 3\}$ for every vertex x of G and

$(a2)$ G has no components of five vertices.

Given a class of graphs $\mathcal{A}$, a graph G is called $\mathcal{A}$–*mininimal* if $G \in \mathcal{A}$ and $G - e \notin \mathcal{A}$ for every $e \in E(G)$. Obviously $\mathcal{G}_r^s$–minimal and $\mathcal{F}_2^3$–minimal graphs exist.

Clearly

2.1 *Let G be a graph.*

$(c1)$ *If $G \in \mathcal{F}_2^3$ ($G \in \mathcal{G}_r^s$) then G has an $\mathcal{F}_2^3$–minimal (respectively, $\mathcal{G}_r^s$–minimal) spanning subgraph F and $\lambda(G) \geq \lambda(F)$.*

$(c2)$ *A graph G is $\mathcal{F}_2^3$–minimal if and only if each edge of G is incident to either a vertex of degree two in G or to a vertex of a 5–leaf of G.*

2.2 *Let $G \in \mathcal{F}_2^3$, Q be a leaf of G. Then $Q \in \mathcal{F}_2^3$, the boundary vertex b of Q is of degree two in Q, and there is a (unique) path–thread $xTy = S(Q)$ in G such that $x = b \in V(Q)$ and $y \in V(G - Q)$.*

If Q is a leaf of a graph $G \in \mathcal{F}_2^3$ then $S(Q)$ is called the *stem* of the leaf Q.
Let $\grave{Q}$ be the graph obtained from $Q \cup S(Q)$ by removing the vertex of degree one.

3 Graph reductions

Our proving strategy is to establish various properties of a minimum counterexample, and finally conclude that it cannot exist. At various stages in this process, we will need some operations that reduce a minimum counterexample to a smaller one providing a contradiction. In this section we present such reductions.

3.1 *Let G be a graph, xTy be a thread of G, $T' = T - \{x, y\}$, and $G' = G - T'$.*

(a1) If $s \geq 4$ and $e(T) \geq 4$, then $\lambda(G') \geq v(G')/(s+1) \Rightarrow \lambda(G) \geq v(G)/(s+1)$.

(a2) If $s = 3$, $e(T) \geq 4$, and $e(T)$ is not divisible by three, then $\lambda(G') \geq v(G')/4 \Rightarrow \lambda(G) \geq v(G)/4$.

Proof Clearly $v(G) = v(G') + v(T')$ and $\lambda(G) \geq \lambda(G') + \lfloor v(T')/3 \rfloor$. Now $\lambda(G') \geq v(G')/(s+1) \Rightarrow \lambda(G) \geq v(G')/(s+1) + \lfloor v(T')/3 \rfloor = v(G)/(s+1) + \gamma_s(T')$ where $\gamma_s(T') = \lfloor v(T')/3 \rfloor - v(T')/(s+1)$. Since $e(T) \geq 4$, clearly $v(T') \geq 3$.

Suppose that $s \geq 4$. Then obviously $\gamma_s(T') \geq 0$.

Now suppose that $s = 3$. Then by our assumption, $e(T)$ is not divisible by three, i.e. $e(T) \in \{3t+1, 3t+2\}$ where $t \geq 1$, and so $v(T') \in \{3t, 3t+1\}$ where $t \geq 1$.

If $v(T') = 3t$ where $t \geq 1$, then $\gamma_3(T') = t - 3t/4 = t/4 > 0$.

If $v(T') = 3t+1$ where $t \geq 1$, then $\gamma_3(T') = t - (3t+1)/4 = (t-1)/4 \geq 0$.

Therefore in both cases $\lambda(G) \geq v(G)/(s+1)$. □

3.2 *Let G be a graph, xTy be a thread of G, and $T' = T - \{x, y\}$. Let G' be obtained from $G - (T - \{x, y\})$ by adding a new vertex z and two new edges edge xz and yz. Suppose that $e(T) = 3t$ where $t \geq 2$. Then $\lambda(G') \geq v(G')/4 \Rightarrow \lambda(G) \geq v(G)/4$.*

Proof Clearly $v(G) = v(G') + v(T') - 1$. Since $e(T) = 3t$ where $t \geq 2$, we have $v(T') = 3t - 1 = 3k + 2$ where $k = t - 1 \geq 1$, and so $\lambda(T') = k$. Let $\mathcal{P}'$ be a maximum Λ–packing in G', and so $\lambda(G') = |\mathcal{P}'|$.

(**p1**) Suppose that $\{xz, yz\} \cap E(\mathcal{P}') = \emptyset$. Then $\lambda(G) \geq \lambda(G') + \lambda(T') = \lambda(G') + k$. Now $\lambda(G') \geq v(G')/4 \Rightarrow \lambda(G) \geq v(G')/4 + k = (v(G) - v(T') + 1)/4 + k = v(G)/4 - (3k+1)/4 + k = v(G)/4 + (k-1)/4 \geq v(G)/4$.

(**p2**) Suppose that $|\{xz, yz\} \cap E(\mathcal{P}')| = 1$, say rxz is a 2–edge path in $\mathcal{P}'$ for some $r \in V(G - T')$. Let $xs \in E(T)$, $\mathcal{P}''$ be a maximum Λ–packing in $T'' = T' - s$, and $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}'' - rxz \cup rxs$. Then $\mathcal{P}$ is a Λ–packing in G, and so $\lambda(G) \geq |\mathcal{P}| = |\mathcal{P}'| + |\mathcal{P}''| = \lambda(G') + \lambda(T'')$. Since $v(T'') = v(T') - 1 = 3k + 1$, we have $\lambda(T'') = k$. Therefore $\lambda(G') \geq v(G')/4 \Rightarrow \lambda(G) \geq v(G')/4 + k = v(G)/4$.

(**p3**) Now suppose that $xz, yz \in E(\mathcal{P}')$, i.e. $xz'y$ is a 2–edge path in $\mathcal{P}'$. Let $st \in E(T - x)$, $\mathcal{P}''$ be a (unique) maximum Λ–packing in $T'' = T' - \{s, t\}$. Let $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}'' - xzy \cup xst$. Then $\mathcal{P}$ is a Λ–packing in G, and so $\lambda(G) \geq |\mathcal{P}| = |\mathcal{P}'| + |\mathcal{P}''| = \lambda(G') + \lambda(T'')$. Since $v(T'') = v(T') - 2 = 3k$, we have $\lambda(T'') = k$. Therefore $\lambda(G') \geq v(G')/4 \Rightarrow \lambda(G) \geq v(G')/4 + k = \lambda(G) \geq (v(G) - v(T') + 1)/4 + k (v(G) - 3k - 1)/4 + k = \geq v(G)/4$ because $k \geq 1$. □

3.3 *Let $a \in V(G)$, $d(a,G) = 3$, A be a leaf of G, and aTb be the stem of A with $a \in V(A)$. Let $\grave{A} = A \cup (T-b)$ and $G' = G - \grave{A}$.*

Then the implication $\lambda(G') \geq v(G')/(s+1) \Rightarrow \lambda(G) \geq v(G)/(s+1)$ holds for $s = 3$ provided $v(\grave{A}) \neq 5$, as well as for every integer $s \geq 4$.

Proof Clearly $v(G) = v(G') + v(\grave{A})$, $v(\grave{A}) \geq 3$, $\lambda(G) \geq \lambda(G') + \lambda(\grave{A})$, and $\lambda(\grave{A}) = \lfloor v(\grave{A})/3 \rfloor$. Since $v(\grave{A}) \geq 3$ and $v(\grave{A}) \neq 5$, we have $\lfloor v(\grave{A})/3 \rfloor - v(\grave{A}))/4 \geq 0$. Therefore $\lambda(G') \geq v(G')/(s+1) \Rightarrow \lambda(G) \geq v(G')/(s+1) + \lfloor v(\grave{A})/3 \rfloor = (v(G) - v(\grave{A}))/(s+1) + \lfloor v(\grave{A})/3 \rfloor = v(G)/(s+1) + [\lfloor v(\grave{A})/3 \rfloor - v(\grave{A}))/(s+1)]$.

Suppose that $s = 3$. Then by our assumption, $v(\grave{A}) \neq 5$. Therefore $\lfloor v(\grave{A})/3 \rfloor - v(\grave{A}))/4 \geq 0$.

Now suppose that $s \geq 4$. Then also $\lfloor v(\grave{A})/3 \rfloor - v(\grave{A}))/4 \geq 0$.

Thus in both cases $\lambda(G) \geq v(G)/(s+1)$. □

3.4 *Let aTb be a thread of G, $a \neq b$, $d(a,G) = d(b,G) = 3$, A be a leaf of G and $a \in V(A)$. Let x_iT_ib, $i = 1,2$, be the two different threads of G distinct from T, $bz_i \in E(T_i)$, and $A' = A \cup T \cup \{bz_1, bz_2\}$. Suppose that $v(\grave{A}) = 5$, i.e. $e(A) + e(T) = 6$ where $e(T) \geq 1$.*

(a1) If $e(T_i) = 2$ for $i = 1,2$ then let $x_iT_ib = x_iz_ib$ and $G' = G - A'$.

(a2) If $e(T_1) = 3$, $e(T_2) = 2$, then let $x_1T_1b = a_1y_1z_1b$, $x_2T_2b = x_2z_2b$.

(a2.1) If $x_1 \neq x_2$, then let G' be obtained from $G - A'$ by adding a new edge $e = y_1x_2$.

(a2.2) If $x_1 = x_2$, then let G' be obtained from $G - A'$ by adding a new edge $e = y_1z_2$.

(a3) If $e(T_1) = 3$, $e(T_2) = 3$, then let $x_1T_1b = a_1y_1z_1b$, $x_2T_2b = x_2y_2z_2b$ and G' be obtained from $G - A'$ by adding a new edge $e = y_1y_2$.

Then $\lambda(G') \geq v(G')/4 \Rightarrow \lambda(G) \geq v(G)/4$.

Proof Clearly $v(G) = v(G') + v(A')$. Since $v(\grave{A}) = 5$, clearly $v(A') = 8$. It is easy to see that $\lambda(A') = 2$. Let $\mathcal{P}'$ be a maximum Λ–packing in G'.

(**p1**) Suppose that $e \notin E(\mathcal{P}')$, i.e. $\mathcal{P}'$ avoids e. Notice that this assumption holds, in particular, in case (a1) because $G' \subset G$, and so there is no e. In case (a2.2), e belongs to a triangle–leaf of G'. Hence there exists a maximum Λ–packing in G' avoiding e. Therefore we can assume that $\mathcal{P}'$ avoids e, and so our assumption holds in this case as well. Then $\lambda(G) \geq \lambda(G') + \lambda(A')$. Therefore $\lambda(G') \geq v(G')/4 \Rightarrow \lambda(G) \geq v(G')/4 + \lambda(A') = (v(G) - 8)/4 + 2 = v(G)/4$.

(**p2**) Suppose that $e \in E(\mathcal{P}')$ and $e(T_1) = 3$, $e(T_2) = 2$, and $x_1 \neq x_2$, and so $e = y_1x_2$.

(**p2.1**) Suppose that $\{x_1y_1, y_1x_2\} \cap E(\mathcal{P}') = y_1x_2$, say $y_1x_2c_2 \in \mathcal{P}'$ where $c_2 \in V(G' - y_1)$. Let $\mathcal{P}_1$ be a maximum Λ–packing of $A_1 = A' \cup y_1z_1 - y_2$ and $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}_1 \cup y_2x_2c_2 - y_1x_2c_2$. Then $\mathcal{P}$ is a Λ–packing of G, $|\mathcal{P}| = |\mathcal{P}'| + |\mathcal{P}_1|$, and $\lambda(G) \geq |\mathcal{P}| = \lambda(G') + \lambda(A_1)$. Obviously $\lambda(A_1) = 2$. Therefore $\lambda(G') \geq v(G')/4 \Rightarrow \lambda(G) \geq v(G')/4 + \lambda(A_1) = (v(G) - 8)/4 + 2 = v(G)/4$.

(**p2.2**) Suppose that $\{x_1y_1, y_1x_2\} \subseteq E(\mathcal{P}')$, i.e. $x_1y_1x_2 \in \mathcal{P}'$. Let $\mathcal{P}_2$ be a maximum Λ–packing of $A_2 = A' - z_1$ and $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}_1 \cup x_1y_1z_1 - x_1y_1x_2$. Then $\mathcal{P}$ is a Λ–packing of G, $|\mathcal{P}| = |\mathcal{P}'| + |\mathcal{P}_2|$, and $\lambda(G) \geq |\mathcal{P}| = \lambda(G') + \lambda(A_2)$. Obviously $\lambda(A_2) = 2$. Therefore $\lambda(G') \geq v(G')/4 \Rightarrow \lambda(G) \geq v(G')/4 + \lambda(A_2) = (v(G) - 8)/4 + 2 = v(G)/4$.

(**p3**) Suppose that $e \in E(\mathcal{P}')$ and $e(T_1) = 3$, $e(T_2) = 3$, and so $e = y_1y_2$. Because of symmetry, we can assume that $x_1y_1y_2 \in \mathcal{P}'$. Let $\mathcal{P}_3$ be a maximum Λ–packing of $A_3 = A' \cup y_2z_2 - z_1$ and $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}_3 \cup x_1y_1z_1 - x_1y_1y_2$. Then $\mathcal{P}$ is a Λ–packing of G, $|\mathcal{P}| = |\mathcal{P}'| + |\mathcal{P}_3|$, and $\lambda(G) \geq |\mathcal{P}| = \lambda(G') + \lambda(A_3)$. Obviously $\lambda(A_3) = 2$. Therefore $\lambda(G') \geq v(G')/4 \Rightarrow \lambda(G) \geq v(G')/4 + \lambda(A_1) = (v(G) - 8)/4 + 2 = v(G)/4$. □

3.5 *Let $s \geq 3$ be an integer, G be a graph, $a \in V(G)$, $d(a, G) = d$, $3 \leq d \leq s$, $\{T_i : i \in I\}$ be the set of threads of G with a common end–vertex a, and $S_a = \cup\{T_i : i \in I\}$.*

Suppose that $I = I_1 \cup I_2 \cup I_3$ is a partition of I such that $I_1 \neq I$ and
$i \in I_1$ implies T_i is a triangle,
$i \in I_2$ implies T_i is a path x_iz_ia, and
$i \in I_3$ implies T_i is a path $x_iy_iz_ia$.

Suppose also that if $|I_3|$ is odd then $I_2 \neq \emptyset$. Let G' be obtained from G as follows:

(*a*1) *Suppose that I_3 is even. Let $S'_a = S_a - \cup\{y_i : i \in I_3\}$* (and so $v(S'_a) = d + 1$). *Partition I_3 into pairs $\{i, i'\}$ and put $G' = G \cup \{y_iy_{i'} : i \in I_3\} - S'_a$. If, in particular, $I_3 = \emptyset$, then $G' = G - S'_a$.*

(*a*2) *Suppose that I_3 is odd, $r \in I_3$ and $j \in I_2$.* (Since I_3 is odd, by our assumption, $I_2 \neq \emptyset$, and so such j exists.) *Partition $I'_3 = I_3 - r$ into pairs $\{i, i'\}$.*

(*a*2.1) *If $x_r \neq x_j$ then let $S'_a = S_a - (\cup\{y_i : i \in I_3 - r\} \cup z_j)$ and $G' = G \cup x_rz_j \cup \{y_iy_{i'} : i \in I'_3\} - S'_a$* (and so $v(S'_a) = d + 1$).

(*a*2.2) *If $x_r = x_j$ then let $S'_a = S_a - (\cup\{y_i : i \in I_3\} \cup z_j)$ and $G' = G \cup y_rz_j \cup \{y_iy_{i'} : i \in I'_3\} - S'_a$* (and so $v(S'_a) = d$).

Then $\lambda(G') \geq v(G')/(s + 1) \Rightarrow \lambda(G) \geq v(G)/(s + 1)$.

Proof We can assume that G is a connected graph. Let $\mathcal{P}'$ be a maximum Λ–packing of G', and so $|\mathcal{P}'| = \lambda(G')$. Obviously, $d \leq v(S'_a) \leq d + 1$ and $v(G) = v(G') + v(S'_a)$.

(**p0**) Consider case (*a*1) with $I_3 = \emptyset$. Then, obviously, $v(G) = v(G') + v(S'_a)$, $\lambda(G) \geq \lambda(G') + \lambda(S'_a)$ $v(S_a) = d + 1)$, and $\lambda(S'_a) = 1$. Therefore $\lambda(G') \geq v(G')/4 \Rightarrow \lambda(G) \geq (v(G) - v(S'_a)/(s + 1) + \lambda(S'_a) = v(G)/(s + 1) - 1 + 1 = v(G)/(s + 1)$.

(**p1**) Consider case (*a*1) with $I_3 \neq \emptyset$, and so $|I_3|$ is even. Let $\mathcal{P}$ be obtained from $\mathcal{P}'$ as follows: if $y_iy_{i'}$ belongs to a 2–edge path $L' \in \mathcal{P}'$ then replace L' in $\mathcal{P}'$ by the 2–edge path L obtained from L' by replacing the edge $y_iy_{i'}$ by the edge y_iz_i from G. Then $\mathcal{P}$ is a Λ–packing in G and $|\mathcal{P}| = |\mathcal{P}'|$. Obviously $v(S'_a) = d + 1$, and so $v(G) = v(G')+d+1$. Let $S''_a = S'_a - V(\mathcal{P})$. Clearly $\lambda(S''_a) = 1$. Then $\lambda(G) \geq |\mathcal{P}|+\lambda(S''_a)$. Therefore $\lambda(G') \geq v(G')/(s + 1) \Rightarrow \lambda(G) \geq \lambda(G') + \lambda(S''_a) \geq v(G')/(s + 1) + 1 = (v(G) - (d + 1))/(s + 1) + 1 \geq v(G)/(s + 1)$.

(**p2**) Consider case (*a*2.1), i.e. I_3 is odd and $x_r \neq x_j$. Let $\mathcal{P}$ be obtained from $\mathcal{P}'$ as follows: If $y_iy_{i'}$, $i \neq r$, in G' belongs to a 2–edge path $L' \in \mathcal{P}'$ then replace L' in $\mathcal{P}'$ by the 2–edge path L obtained from L' by replacing the edge $y_iy_{i'}$ by the edge y_iz_i from G. If x_rz_j in G' belongs to a 2–edge path $P' \in \mathcal{P}'$ and $P' \neq x_rz_jx_j$ then replace P' in $\mathcal{P}'$ by the 2–edge path P obtained from P' by replacing the edge x_rz_j by the edge x_ry_r in G. If $P' = x_rz_jx_j \in \mathcal{P}'$ then replace P' in $\mathcal{P}'$ by the 2–edge path $P = x_ry_rz_r$. Then $\mathcal{P}$ is a Λ–packing in G and $|\mathcal{P}| = |\mathcal{P}'|$. Obviously $v(S'_a) = d + 1$, and so $v(G) = v(G') + d + 1$. Let $S''_a = (S'_a \cup z_j) - V(\mathcal{P})$. Clearly $\lambda(S''_a) = 1$. Then

$\lambda(G) \geq |\mathcal{P}| + \lambda(S''_a)$. Therefore $\lambda(G') \geq v(G')/(s+1) \Rightarrow \lambda(G) \geq \lambda(G') + \lambda(S''_a) \geq v(G')/(s+1) + 1 = (v(G) - (d+1))/(s+1) + 1 \geq v(G)/(s+1)$.

(**p3**) Consider case (a2.2), i.e. I_3 is odd and $x_r = x_j$. Since $y_r z_j$ belongs to a triangle–bock in G', there exists a maximum Λ–packing $\mathcal{P}'$ in G' that avoids $y_r z_j$. Let $\mathcal{P}$ be obtained from $\mathcal{P}'$ as follows:

(b1) If $y_i y_{i'}$, $i \neq r$, in G' belongs to a 2–edge path $L' \in \mathcal{P}'$ then replace L' in $\mathcal{P}'$ by the 2–edge path L obtained from L' by replacing the edge $y_i y_{i'}$ by the edge $y_i z_i$ from G.

(b2) If $y_r z_j$ in G' belongs to a 2–edge path $P' \in \mathcal{P}'$ and $P' \neq x_r z_j x_j$ then replace P' in $\mathcal{P}'$ by the 2–edge path P obtained from P' by replacing the edge $x_r z_j$ by the edge $x_r y_r$ from G.

Then $\mathcal{P}$ is a Λ–packing in G and $|\mathcal{P}| = |\mathcal{P}'|$. Obviously $v(S'_a) = d$, and so $v(G) = v(G') + d$. Let $S''_a = S'_a - V(\mathcal{P})$. Clearly $\lambda(S''_a) = 1$. Then $\lambda(G) \geq |\mathcal{P}| + \lambda(S''_a)$. Therefore $\lambda(G') \geq v(G')/(s+1) \Rightarrow \lambda(G) \geq \lambda(G') + \lambda(S''_a) \geq v(G')/(s+1) + 1 = (v(G) - d)/(s+1) + 1 \geq v(G)/(s+1)$. □

3.6 *Let $a \in V(G)$, $d(a, G) = 3$, $aT_i x_i$ be the three threads ending at a. Suppose that x_j belongs to a 5–leaf X_j, $e(T_j) = 1$ for $j = 1, 2$. Let $X = X_1 \cup X_2 - \{x_1, x_2\}$. Let G' be obtained from $G - X$ by adding a new edge $x_1 x_2$. Then $\lambda(G') \geq v(G')/4 \Rightarrow \lambda(G) \geq v(G)/4$.*

Proof Obviously, $v(G) = v(G') + v(X)$, $v(X) = 8$, $\lambda(X) = 2$, and $\lambda(G) \geq \lambda(G') + \lambda(X)$. Therefore $\lambda(G') \geq v(G')/4 \Rightarrow \lambda(G) \geq v(G')/4 + \lambda(X) = (v(G) - 8)/4 + 2 = v(G)/4$. □

4 Constructions of extremal graphs

In this section we give constructions providing infinitely many connected graphs for which the bounds in **1.6**, **1.7**, and **1.9** are attained.

Given a tree T, let, as above, $L(T)$ denote the set of leaves of T. Let $\mathcal{T}_s$ denote the set of trees T such that if $x \in V(T) \setminus L(T)$ then $d(x, T) = s$. Let T_k be obtained from $T \in \mathcal{T}_s$ by subdividing each non-dangling edge of T by k vertices and each dangling edge by $k - 1$ vertices. Let $\mathcal{T}_{s,k} = \{T_k : T \in \mathcal{T}_s\}$.

A Λ_k–*packing* N is a subgraph of G such that every component of N is a k–edge path. Let $\lambda_k(G)$ denote the maximum number of disjoint k–edge paths in G. Obviously $\lambda_1(G) = \tau_1(G)$ is the size of a maximum matching in G, $\lambda(G) = \lambda_2(G) = \tau_2(G)$, and $\lambda_k(G) \leq \tau_k(G)$.

4.1 *If $T \in \mathcal{T}_s$ and $v(T) \geq s + 1$ then $\lambda_k(T_k) = \tau_k(T_k) = (v(T_k) - k)/(sk - k + 1)$.*

Proof Let $T \in \mathcal{T}_s$. We prove our claim on $\lambda_k(T_k)$ by induction on $v(T)$. The proof on $\tau_k(T_k)$ is similar.

If $v(T) = s + 1$, then $T = K_{1,s+1}$ and our claim is obviously true. So suppose $v(T) > s + 1$. Let L^x denote the set of leaves in T adjacent to a vertex x. Since $T \in \mathcal{T}_s$ and $v(T) > s + 1$, it follows that T has a vertex z with $|L^z| = s - 1$. Let $T' = T - L^z$. Then obviously $v(T') = v(T) - (s - 1)$ and $T' \in \mathcal{T}_s$. Let Y^z be the

subgraph of T induced by $z \cup L^z$. Clearly Y^z is isomorphic to $K_{1,s-1}$. Let Y^z_k be obtained from Y^z by subdividing each edge by $k-1$ vertices. Then $T'_k) = T_k - Y^z_k$. It is easy to see that $\lambda_k(Y^z_k) = 1$, $v(T_k) = v(T'_k) + v(Y^z_k)$, $v(Y^z_k) = sk - k + 1$, and T_k has a maximum Λ_k–packing $\mathcal{P}$ that contains exactly one k–edge path from Y^z_k. Hence $\lambda_k(T_k) = \lambda_k(T'_k) + \lambda_k(Y^z_k)$. Clearly $v(D') < v(D)$. Therefore by the induction hypothesis, $\lambda_k(T'_k)) = (v(T'_k) - k)/(sk - k + 1)$. Then $\lambda_k(T_k)) = (v(T'_k) - k)/(sk - k + 1) + \lambda_k(Y^z_k) = (v((T_k)) - (sk - k + 1) - k)/(sk - k + 1) + 1 = (v(T_k) - k)/(sk - k + 1)$. □

Let Y denote the graph obtained from three disjoint cycles A_i of five vertices by adding a new vertex a and three edges $\{aa_1, aa_2, aa_3\}$ where $a_i \in V(A_i)$, $i = 1, 2, 3$. It is easy to see that $\lambda(Y) = v(Y)/4 = 4$. Here is a more general construction of extremal graphs.

4.2 *Let H be a graph with possible loops and parallel edges such that each vertex of H is of degree at least three and at most s. Let G_k be a graph obtained from H by subdividing every edge by exactly $k \geq 1$ vertices. Then $\lambda_k(G_k) = \tau_k(G_k) = v(H) \leq v(G_k)/(s+1)$ and the equality holds if and only if H is an s–regular graph.*

Proof Obviously $\lambda_k(G) \leq \tau_k(G)$. Let $\mathcal{P}$ be a Λ_k–packing in G. Then each k–edge tree in $\mathcal{P}$ contains a vertex from $V(H)$. Therefore $\lambda_k(G) \leq \tau_k(G_k) \leq v(H)$. We can assume that H is connected. Let T be a spanning tree of H. Obviously there is an edge $e = xy \in E(H) - E(T)$. Let $D = T \cup e$ and D_x be the directed graph obtained from D by directing each edge of $T \subset D$ towards x and by directing e from x to y. For $z \in D_x$, let $p(z)$ denote the edge in D_x with the tail z. Clearly $p : V(D'_x) \to E(D'_x)$ is a bijection. Let $\ddot{D}$ be obtained from D by subdividing each edge of D by exactly k vertices. We can assume that $\ddot{D} \subseteq G_k$. Let $\mathcal{P}$ denote the set of k–edge paths zPt in $\ddot{D}$ such that $z \in V(D)$ and $V(P - z)$ is the set of vertices of $\ddot{D}$ that subdivide $p(z)$. Then clearly $\mathcal{P}$ is a Λ_k–packing in $\ddot{D}$, and therefore in G_k, and $|\mathcal{P}| = v(T) = v(H)$. Therefore $\lambda_k(G) = \tau_k(G_k) = |\mathcal{P}| = v(H)$.

Obviously, $2e(H) \leq sv(H)$ and $v(G) = v(H) + 2e(H) \leq (s+1)v(H)$. Therefore $\lambda(G_k) = \tau_k(G_k) \leq v(G_k)/(s+1)$ and the equality holds if and only if H is an s–regular graph. □

¿From **4.2** we have, in particular, for graphs in $\mathcal{G}^3_2$:

4.3 *Let $k \geq 1$ be an integer, H be a cubic graph with possible loops and parallel edges. Let F_k be a graph obtained from H as follows:*

if $e \in E(H)$ and e is not adjacent to a loop then subdivide e by exactly k vertices,

if l is a loop in H rooted in a vertex x and $e = xy$ is an edge incident to x, then subdivide l by $k(l) \geq 2$ vertices and e by $k(e) \geq 0$ vertices such that $k(l) + k(e) = 2k$.

Then $\lambda_k(F_k) = \tau_k(F_k) = v(H) = v(F_k)/4$.

Proof Let G_k be the graph obtained from H as described in **4.2**. It is easy to see that $\lambda_k(F_k) = \lambda_k(G_k)$. Therefore our claim follows from **4.2**. □

5 Proof of Theorem 1.6

First we will establish some properties of a hypothetical minimum counterexample to theorem **1.6**. We will use these properties to proof theorem **1.6** by showing that no counterexamples exist.

In all claims below, except for **1.6**, we assume that G is a connected graph satisfying the following conditions:

$(a1)$ G is an $\mathcal{F}_2^3$–minimal graph,

$(a2)$ $\lambda(G) < v(G)/4$, and

$(a3)$ G has the minimum number of vertices among all graphs satisfying $(a1)$ and $(a2)$.

5.1 *Let aTb be a thread of G. If neither a nor b belongs to a 5–leaf, then $e(T) \in \{2, 3\}$.*

Proof (uses **3.1** and **3.2**). Suppose, on the contrary, that $e(T) \notin \{2, 3\}$. Obviously $e(T) \geq 1$. If T has exactly one edge u, then $G - u \in \mathcal{F}_2^3$, and so G is not $\mathcal{F}_2^3$–minimal, a contradiction. Therefore we have exactly one of the following two cases:

$(c1)$ $e(T) \in \{3t + 1, 3t + 2\}$ where $t \geq 1$ and

$(c2)$ $e(T) = 3t$ where $t \geq 2$.

(**p1**) Suppose that $e(T) \in \{3t + 1, 3t + 2\}$ where $t \geq 0$. Let, as in **3.1**, $G' = G - (T - L(T))$. Since $G \in \mathcal{F}_2^3$ and neither a nor b belongs to a 5–leaf, we have $G' \in \mathcal{F}_2^3$. Clearly $v(G') < v(G)$. By the minimality of G, $\lambda(G') \geq v(G')/4$. Then by **3.1**, $\lambda(G) \geq v(G)/4$, a contradiction.

(**p2**) Now suppose that $e(T) = 3t$ where $t \geq 2$. Let G' be the graph defined in **3.2**. Since $G \in \mathcal{F}_2^3$, obviously $G' \in \mathcal{F}_2^3$. Clearly $v(G') < v(G)$. By the minimality of G, $\lambda(G') \geq v(G')/4$. Then by **3.2**, $\lambda(G) \geq v(G)/4$, a contradiction. □

5.2 *G has no 5–leaves.*

Proof (uses **3.3**, **3.4**, **3.6**, and **5.1**). Suppose, on the contrary, that G has a 5–leaf X_1. Let aT_1x_1 be a (unique) thread in G such that $x_1 \in V(X_1)$.

(**p1**) Suppose that x_1 belongs to a cycle–leaf. Then, obviously, $\lambda(G) \geq v(G)/4$, a contradiction.

(**p2**) Suppose that $e(T_1) \geq 2$. Let, as in **3.3**, $G' = G - \grave{A}$. Since $G \in \mathcal{F}_2^3$ and b belongs to no 5–leaf, we have $G' \in \mathcal{F}_2^3$. Clearly $v(G') < v(G)$. By the minimality of G, $\lambda(G') \geq v(G')/4$. Then by **3.3**, $\lambda(G) \geq v(G)/4$, a contradiction.

(**p3**) Let aT_2x_2 and aT_3x_3 be the threads ending at a and distinct from aT_1x_1. By (**p1**), x_1 belongs to no 5–leaves. Therefore $T_2 \neq T_3$. We can assume that $e(T_2) \leq e(T_3)$.
If x_i belongs to no 5–leaf then by **5.1**, $e(T_i) \in \{2, 3\}$.

(**p3.1**) Suppose that neither x_2 no x_3 belongs to a 5–leaf. Then $e(T_i) \in \{2, 3\}$ for $i = 1, 2$. Let G' be the graph defined in **3.4**. Then $G \in \mathcal{F}_2^3$ implies $G' \in \mathcal{F}_2^3$. Clearly $v(G') < v(G)$. By the minimality of G, $\lambda(G') \geq v(G')/4$. Then by **3.4**, $\lambda(G) \geq v(G)/4$, a contradiction.

(**p3.2**) Now suppose that x_i belongs to a 5–leaf for some $i \in \{2,3\}$. Then by (**p2**), $e(T_i) = 1$, and so we can assume that $i = 2$. Let G' be the graph defined in **3.6**. Then $G \in \mathcal{F}_2^3$ implies $G' \in \mathcal{F}_2^3$. Clearly $v(G') < v(G)$. By the minimality of G, $\lambda(G') \geq v(G')/4$. Then by **3.6**, $\lambda(G) \geq v(G)/4$, a contradiction. □

5.3 *Every path–thread in G has exactly three edges.*

Proof (uses **3.5**, **5.1**, and **5.2**). By **5.2**, G has no 5–leaves. Let aT_1x_1 be a path–thread in G. By **5.1**, $e(T_1) \in \{2,3\}$. Suppose, on the contrary, that $e(T_1) = 2$. If each end–vertex of T belongs to a cycle–leaf then, obviously $\lambda(G) \geq v(G)/4$. Therefore we can assume that one end–vertex, say a, of T_1 belongs to no cycle–leaf. Let aT_1x_1, aT_2x_2, and aT_3x_3 be the three threads ending at a.

Since G has no 5–leaves, by **5.1**, $e(T_2) \in \{2,3\}$ and $e(T_3) \in \{2,3\}$. Let G' be the graph defined in **3.5**.

(**p1**) Suppose that $x_1 \neq x_2 = x_3$ and $e(T_2) = 2$ and $e(T_3) = 3$. Then, obviously, $\lambda(G) \geq v(G)/4$. Since $G \in \mathcal{F}_2^3$, we have $G' \in \mathcal{F}_2^3$. Clearly $v(G') < v(G)$. By the minimality of G, $\lambda(G') \geq v(G')/4$. Then by **3.5**, $\lambda(G) \geq v(G)/4$, a contradiction.

(**p2**) Suppose that $e(T_2) = e(T_3) = 2$. Then $G' = G - T_1 \cup T_2 \cup T_3 - \{x_1, x_2, x_3\}$. We know that $G \in \mathcal{F}_2^3$ and that neither x_1 nor x_2 nor x_3 belongs to a 5–leaf.

Suppose that G' has a 5–component C. Since G has no 5–leaves, $V(C) \cap \{x_1, x_2, x_3\}| \geq 2$. Since $e(L) \in \{2,3\}$ for every path–thread L in G, we have $V(C) \cap \{x_1, x_2, x_3\}| = 2$, say $x_1, x_2 \in V(C)$, and $x_1x_2 \notin E(G)$. Then we have a contradiction with (**p1**) if we put $a := x_1$ and $x_2 = x_3 := x_2$. Therefore G' has no 5–components. Now since $G \in \mathcal{F}_2^3$, we have $G' \in \mathcal{F}_2^3$. Clearly $v(G') < v(G)$. By the minimality of G, $\lambda(G') \geq v(G')/4$. Then by **3.5**, $\lambda(G) \geq v(G)/4$, a contradiction. □

5.4 *Every leaf of G is a triangle.*

Proof Let A be a leaf of G and T be the stem of A.

Suppose that $v(\grave{A}) \neq 5$. Let, as in **3.4**, $G' = G - \grave{A}$. By **5.2**, G has no 5–leaves. Therefore $G \in \mathcal{F}_2^3$ implies $G' \in \mathcal{F}_2^3$. Clearly $v(G') < v(G)$. By the minimality of G, $\lambda(G') \geq v(G')/4$. Then by **5.2**, $\lambda(G) \geq v(G)/4$, a contradiction.

Now suppose that $v(\grave{A}) = 5$, i.e. $v(A) + e(T) = 6$. By **5.3**, $e(T) = 3$. Therefore $v(A) = 3$. □

Now we are ready to prove our first main theorem:

1.6 *Let $G \in \mathcal{F}_2^3$. Then $\lambda(G) \geq v(G)/4$.*

Proof (uses **2.1**, **4.2**, **5.3**, and **5.4**). Suppose, on the contrary, that our claim is not true. Obviously, a triangle is the minimum graph in $\mathcal{F}_2^3$ and our claim is true for a triangle. Let G be a vertex minimum counterexample. By **2.1**, we can assume that G is $\mathcal{F}_2^3$–minimal. In other words, G is a graph such that

($a1$) G be an $\mathcal{F}_2^3$–minimal graph,

($a2$) $\lambda(G) < v(G)/4$, and

($a3$) G has the minimum number of vertices among all graphs satisfying ($a1$) and ($a2$).

By **5.3**, each path–thread of G has exactly three edges. By **5.4**, each leaf of G is a triangle. Then by **4.2**, $\lambda(G) = v(G)/4$, a contradiction. □

6 Proof of Theorem 1.7

In all claims below, except for **1.7**, we assume that G is a connected graph satisfying the following conditions:

$(a1)$ G is an $\mathcal{G}_2^s$–minimal graph, $s \geq 4$,

$(a2)$ $\lambda(G) < v(G)/(s+1)$, and

$(a3)$ G has the minimum number of vertices among all graphs satisfying $(a1)$ and $(a2)$.

6.1 *If T is a path–thread of G, then $e(T) \in \{2,3\}$. If T is a cycle–thread of G, then $e(T) = 3$, i.e. T is a triangle.*

Proof (uses **3.1**). Suppose, on the contrary, that $e(T) \notin \{2,3\}$. Obviously $e(T) \geq 1$. If T has exactly one edge u, then $G - u \in \mathcal{G}_2^s$, and so G is not $\mathcal{G}_2^s$–minimal, a contradiction. Therefore $e(T) \geq 4$. Let, as in **3.1**, $G' = G - (T - L(T))$. Obviously $G \in \mathcal{G}_2^s$ implies $G' \in \mathcal{G}_2^s$. Clearly $v(G') < v(G)$. Therefore by the minimality of G, $\lambda(G) \geq v(G)/(s+1)$. Then by **3.1**, $\lambda(G) \geq v(G)/(s+1)$, a contradiction. □

6.2 *Every thread of G has exactly three edges.*

Proof (uses **3.5** and **6.1**). If T is a cycle–thread of G then by **6.1**, T is a triangle, and so $e(T) = 3$. So let us consider a path–thread $a_1T_1x_1$. By **6.1**, $e(T_1) \in \{2,3\}$.

Suppose, on the contrary, that $e(T_1) = 2$. Let $\{T_i : i \in \{1,\ldots,k\}\}$ be the set of threads of G with a common end–vertex a. Let G' be a graph defined in **3.5**. Then $G \in \mathcal{G}_2^s$ implies $G' \in \mathcal{G}_2^s$. Clearly $v(G') < v(G)$. Therefore by the minimality of G, $\lambda(G) \geq v(G)/(s+1)$. Then by **3.5**, $\lambda(G) \geq v(G)/(s+1)$, a contradiction. □

Now we are ready to prove theorem

1.7 *Let $G \in \mathcal{G}_2^s$ and $s \geq 4$. Then $\lambda(G) \geq v(G)/(s+1)$.*

Proof (uses **2.1**, **4.2**, and **6.2**). Suppose on the contrary, that our claim is not true. Obviously, a triangle is the minimum graph in $\mathcal{G}_2^s$ and our claim is true for a triangle. Let G be a vertex minimum counterexample. By **2.1**, we can assume that G is $\mathcal{G}_2^s$–minimal. In other words, G is a graph such that

$(a1)$ G be an $\mathcal{G}_2^s$–minimal graph,

$(a2)$ $\lambda(G) < v(G)/(s+1)$, and

$(a3)$ G has the minimum number of vertices among all graphs satisfying $(a1)$ and $(a2)$.

By **6.2**, each thread of G has exactly three edges. By **4.2**, $\lambda(G) = v(G)/(s+1)$, a contradiction. □

7 Proof of Theorem 1.9

A subtree B of a tree is called a *branch* of T if either $B = T$ or $T - B$ is also a tree. If $B \neq T$ then let $r(B)$ be the vertex of B adjacent to a vertex in $T - B$. We call $r(B)$ the *root* of the branch B.

Let $\tau_k(G)$ denote the maximum number of disjoint trees of k edges in G.

7.1 *Let T be a tree, $T \in \mathcal{G}_1^s$, and $v(T) \geq k+1$. Then T has a branch B such that $v(B) \leq (s-1)k+1$ and $\tau_k(B) = 1$.*

Proof A branch B of a tree $T \in \mathcal{G}_1^s$ is called *k–good* if $v(B) \leq (s-1)k+1$ and $\tau_k(B) = 1$. We prove our claim by induction on $v(T)$. If $v(T) = k+1$, then our claim is obviously true. Therefore let $v(T) > k+1$.

Let x be a leaf of T and $T' = T - x$. Clearly T' is a tree, $T' \in \mathcal{G}_1^s$, and $v(T') < v(T)$. Therefore by the induction hypothesis, T' has a k–good branch B'. Let $r' = r(B')$ and $xy \in E(T)$. Let us assume that B' is a k–good branch in T of the minimum number of vertices. If $y \in V(T - (B' - r')$ then B' is a required branch of T. Therefore let $y \in V(B' - r')$. Since $\tau_k(B') = 1$ and B' is a vertex minimum k–good branch in T', clearly $\tau_k(C') = 0$ for every component C' of $B' - r'$, and so $v(C') \leq k$. Therefore every subtree S in B' of $k+1$ vertices contains r'. Let X' be the component of $B' - r'$ containing x. Let $X = X' \cup xy$ and $B = B' \cup xy$. Clearly B is a branch of T and $r' = r$ is the root of B.

Suppose that $v(X') < k$. Then $v(B') < (s-1)k+1$. Hence $v(X) \leq k$ and $v(B) = v(B') + 1 \leq (s-1)k+1$. Since $\tau_k(B') = 1$, clearly $\tau_k(B) \geq 1$, and so B has a subtree of k edges. Since $v(C) \leq k$ for every component C of $B - r$, every subtree of k edges in B contains r, and so $\tau_k(B') = 1$. Therefore B is a k–good branch in T.

Now suppose that $v(X') = k$. Then X is a subtree of T having k edges, X is a branch of T and $v(X) \leq v(B')(s-1)k+1$. Therefore X is a k–good branch in T. □

Now we are ready to prove theorem

1.9 *Let s and k be integers, $s \geq 3$ and $k \geq 1$, $G \in \mathcal{G}_1^s$, and G has no k–vertex components. Then $\tau_k(G) \geq (v(G) - k)/(sk - k + 1)$.*

Proof (uses **7.1**). Let $G \in \mathcal{G}_1^s$. We can assume that G is connected. Since G has no k–vertex component, we have $v(G) \geq k+1$. Let T be a spanning tree of G. Clearly $T \in \mathcal{G}_1^s$ and $\lambda_k(G) \geq \lambda_k(T)$. Therefore it is sufficient to proof our theorem for every tree T in $\mathcal{G}_1^s$.

We prove our claim for trees by induction on $v(T)$. If $v(T) = k+1$, then our claim is obviously true. So we assume that $v(T) > k+1$. By **7.1**, T has a branch B such that $v(B) \leq (s-1)k+1$ and $\tau_k(B) = 1$. Let $T' = T - B$. Obviously $T' \in \mathcal{G}_1^s$, $v(T') = v(T) - v(B) < v(T)$, and $\tau_k(T) \geq \tau_k(T') + \tau_k(B)$. By the induction hypothesis, $\tau_k(G') \geq (v(G') - k)/(sk - k + 1)$. Therefore $\tau_k(G) \geq (v(G') - k)/(sk - k + 1) + 1 = (v(T) - v(B) - k)/(sk - k + 1) + 1$. Since $v(B) \leq (s-1)k+1$, we have $\tau_k(G) \geq (v(T) - (s - k + 1) - k)/(sk - k + 1) + 1 = (v(G) - k)/(sk - k + 1)$. □

In Section 4 we gave a construction that provides infinitely many trees T in $\mathcal{G}_1^s$ such that $\tau_k(G) \geq (v(G) - k)/(sk - k + 1)$ (see **4.1** above).

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