DIMACS Technical Report 2000-42
December 2000

Perfect Dominating Sets of Grids

by

Italo J. Dejter [1]
Department of Mathematics and Computer Science
University of Puerto Rico, Rio Piedras, PR 00931-3355

[1]This work was performed while the author was on leave from University of Puerto Rico and visiting DIMACS and Denmark Technical University during the academic year 1999-2000.

ABSTRACT

Let m and n be positive integers. The algorithmic search for perfect dominating sets of the rectangular grid $\mathcal{G}_{m,n}$ with an initial condition S' defined as an admissible subset of a side $\mathcal{G}_{m,1}$ of $\mathcal{G}_{m,n}$ is considered. A binary decision algorithm that generates all perfect dominating sets of $\mathcal{G}_{m,n}$ satisfying (or containing) such S' is presented, and some related questions and conjectures are posed, leading to consideration of their periodic extendibility to the plane lattice as well as to extremal properties of the grid depth n for a fixed grid width m under the presence of such dominating sets.

1 Introduction

1.1 Preliminaries

A vertex subset S of a graph G is called a *perfect dominating set* (PDS) of G if any vertex v of G not in S is adjacent to exactly one element of S. An isolated PDS of G is called an *efficient dominating set*, ([6, 9]), or a 1-*perfect code*. [10] proved that the problem of finding a 1-perfect code of a graph is NP-complete. The study of PDS's of n-cubes was addressed in [12, 7, 3, 5, 1, 2, 11, 4]. On the other hand [8] proved that the problem of finding a minimal PDS of a planar graph is NP-complete, (where a different terminology is used: what here is called a PDS is referred as a semiperfect dominating set, while the name 'PDS' is reserved for 1-perfect codes).

Let the *plane lattice* $\mathcal{L}$ be the graph with vertex set $\{(i,j) : i,j \in \mathcal{Z}\}$ and such that any two vertices are adjacent if and only if their Euclidean distance is 1. It is easy to see that $\mathcal{L}$ contains an uncountable family of PDS's, namely in one-to-one correspondence with the set of infinite $\{0,1\}$-sequences. Thus there is no algorithmic characterization of PDS's for $\mathcal{L}$.

A path P_r of length $r-1$, where $1 \leq r \in \mathcal{Z}$, is a connected graph containing r vertices and $r-1$ edges. The notation P_r extends to $P_1 = \{v\}$, formed by a single vertex v, and to the infinite path P_∞, a connected graph with one vertex u of degree 1 and such that the vertex set of $P_\infty - \{u\}$ is enumerable with all its vertices having degree 2 in P_∞. If $r = 1$ (respectively $r = \infty$) we say that P_r is a path of length 0 (respectively ∞).

Proposition 1.1 *Each connected component of the subgraph $\mathcal{L}[S]$ induced by a* PDS *S of $\mathcal{L}$ is a product subgraph $P_r \times P_s$.*

Proof: Assume that three vertices u, v, w of $Q = \{(i,j), (i+1,j), (i,j+1), (i+1,j+1)\}$ are members of a PDS S of $\mathcal{L}$. Clearly, u, v, w are in a common component $\mathcal{C}$ of $\mathcal{L}[S]$. Then the remaining vertex of Q is also a member of $\mathcal{C}$ because of the definition of PDS. By reiterating this argument, the conclusion follows. ∎

Remark: A similar conclusion can be drawn for the case of higher dimensional lattices. Compare with the conclusion of [12] that the components of the induced subgraph of a PDS's of the n-cube are subcubes. ▬

Let $\mathcal{G}_{m,n} \subseteq \mathcal{L}$ be the graph product $P_m \times P_n$, where $1 \leq m \in \mathcal{Z}$ and $n \in \{0, 1, 2, \ldots, \infty\}$. A *side* of $\mathcal{G}_{m,n}$ is a subgraph $P_m \times \{v\}$ or $\{u\} \times P_n$ of $\mathcal{G}_{m,n}$, where v (respectively u) is an endvertex of P_n (respectively P_m). We say that the lengths of P_m and P_n are respectively the *width* and the *depth* of $\mathcal{G}_{m,n}$. If $1 < n \in \mathcal{Z}$ then $\mathcal{G}_{m,n}$ is a finite rectangular grid. If $n = 1$ then $\mathcal{G}_{m,n}$ is a path of length $m-1$. If $n = \infty$ then $\mathcal{G}_{m,n}$ is an extended rectangular grid with one side of width $m-1$ and two infinite parallel sides.

Corollary 1.1 *Each component of the subgraph $\mathcal{G}_{m,n}[S]$ induced by a* PDS *S of $\mathcal{G}_{m,n}$, where $1 \leq m$ and $1 \leq n \in \mathcal{Z}$ or $n = \infty$, is a product graph $P_r \times P_s$, where $1 \leq r \leq m$ and $1 \leq s \leq n$ or $n = \infty$.*

Without loss of generality, we identify the vertex set of $\mathcal{G}_{m,n}$ with the set of points in the Cartesian plane having coordinates (i,j) such that $0 \leq i < m$, $0 \leq j$ and if n is finite then

$j < n$. Thus we have the following graph containments:

$$\mathcal{G}_{m,1} \subset \mathcal{G}_{m,2} \subset \mathcal{G}_{m,3} \subset \ldots \mathcal{G}_{m,n} \subset \ldots \subset \mathcal{G}_{m,\infty}.$$

1.2 A Natural Question

It is natural to consider problems of existence of certain objects given some initial condition(s). For example, in looking for PDS's of a graph G with a distinguished subgraph G' such an initial condition could be given as a PDS S' of a subgraph G'' of G' so that we could consider the problem of existence of a PDS S of G containing S'. Say $G = \mathcal{G}_{m,n}$ with $m > 1$, $n > 1$ and $G' = \mathcal{G}_{m,1} = P_m \times \{0\}$ (viewed as the 'top' side of G so points in the Cartesian plane with positive second coordinates are represented below $\mathcal{G}_{m,1}$); in this case S' is the only PDS of G'': there are no other PDS's of G'' other than S'. The following question arises.

Question 1.1 *Given a graph G, a distinguished subgraph $G' \subseteq G$ and a unique* PDS *S' of a subgraph G'' of G', does there exist a* PDS *S of G containing S'?*

Question 1.1 addresses the problem of determining a PDS of G given an initial condition constituted by the vertex subset S' of $G' \subseteq G$. The following proposition and subsequent corollary show that having the proposed initial condition as a PDS S' of a subgraph G'' of the subgraph $G' \subset G$ is the adequate setting for the problem.

Proposition 1.2 *A necessary condition for the existence of S in* Question 1.1 *is that there are no two components of the induced subgraph $G'[S'] = G[S']$ of G' or G at distance* 2.

Proof: Assuming that two components $\mathcal{C}_1, \mathcal{C}_2$ of $G'[S']$ are at distance 2, a vertex of $G' \subseteq G$ internal in a path realizing that distance is dominated by respective boundary vertices in $\mathcal{C}_1, \mathcal{C}_2$, which are vertices of any PDS S of G containing S', contrary to the definition of a PDS. ∎

A vertex subset S' of G' satisfying the condition in Proposition 1.2 will be said to be *admissible*.

Corollary 1.2 *A vertex subset S' of G' is admissible if and only if $G'[S']$ has its components at distances ≥ 3. This is insured by our initial assumption that S' be a* PDS *of a subgraph G'' of G'.*

1.3 The Case For $G = \mathcal{G}_{m,n}$ and $G' = \mathcal{G}_{m,1}$

Proposition 1.3 *Let $G' = \mathcal{G}_0 = \mathcal{G}_{m,1}$. Then S' is a* PDS *of a subgraph of G' if and only if S' is a vertex subset of $G' = \mathcal{G}_{m,1}$ such that the distance between components of the induced subgraph of S' is ≥ 3.*

Proof: $G' = \mathcal{G}_0 = \mathcal{G}_{m,1}$ is a path graph. Thus, if S' is a PDS of a subgraph of G', then no two components of the subgraph induced by S' can be at distance < 3. Conversely, if the distance between such components is ≥ 3, then S' is the unique PDS of a well-defined subgraph of G'. ∎

¿From now on we restrict to the case $G' = \mathcal{G}_0 = \mathcal{G}_{m,1} \subseteq G = \mathcal{G}_{m,n}$. and ask for the existence of a PDS $S = S(m,n,S')$ of $\mathcal{G}_{m,n}$ containing S'.

Extending the definition of $\mathcal{G}_0$, let the *j-level* $\mathcal{G}_j$ of $\mathcal{G}_{m,n}$ be composed by those vertices of $\mathcal{G}_{m,n}$ whose second coordinate is j, for $0 \leq j < n$.

Elementary instances in the problem of determining a PDS of $\mathcal{G}_{m,n}$ containing an initial condition S' in $\mathcal{G}_0$ are given in the following two propositions.

Proposition 1.4 *If S' coincides with the vertex subset of $G' = \mathcal{G}_0$ and there exists a proper* PDS *$S = S(m,n,S')$ of $\mathcal{G}_{m,n}$ containing S' then $\mathcal{G}_{m,n}[S]$ has as its components those $\mathcal{G}_j$ for which j is divisible by 3. If in addition n is finite then either $n-2$ or $n-1$ is divisible by 3.*

Proof: It follows from Corollaries 1.1 and 1.2, noticing that if $n-1$ is divisible by 3 then the last level $\mathcal{G}_n \subseteq S$ and if $n-2$ is divisible by 3 then $\mathcal{G}_n$ is disjoint from S. ∎

Two components $\mathcal{C}_1, \mathcal{C}_2$ of the induced subgraph $\mathcal{G}_0[S']$ of an admissible vertex subset S' of $\mathcal{G}_0$ are said to be *contiguous* if any path contained in $\mathcal{G}_0$ and having an endvertex c_i in C_i, for $i = 1,2$, does not contain vertices of $\mathcal{G}_0 - (\mathcal{C}_1 \cup \mathcal{C}_2)$.

Proposition 1.5 *If the vertex subset S' of $\mathcal{G}_0$ is such that*

(a) *each two contiguous components of $\mathcal{G}_0[S']$ are at distance 3 and*

(b) *the endvertices of $\mathcal{G}_0$ are at distance ≤ 1 from S',*

then

1. *there exists only one* PDS *$S = S(m,n,S')$ of $\mathcal{G}_{m,n}$ containing S' and*

2. *the components of $\mathcal{G}_{m,n}[S]$ are the products of the components of $\mathcal{G}_0[S']$ by P_n.*

Proof: this follows from Corollaries 1.1 and 1.2. ∎

An admissible vertex subset S' of $\mathcal{G}_0$ is said to be *incomplete* if it is not as in the premises of Propositions 1.4 and 1.5.

Now we present an answer to Question 1.1 when $G = \mathcal{G}_{m,n}$, $G' = \mathcal{G}_0 = \mathcal{G}_{m,1}$ and S' is an *incomplete admissible initial condition*, that is: an incomplete admissible vertex subset of $\mathcal{G}_0$.

Theorem 1.1 *Let m be a fixed positive integer and S' be an incomplete admissible vertex subset of $G' = \mathcal{G}_0 = \mathcal{G}_{m,1}$. Then there exists a binary decision algorithm Θ that produces all sets $S(m,n,S')$ for $0 < n$.*

Proof: The binary decision algorithm Θ is presented in Section 2. The proof of the theorem is completed in Section 4, Corollary 4.1. ∎

2 Binary Decision Algorithm

The binary decision algorithm Θ in the statement of Theorem 1.1 can be expressed as $\Theta = \Theta(\alpha, \beta)$, where α, β are the two possible choices for the continuation of Θ at each binary decision instance of its development, so that a binary decision tree T_Θ is generated. Each one of the internal nodes of T_Θ, representing a partial solution to the determination of an $S(m, n, S')$, descends say to the left for α and to the right for β. The algorithm Θ is conceived so that m and S' are fixed, with n increasing one level at a time for each one of the branchings of T_Θ and termination in any branch that completes an $S(m, n, S')$. Some branches may grow forever, in which case we would have sets $S(m, \infty, S')$ showing periodicity in their stepwise generation, (see Subsection 5.2).

Algorithm $\Theta = \Theta(\alpha, \beta)$ is given in the following four subsections.

2.1 Vertex-Labeling Initialization

A function f from the set of vertices v of $\mathcal{G}_{m,n}$ onto $\{0, 1, 2, 3, 4\}$ is defined. We initialize f by $f(v) := 0$ for every such vertex v. Then, f is redefined on $\mathcal{G}_0$ as follows:

A. if $v \in S'$, then $f(v) := 2$;

B. if $v \notin S'$ is adjacent on its right, (left), to the leftmost, (rightmost), vertex of a component of $\mathcal{G}_0[S']$, then $f(v) := 1$, ($f(v) := 3$);

Proposition 2.1 *An admissible vertex subset S' of $\mathcal{G}_0$ is incomplete if and only if $f(v) = 0$ for some vertex v of $\mathcal{G}_0$.*

Proof: Observe that $f(v) = 0$ if and only if either there exist two contiguous components of $\mathcal{G}_0[S']$ at distance > 3 or at least one endvertex of $\mathcal{G}_0$ at distance > 1 from S'. ∎

We require that S' be an incomplete admissible vertex subset of $\mathcal{G}_0$ in order to avoid PDS's S of $\mathcal{G}_{m,n}$ with all the components of $\mathcal{G}_{m,n}[S]$ being products of corresponding components of S' with a fixed P_n or P_∞, a simple situation.

The sequence of steps in Subsections 2.2 to 2.4 below must be applied initially for $j = 0$. This produces a redefinition of f on $\mathcal{G}_1$ using effectively labels $f(v) \in \{0, 1, 2, 3, 4\}$ not all equal to 0, since S' is assumed to be incomplete. Assuming that such redefinition of f was already done for all $\mathcal{G}_\ell$'s, for $0 \le \ell \le j < n$, a further application of the sequence of steps redefines f on $\mathcal{G}_{j+1}$. Then, in the union $\mathcal{G}_{m,j+2}$ of all $\mathcal{G}_\ell$'s, for $0 \le \ell \le j + 1$, some vertices v admit labels $f(v)$ in $\{0, 1, 2, 3\}$, with the same meaning as in the redefinition of f on $\mathcal{G}_0$; if there are remaining vertices $w = (i, j + 1)$, they are labeled $f(w) = f(i, j + 1) = 4$, meaning that $f(i, j) = 2$ and (i, j) is the only vertex dominating w among those labeled with 2, which eventually could form part of a PDS. If $\mathcal{G}_{j+1}$ is incomplete, that is if it has some v's with $f(v) = 0$, then the set S of vertices w of $\mathcal{G}_{m,j+2}$ with $f(w) = 2$ is called a *quasiperfect dominating set* (QPDS) of $\mathcal{G}_{m,j+2}$; in fact, those v's are not dominated by S. In this case, the sequence of steps proceed with a redefinition of f on $\mathcal{G}_{j+2}$ and so on.

Assume now that f is already defined for all vertices on $\mathcal{G}_\ell$, for $0 \le \ell \le j$.

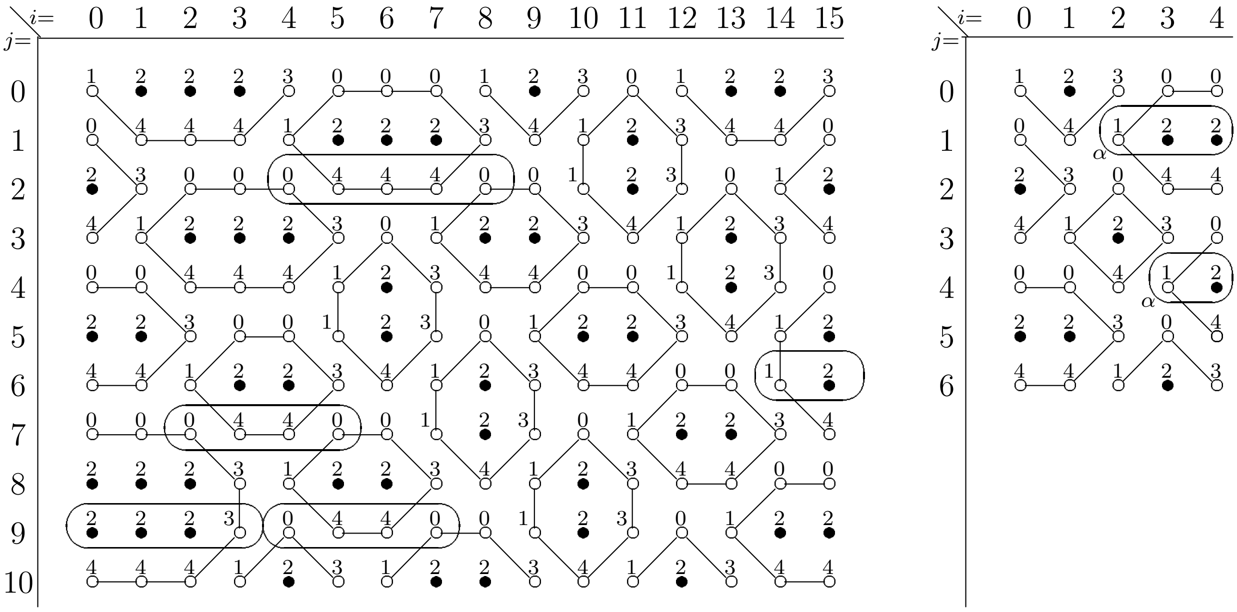

Figure 1: Two applications of Θ

2.2 Labeling Progressively Vertices of $\mathcal{G}_{j+1}$

1. For $i := 0$ to $m-1$ do if $f(i,j) = 0$ then

   (A) if $i > 0$ and $f(i-1,j) > 0$ then: **(a)** $f(i-1,j+1) := 1$; **(b)** if $i > 1$ then: **(b1)** $k := i-2$; **(b2)** while $f(k,j) = 2$ do: **(i)** $f(k,j+1) := 4$; **(ii)** $k := k-1$;

   (B) if $i < m-1$ and $f(i+1,j) > 0$ then: **(a)** $f(i+1,j+1) := 3$; **(b)** if $i < m-2$ then: **(b1)** $k := i+2$; **(b2)** while $f(k,j) = 2$ do: **(i)** $f(k,j+1) := 4$; **(ii)** $k := k+1$;

   (C) $f(i,j+1) := 2$;

2. For $i := 0$ to $m-3$ do if $f(i+k,j) = k+1$ and $f(i+k,j+1) = 0$ for all $k = 0,1,2$ then for $k := 0$ to 2 do $f(i+k,j+1) := k+1$;

2.3 Decision Between Choices α, β

3. If $f(0,j) = 2$ and $f(0,j+1) = 0$ then: **(a)** $k := 0$; **(b)** while $f(k,j) = 2$ and $f(k,j+1) = 0$ do $k := k+1$; **(c)** either

   (α) for $\ell := 0$ to $k-1$ do $f(\ell,j+1) := 4$; or

   (β) **(i)** for $\ell := 0$ to $k-1$ do $f(\ell,j+1) := 2$; **(ii)** $f(k,j+1) := 3$;

   (This will be referred to as a binary *outer* decision or BOD, to be used in Section 5).

4. For $i := 0$ to $m-2$ do if $f(i,j) = 1$ and $f(i+1,j+1) = 0$ then: **(a)** $k := i+1$; **(b)** while $f(k,j) = 2$ and $f(k,j+1) = 0$ do $k := k+1$; **(c)**

   if $k \leq m-1$ then: if $f(k,j+1) = 0$ then either

(α) for $\ell := i+1$ to $k-1$ do $f(\ell, j+1) := 4$; or

(β) **(i)** $f(i,j{+}1) := 1$; **(ii)** $f(k,j{+}1) := 3$; **(iii)** for $\ell := i{+}1$ to $k{-}1$ do $f(\ell,j{+}1) := 2$;

(This will be referred to as a binary *inner* decision or BID).

else, (i.e. if $k = m$), either

(α) for $\ell := i+1$ to $m-1$ do $f(\ell, j+1) := 4$; or

(β) **(i)** $f(i, j+1) := 1$; **(ii)** for $\ell := i+1$ to $m-1$ do $f(\ell, j+1) := 2$;

(This will again be referred to as a binary *outer* decision or BOD).

2.4 Checking PDS Formation

5. Let $\tau(j) = |\{i \in \{1,\ldots,m{-}1\}; f(i,j{+}1) = 0\}|$; if $\tau(j) = 0$, then the vertices $v = (i,\ell)$ for which $0 \le \ell \le j{+}1$ and $f(i,\ell) = 2$ form by a PDS S of $\mathcal{G}_{m,j+2}$; otherwise, if $j < n$: **(a)** $j := j+1$; **(b)** go to step 1.

3 Examples

In order to illustrate Θ, we represent $\mathcal{G}_{m,n}$ orthogonally, with columns indicated from left to right for indices $i = 0,\ldots,m-1$ and rows representing levels $\mathcal{G}_j$ indicated from top to bottom for $j = 0,\ldots,n$. The vertices of a resulting QPDS or PDS S are represented by black nodes and the remaining ones by white nodes. Figure 1 shows two PDS's of $\mathcal{G}_{16,11}$ and $\mathcal{G}_{5,7}$, respectively, obtained via Θ, where ovals encircle those vertex subsets that are affected by binary decisions between choices α, β in Subsection 2.3.

Notice that on the left of Figure 1, $\mathcal{L}[S']$ is formed by a component induced by each one of the three vertex subsets:

$$\{(1,0),(2,0),(3,0)\},\ \{(9,0)\},\ \{(13,0),(14,0)\}.$$

Thus, after the initialization of f, the successive values of f in $\mathcal{G}_0$ for $i = 0,1,\ldots,15 = m-1$ are: 1222300012301223. The successive values of f in $\mathcal{G}_1$ after step 1 are: 0444122234123440, where the zeros are remnants of the initialization of f. Steps 2 to 5 are passed through with no variations and in step 5, after setting $j = 1$, the process is sent to step 1 with $j = 2$. After step 1, f on $\mathcal{G}_2$ looks like: 2300?444?0???012, where ? stands for those values of f equal to 0 that may still vary in the future. After step 2, f on $\mathcal{G}_2$ is: 2300?444?0123012. A first decision must be taken now according to step 4, for $i = 4$ and $k = 8$; Figure 1 shows that α was taken, yielding the 2-level: 2300044400123012. The rest of Θ is applied similarly. Figure 1 shows that the successive decisions happened at j =2,6,7,9,9 for respective steps 4(BID),4(BOD),4(BID),3(BOD),4(BID) with $(i,k) = (4,8)$, $k = 14$, $(i,k) = (2,5)$, $k = 3$, $(i,k) = (4,6)$ and that the choices were $\alpha,\beta,\alpha,\beta,\alpha$. We leave for the reader to check the right example in Figure 1.

To ease the comprehension of Θ, Figure 1 contains some additional segments η between white nodes, composing closed polygons π going around each component of the induced subgraph of S, unless such a component touches the boundary $\mathcal{B}$ of $\mathcal{G}_{m,n}$, in which case π remains open in its intersection with $\mathcal{B}$. Each segment η may be horizontal, vertical or a diagonal ξ at an angle of ± 45 deg. Each diagonal ξ_0 is incident in its corresponding π to at most one horizontal η, at most one vertical η and at most two ξ's perpendicular to ξ_0. This is due to the product-graph structure of each component seen in Corollary 1.1. As a result, the π's involved have special shapes: octagons, hexagons, rhombi (or tilted squares, as visible in $\mathcal{G}_{5,7}$, Figure 1), or intersections of these with $\mathcal{B}$. Figure 1 has the representation of each vertex v of $\mathcal{G}_{m,n}$ labeled with an integer $f(v)$ in $\{0,1,2,3,4\}$, according to the following rule: if v is: **(a)** black, then $f(v)=2$; **(b)** white ($\Rightarrow$ contained in a π as above), then $f(v)=0$ (respectively 4, 1, 3) if it is above (respectively below, at left, at right) of the component of $\mathcal{G}_{m,n}[S]$ enclosed by π.

4 PDS Generation and Exhaustion

Theorem 4.1 *If there exists $0<j\in\mathcal{Z}$ such that $\tau(j)=0$ in step* 5, Subsection 2.4, *then Θ yields an $S(m,j+2,S')$ formed by those vertices v for which $f(v)=2$.*

Proof: Assume the f-labeling at levels $\mathcal{G}_0,\mathcal{G}_1,\ldots,\mathcal{G}_j$ is completed. Then step 1 forces the generation of new values of f at level $\mathcal{G}_{j+1}$ with a widening "bell" effect: every component C^0 of the subgraph induced by the 0-labeled vertices at level $\mathcal{G}_j$ with vertex set $\{(i,j),\ldots,(k,j)\}$, $(0\le i\le k<m)$, has a corresponding component C^2 in the subgraph induced by the new 2-labeled vertices at level $\mathcal{G}_{j+1}$, with vertex set $\{(i,j+1),\ldots,(k,j+1)\}$, plus a 1- and a 3-labeled vertices at positions $(i-1,j+1)$ and $(k+1,j+1)$, respectively if $i\ge 1$ and $k+1<m$, as in the two upper levels of each polygon in Figure 1. This bell effect creates restrictions on the selection of f for the remaining values at $\mathcal{G}_{j+1}$. The attribution of label 4 to the sides of each bell "island", (or sequence of contiguous values 122...223), in $\mathcal{G}_{j+1}$ and steps 2,3,4 handle the effects of these restrictions. Some of the attributions of label 4 allow the formation of rhombi around each isolated 2-labeled vertex v, because of bell effect initiated contiguously to v. Otherwise, step 2 is in charge of copying subsequences 123 from positions $(i,j),(i+1,j),(i+2,j)$ to positions $(i,j+1),(i+1,j+1),(i+2,j+1)$, $(0\le i<m-2)$. (Here an alternative choice of labels 040 for $(i,j+1),(i+1,j+1),(i+2,j+1)$ would yield two labels 2 at distance 2 on level $\mathcal{G}_{j+2}$, a contradiction to the definition of a PDS). Now, in each one of the three decision circumstances indicated in steps 3 and 4, there are two possible options to follow admissibly with the construction of a PDS, namely for choices α and β. The absence of vertices with label 0 at some level $j+1$, signaled by the nullity of $\tau(j)$ in step 6, yields a PDS of $\mathcal{G}_{m,j+2}$ formed by the vertices that have label 2. If not, then steps 1 to 5 must be repeated with $j:=j+1$. ∎

Theorem 4.2 *If n is finite, then an exhaustive search $\mathcal{T}$ of T_Θ yields all the* PDS*'s of $\mathcal{G}_{m,j}\subseteq\mathcal{G}_{m,n}$ with fixed admissible initial condition S'.*

Proof: Θ can be applied as a subprocedure of an exhaustive search $\mathcal{T}$ of T_Θ. In each branching of $\mathcal{T}$, a decision is taken which is either a choice α or β, and both decisions are considered in their order of the exhaustive search. Eventually, step 5 of the subprocedure

Θ yields $\tau = 0$, thus yielding a leaf of T_Θ. This way, $\mathcal{T}$ produces all PDS's containing S' of subgraphs $\mathcal{G}_{m,j}$ of $\mathcal{G}_{m,n}$ with the common side $\mathcal{G}_0$. ∎

Corollary 4.1 *If n is finite, then $\mathcal{T}$ in* Theorem 4.2 *yields all* PDS*'s of $\mathcal{G}_{m,n}$ with fixed S'.*

Proof: Since n is finite, the PDS's cited in the statement are given by those leaves of T_Θ at the n-th tree level. ∎

Proposition 4.1 *Θ spends $O(m \times n)$ time to reach a* PDS *as in* Theorem 4.1*, if all the previous successive choices α, β were taken arbitrarily.*

Proof: No more than five composed complete passes of each $\mathcal{G}_j$ in steps 1 to 5 are performed by Θ, even with the arbitrary selections of α, β. ∎

Corollary 4.2 *The running time of $\mathcal{T}$ in* Theorem 4.2 *is $O(2^{m+n})$.*

Proof: An upper bound on the size of the binary decision tree $\mathcal{T}_\Theta$ is 2^{m+n}. This and Proposition 4.1 yield the assertion. ∎
Remark: In practice, linear choice strategies, like the α-, β-, β_α- and $\alpha_{in}\beta_{out}$-strategies in Section 5 below, allow to pose some conjectures, shown below. ▬

5 Open Problems, Strategies and Conjectures

5.1 Constant-Choice Strategies and Variations

The following question remains unanswered.

Question 5.1 *Given $1 \leq m \in \mathcal{Z}$ and an incomplete admissible $S' \subset \mathcal{G}_0$, does there exist an n, $(1 < n \in \mathcal{Z})$, such that $\mathcal{G}_{m,n}$ contains a* PDS *S whose intersection with $\mathcal{G}_0$ equals S'?*

In order to deal with Question 5.1, a TurboPascal implementation θ of Θ was built, see the procedure pdsg in the program ad.pas contained in **http://home.coqui.net/dejterij/pdsg.zip**. Let $\phi : \mathcal{G}_0 \to \mathcal{G}_0$ be the bijection given by $(i, 0) \leftrightarrow (m - 1 - i, 0))$. A pair $\mathcal{P} = \{S'_1, S'_2\}$ of incomplete admissible vertex subsets of $\mathcal{G}_0$ is *symmetrical* if $\phi(S'_1) = S'_2$. Four particular strategies were applied to θ, for a representative of each possible symmetrical pair $\mathcal{P}$, for all values of $m < 18$, namely:

1. *α-strategy*: all binary decisions of Θ are made with choice α; also called the *thin strategy* (producing *thin* PDS's S), because each component of a resulting $\mathcal{G}_{m,n}[S]$, $(n \in \mathcal{Z}$ or $n = \infty)$, is either formed by an isolated vertex or is a path either horizontal or vertical;

2. *β-strategy*: all binary decisions of Θ are made with choice β; also called the *thick strategy*, for it allows the largest integers s for components $P_r \times P_s$ of a resulting $\mathcal{G}_{m,n}[S]$, $(n \in \mathcal{Z}$ or $n = \infty)$;

3. β_α*-strategy*: a binary decision of Θ is made with choice β, unless this is not the first decision and the previous one was made with choice β, in which case, choice α is selected; the result is a PDS S as required, but the components of the subgraph induced by S do not intersect more than two contiguous levels $\mathcal{G}_\ell$ and $\mathcal{G}_{\ell+1}$.

4. $\alpha_{in}\beta_{out}$*-strategy*: a binary inner, respectively outer, decision of Θ is made with choice α, respectively β; we consider the possibility that for $m > 7$ there is always a PDS of a finite grid with width $m-1$ having maximum depth as cited in relation to Questions 5.3 and 5.4 below and obtained by means of the $\alpha_{in}\beta_{out}$-strategy.

The following conjectures arose from the output results.

Conjecture 5.1 *The answer to the question above is affirmative.*

Let a symmetrical pair $\mathcal{P}$ or one of its S''s be called *x-hard* if the x-strategy, ($x = \alpha, \beta, \beta_\alpha, \alpha_{in}\beta_{out}$), does not yield a PDS of a bounded $\mathcal{G}_{m,n}$. The *initial f-vector* of an admissible vertex subset S' of $\mathcal{G}_0$ is the sequence $(f(i,0))_{i=0}^{m-1}$. The β-strategy allowed to see that Conjecture 5.1 holds for $m < 15$. Moreover, for $m < 18$ there is a small amount of β-hard symmetrical pair $\mathcal{P}$'s: none for $m < 15$; four for $m = 15$, namely those with initial f-vectors:

123012312312230 123012231231223 123123012231223 123012312231223;

27 cases for $n = 16$, namely those with initial f-vectors:

2312231231230012	2312312231230012	1230123123123012	2312312230012312
1231230123012312	1231231223012312	2312230012312312	1230123012312312
1231223012312312	2312300122312312	2300123122312312	2312312300122312
2300123123122312	1230123123122300	2312312301231223	2312301231231223
0123123012231223	1230012312231223	1230123123012230	1230123123012223
1231230012312223	0123123012312223	1230012312312223	1230122312312223
1231230122312223	1230123122312223	1230123123122230	

and 57 cases for $n = 17$. For each of these remaining $\mathcal{P}$'s, there exists a pair of values j, k with $0 < j < k$ such that a resulting PDS S intersects $\mathcal{G}_j$ and $\mathcal{G}_k$ on the same subset, which does not project isomorphically onto S'. (Compare with Conjecture 5.3 below). However, we have the following.

Conjecture 5.2 *Let S' have the components of $\mathcal{G}_0[S']$ at distances > 3. Then there exists $0 < n \in \mathcal{Z}$ and a* PDS *S of $\mathcal{G}_{m,n}$ obtained via the β-strategy and whose intersection with $\mathcal{G}_0$ equals S'. In this case the effects of the β- and the β_α-strategies coincide.*

Examples. Given a graph G and a vertex subset S of G, let the *complement* $G - S$ of S in G be obtained by deleting S from G. The motivation of [12] to study a PDS S of an r-regular graph G, (e.g. the r-cube, see [3, 5]) was that $G - S$ is $(r-1)$-regular. In the running of θ and its α-, β-, and β_α-strategies it was possible to observe how complements of QPDS's and PDS's are formed level by level by replacing entries '0', '1', '2', '3' and '4', by ASCII

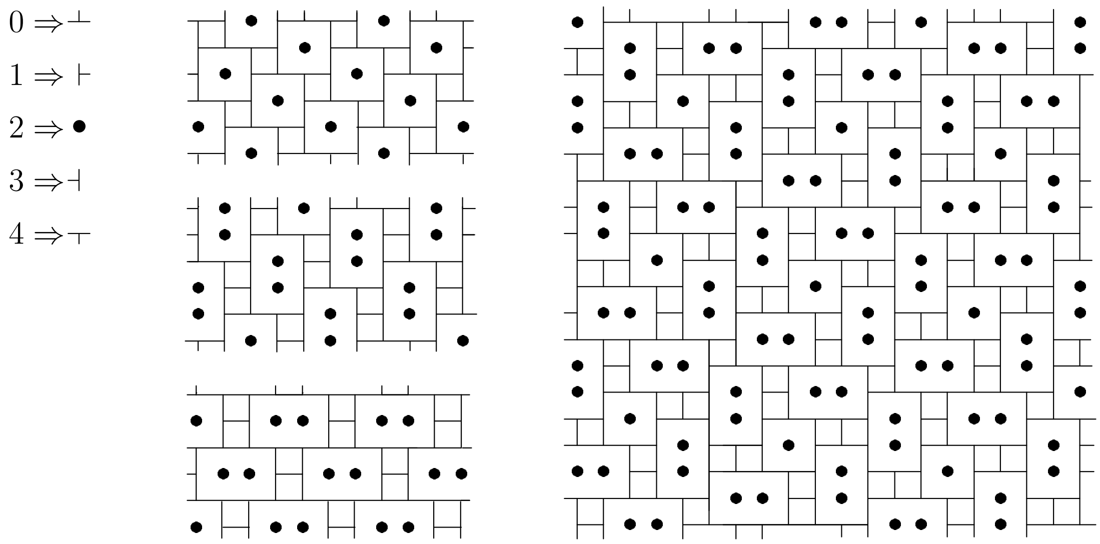

Figure 2: Some complements of PDS's in $\mathcal{L}$

symbols chr(193), chr(180), chr(254), chr(193) and chr(194), respectively, as suggested in the assignment at left in Figure 2; in practice, chr(abc) is obtained by typing abc on the right side of standard PC keyboards while pressing the 'alt' key. The rest of Figure 2 contains four partial representations of complements of PDS's S of $\mathcal{L}$. We selected these representations to be extendible to the whole plane by double periodicity. Only two of the PDS's suggested in Figure 2, namely the upper-left and the right ones, contain isolated vertices w; each of these w's is the center of symmetry of a $\mathcal{Z}_4$-action on the plane given by quarter turns; moreover, these w's are the vertices of a (tilted) orthogonal lattice with minimal squares that have as sides the diagonals of (1×2)- and (1×6)-rectangular grids, respectively. ▪

5.2 Periodic Extendibility

Clearly, an initialization for an instance of Θ can be given by means of an admissible initial f-vector, that is the f-vector of an admissible initial S'. In order to present a modification Θ' of Θ, we redefine an initial f-vector to be an m-sequence with entries in $\{0, 1, 2, 3, 4\}$ and obtained from the f-vector of an admissible initial S' by replacing some of its entries 0 by 4. In this case we say that f is *generalized*. In such a context, a PDS of $\mathcal{G}_{m,\infty}$ is *periodic* if there exists a *period* j, that is an integer $j > 0$ for which $\mathcal{G}_j$ intersects a resulting QPDS S so that $\mathcal{G}_j \cap S$ projects isomorphically onto S'. Then Θ' is obtained from Θ by replacing step 5 in Subsection 2.4 by:

5'. Let $\tau'(j) = |\{i \in \{1, \ldots, m-1\}; f(i, j+1) = f(0, j+1)\}|$; if $\tau'(j) = m$ then the vertices $v = (i, \ell)$ for which $0 \leq \ell \leq j+1$ and $f(i, \ell) = 2$ form a periodic PDS S of $\mathcal{G}_{m,\infty}$ with period j; otherwise: **(a)** $j := j + 1$; **(b)** go to step 1.

Periodic PDS's leading to extensions of the upper-, middle-, lower-left and right partial PDS representations in Figure 2 to the whole plane can be obtained from Θ' via the α-

strategy and the following corresponding initial f-vectors:

$$12304 \quad 12312304 \quad 1223 \quad 1231223044004122312300412341230123044.$$

One more observation on the original initial f-vectors, (i.e. with no entry 4): let an initial f-vector be called *trivial* if it is a subsequence of contiguous appearances of the repetition sequence $\dots 123 \dots 123 \dots$ of period 123; among initial f-vectors associated to (non-in)complete admissible S''s, the trivial ones terminate with $j = 1$; any other one of them terminates with $j = 1$ for choice β, yielding however for choice α a periodic PDS of period 4, as shown for initial f-vector 1223 in Figure 2.

Based on Θ', the following was verified for $m < 18$.

Conjecture 5.3 *For each generalized initial f-vector g, the α-strategy for Θ' yields a periodic* PDS *S restricting to $S' = \{(i,0) \in \mathcal{G}_0; g(i,0) = 2\}$. As a result, for every $m > 0$ and generalized initial f-vector g, there is a cylindrical grid $P_m \times C_n$ obtained as a quotient graph of $\mathcal{G}_{m,n}$ and containing a* PDS *S whose intersection with (the image of) $\mathcal{G}_0$ equals (the image of) S', where C_n is a cycle on n vertices.*

If Conjecture 5.3 holds, then there is a function Ψ, from the set of nontrivial generalized initial f-vectors g into $\mathcal{Z}$, that assigns to each g the smallest period j of a periodic PDS as in the conjecture. The assignments of Ψ would constitute an invariant of generalized initial f vectors given by the minimal period of a thin periodic PDS's.

Remark: If Conjecture 5.3 holds for a generalized initial f-vector g which is extendible periodically, (i.e. g extends to $\mathcal{L}[\mathcal{Z} \times \{0\}]$ via the translation $i \to m+i$), then the restrictions of f to $\mathcal{L}[\{0\} \times \mathcal{Z}]$ and to $\mathcal{L}[\{m-1\} \times \mathcal{Z}]$ either coincide or differ by a translation $j \to p+j$, where $0 < p < n$. In the former case, S is extendible to a doubly-periodic PDS of the plane, which yields, as a quotient graph, a grid $C_m \times C_n$ on the torus, containing an image PDS of S. ∎

Conjecture 5.4 *Is the former case considered in the previous Remark always the case?*

Another problem here involves establishing for which values of m and g the α-strategy for Θ' determines QPDS's extendible to PDS's S of $\mathcal{L}$ possessing $\mathcal{Z}_4$-actions of the plane given by quarter turns, (as in the upper-left and right representations of Figure 2), with isolated vertices as centers of symmetry forming themselves tilted orthogonal lattices. It turns out that if $\mathcal{L}[S]$ has components with no more than two vertices, then such tilted lattices have minimal squares whose sides are diagonals of $(1 \times (2+4k))$-rectangular grids, $(0 \le k \in \mathcal{Z})$, and minimal generalized initial f-vectors generating them have $1 + (2+4k)^2$ entries.

5.3 Extremal Behavior

C. Thomassen posed the following interesting conjecture.

Conjecture 5.5 *Given $1 \le m \in \mathcal{Z}$ and an incomplete admissible $S' \subset \mathcal{G}_0$, there exists $n \in \mathcal{Z}$ with $n > 1$ depending only on m and not on S' such that $\mathcal{G}_{m,n}$ contains a* PDS *$S(m,n,S')$.*

Inspired by Thomassen's conjecture, the following extremal problem was conceived. Let $m \in \mathcal{Z}$ with $3 < m$ be a fixed width. Consider the chain of graphs under containment shown at the end of Subsection 1.1. The first part of the problem asks the following.

Question 5.2 *Given $\mathcal{G}_{m,\infty}$ and given a unique PDS S' of a subgraph G'' of $\mathcal{G}_0$, which is the minimum finite depth $n = n(S') \in \mathcal{Z}$, if any, such that there exists a PDS $S(m, n, S')$?*

In other words Question 5.2 asks for the length of the shortest finite maximal path of the decision tree associated with Θ. If such an n does not exists, then we set $n = \infty$. The problem itself consists of the following important question.

Question 5.3 *Is there a finite maximum depth $N(m)$ of the minima $n = n(S')$ when S' varies over all unique PDS's of subgraphs G'' of $\mathcal{G}_0$?*

Question 5.4 *If the answer to Question 5.3 is yes, which is the value of $N(m)$?*

The programs in the zipped internet link above mentioned, including the cited ad.tex, yield answers to Question 5.4 for $4 \leq m \leq 17$. This link also contains some resulting extremal conclusions, including associated representations, like those in Figure 2. A synopsis of the results is present below, with each line composed from left to right by: a value for m; an initial f-vector $f(S')$ of an incomplete compatible S' of $\mathcal{G}_0$ for which $N(m)$ is realized; the actual value of $N(m)$; and the corresponding sequence $\mathcal{S}$ of decisions in $\{\alpha, \beta\}$, where α and β are replaced respectively by 0 and 1.

m	$f(S')$	$N(m)$	$\mathcal{S}$	m	$f(S')$	$N(m)$	$\mathcal{S}$
4	1230	3	1	13	2312222300012	12	0001
5	12230	4	0	13	2230001230123	12	01111
6	230123	5	11	13	2230001223123	12	0111
7	2312230	6	01	14	12230001230123	14	01111
8	23000123	8	011	14	12230001223123	14	0111
9	230123123	8	01	14	12230122230123	14	10111
9	223000123	8	011	14	12230122223123	14	0111
10	1223000123	10	011	15	223000001230123	15	010111
10	1222230123	10	101	15	223000001223123	15	00111
11	22300000123	11	0011	16	2312300000122223	16	100011
12	122301231230	11	0011	16	2312300012222223	16	100101
12	122223000123	11	00110	17	22300000001230123	18	000111
12	222300000123	11	0011	17	22300000001223123	18	000111

Question 5.5 *Is there an underlying unifying pattern structure for the PDS's associated to the maxima $N(m)$ of minima $n(S')$ in Question 5.3?*

Let $0 \leq k \leq 4$. A subvector of $f(S')$ is said to be *maximal contiguously-k-valued* if the induced subgraph of the vertices it represents is a path and the associated value $f(i, 0)$ to each of these vertices is equal to k. Then the vectors $f(S')$ realizing the maximum $N(m)$ above seem to suggest that their maximal contiguously-0-valued subvectors must have odd width.

Question 5.6 *Is it true that the maximal contiguously-0-valued subvectors in those $f(S')$ realizing the maximum $N(m)$ must have odd width, for every $m > 3$ for which $N(m)$ is well-defined?*

Question 5.7 *Is it true that for the numbers $m > 12$ for which $N(m)$ is well-defined the contiguously-0-valued subvectors in Question 5.6 are interior to G_0?*

Remark: Assuming that the answer to Question 5.3 is yes, one possible resulting behavior is that for $m > 6$, ($m > 12$), the vectors $f(S')$ realizing the depth $N(m)$ contain more than one, (two), maximal contiguously-2-valued subvectors, as it seems feasible according to the synopsis of results above. ▬

Question 5.8 *Should the behavior noticed in the Remark above constitute part of a general rule or is it that for each number $m > 12$ for which $N(m)$ is well-defined the $f(S')$'s realizing the depth $N(m)$ contain exactly three maximal contiguously-2-valued subvectors?*

Question 5.9 *Is it true that for each $m > 12$ for which $N(m)$ is well-defined the maximal contiguously-2-valued subvectors have either unit or even width?*

Moreover, we suspect the following.

Conjecture 5.6 *For $m > 7$ there is always a PDS of a finite rectangular grid of width $m-1$ having maximum depth $N(m)$ and obtained by means of the $\alpha_{in}\beta_{out}$-strategy of Subsection 5.1.*

In fact, our suspicion is true for $7 < m \leq 17$, as we comment in the file readme.ps of the electronic zipped directory above mentioned.

5.4 Grids of Dimensions > 2

Finally, it remains to consider extending Θ and Θ' to r-dimensional lattices with $r > 2$, with related considerations. If $r = 3$, the value set for a function f as in the presentation of Θ above would be enlarged to $\{0, 1, 2, 3, 5, 6\}$, with one of these values taken for the vertices of a PDS and the others for the left, right, top, bottom, front and back neighbors of each one of the components (products of three paths) of its induced subgraph.

Acknowledgement. The author would like to thank C. Thomassen for his hospitality and enlightening comments.

References

[1] J. Borges, *On Perfect Dominating Sets*, preprint.

[2] J. Borges, *Codis Perfectes, Particions i Generalitzations*, Ph.D. Thesis, Dept. d'Informàtica, Universitat Autònoma de Barcelona, 1998.

[3] J. Borges and I. J. Dejter, *On Perfect Dominating Sets in Hypercubes and Their Complements*, Journal of Combinatorial Mathematics and Combinatorial Computing, **20**(1996) 161-173.

[4] I. J. Dejter and K. T. Phelps, *Ternary Hamming and Binary Perfect Covering Codes*, in A. Barg and A. Lytsin eds., *Codes and Association Schemes*, DIMACS Series, Amer. Math. Soc., to appear.

[5] I. J. Dejter and J. Pujol, *Perfect Domination and Symmetry*, submitted to Congr. Numer., **111**(1995) 18-32.

[6] I. J. Dejter and O. Serra, *Efficient Dominating Sets in Cayley Graphs*, Tech. Rep., DIMACS, Rutgers University,

[7] I. J. Dejter and P. M. Weichsel, *Twisted Perfect Dominating Subgraphs of Hypercubes*, Congr. Numer., 94(1993), 67-78.

[8] M. R. Fellows and M. N. Hoover, *Perfect Domination*, Australian J. of Combinatorics, **3**(1991), 141-150.

[9] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in Graphs, M. Dekker Inc., 1998.

[10] J. Kratochvil and M. Krivánek, *On the Computational Complexity of Codes in Graphs*, MFCS 1988, 396-404.

[11] P. R. J. Östergård and W. D. Weakley, *Constructing Covering Codes with Given Automorphism*, Designs, Codes and Cryptography, **16**(1999), 65-73.

[12] P.M. Weichsel, *Dominating Sets of n-Cubes*, J. Graph Theory, 18 (1994), 479-488.