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Resource Allocation in a Distributed Network

by

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ABSTRACT

This paper discusses distributed resource allocation in a communication system. Some recent approaches to decentralized resource allocation in a communication system are summarized. A characterization for resource control and pricing in a distributed wireless network with transmit power control is presented.

Power control is modelled as a noncooperative externality game between the users. In the absence of a price each user has the incentive to increase the transmit power to increase the quality-of service as measured by the signal-to-noise ratio. However, the increase in transmission energy of a representative user by definition deteriorates the signal-to-noise ratio for all other users, thus imposing an externality cost. The Nash equilibrium in an uncontrolled distributed network is one where the system fails due to excessive congestion. However, by introducing a simple linear pricing penalty, the distributed system can be made to achieve the Pareto-optimal quality of service.

Convergence to an efficient Nash equilibrium under pricing is observed in a numerical example assuming that the users apply a standard learning automaton.

1 Introduction

Resource allocation in the new distributed communication systems, such as the Internet and future wireless networks, is a challenge. In these systems the resource allocation problem is characterized by incomplete information and asynchronous decisions exacerbating efficient approaches to efficient system utilization and congestion control.

Recently, Kelly has suggested in [10] that the centrally optimal resource allocation in a communication system can be decomposed into user's utility maximization and network's revenue maximization subproblems. It is shown that there exists a pricing rule based on resource pricing such that distributed resource allocation achieves the same efficiency and fairness as a centralized solution (ibid.).

The paper is organized as follows. A model for distributed resource allocation in a communication network is presented in section 2. Section 3 addresses stability in the distributed resource allocation. Section 4 is a conclusion.

2 Optimizing Utility Functions in a Distributed Communication System

This section studies distributed resource allocation for communication networks such as the Internet. The framework also allows to address other communication systems, in particular wireless networks. In a wireless network as defined by [13] the control variable is typically the transmit power. Mechanisms other than power control (such as coding) are not considered for simplicity.

2.1 Objective Functions for Communication Networks

In a distributed network the users decide on optimal resource usage independently of each other. Let each user $i = 1, ..., m$ maximize a concave objective (utility) function u_i (strictly increasing in its argument) subject to a pricing penalty (cf. [10], [8]):

$$\max r_i = \max u_i\left(\frac{y_i}{\sum_{j \neq i} y_j}\right) - P_i y_i, \tag{1}$$

Here $y_i = g_{ii} x_i$ denotes the signal power received at receiver i due to user i's transmit power x_i, and P_i the price for y_i. The term g_{ii} denotes the link fading coefficient and constitutes the iith coefficient in the $m \times m$ link gain matrix $\mathbf{G}$ as in [13]. Furthermore $I_i = \sum y_{j \neq i}$ denotes the total interference to user i. The argument in the utility function is $\frac{y_i}{I} = \alpha_i$; this is the signal-to-noise ratio SNR of user i [13], denoted by α_i.

To see the analogy between resource allocation problems in wireless and wired networks consider the problem

$$\max r_i = \max_{y_i} u_i\left(\frac{y_i}{y_i + \sum_{j \neq i} y_j}\right) - y_i, \tag{2}$$

due to [4], which captures the objective function of a representative user in the Internet. This is effectively the same as (1) when $P_i = 1 \;\forall\; i$ and when for simplicity $I = \sum_{i=1}^{m} y_i$, an approximation adopted also in [6]. Thus, not only are the centralized resource allocation problems similar for a wireless and wireline network [8] but the same holds for the decentralized allocation of resources. Here each user also causes interference to himself. In [4] $u_i(y_i) = a_i y_i \;\forall\; i$. In what follows, concentrate on the simplified (linearized) objective function for a representative user (2).

In a centralized system the resource allocation problem solves

$$\max \sum u\left(\frac{y_i}{\sum y_i}\right) \;\; s.t. \;\; \mathbf{G}\mathbf{x} = \mathbf{y}, \;\; \sum x_i \le R \tag{3}$$

where the constraints express resource constraints, $\mathbf{G}$ is a network matrix [10] and $\mathbf{y}$ denotes the vector of received rates. The Pareto-optimal solution is to allocate each user the same SNR: the first order optimality condition is to set $u'(\alpha_i) = \lambda = P \;\forall i = 1,..,m$, where λ denotes the resource shadow cost. This paper discusses a pricing mechanism that allows the distributed system described by (1) achieve the same efficiency as the centralized system. Without a price $P_i = 0$ and optimization according to (1) leads to each user i setting $y_i^* = \infty$.

2.2 Distributed Network Game

This section derives a distributed solution to the allocation of resources in a communication network with users individually maximizing utility functions with respect to received signals; In a distributed communication system in general and also in a distributed wireless system, user's utility is derived from received signal y.

Here the externality game in a general communication system (Internet or a wireless network) is defined as a noncooperative game where the players are the users. The stability of an externality game for a (mobile) distributed network is discussed in section 3.

A distributed solution is defined in terms of game theory, see e.g. [12]. Previously game theory has been applied on the study of resource allocation in a communication system in e.g. [9, 7, 5]

Definition 2.1. *For a game with m players, the* normal form representation $\Gamma = [m, \{S_i\}, \{r_i\}]$ *specifies for each player i a set of strategies S_i (with $y_i \in S_i$) and a payoff function $r_i(y_1, ..., y_m)$ giving the utility levels associated with the outcome arising from the strategies $y_1, ..., y_m$.*

In this paper define the game with users as the players, individually choosing received signal level y to maximize utility. From the user's maximization problem (1), the optimal received signal y_i satisfies the first order condition $u_i'(y_i) = P_i \;\; \forall\; i = 1,..,m$. The main solution concept in noncooperative game theory is a Nash-equilibrium:

Definition 2.2. *A strategy profile $\mathbf{y} = (y_1, ..., y_m)$ constitutes a Nash-equilibrium of game Γ if for every $i = 1,..,m$,*

$$r_i(y_i, y_{-i}) \ge r_i(y_i', y_{-i}), \;\forall\; y_i' \in S_i. \tag{4}$$

When $u_i = \frac{a_i y_i}{\sum y_i}$ the solution to utility maximization is as in [4]:

$$y_i = \bar{X}[1 - \frac{\bar{X}}{a_i}]^+, \tag{5}$$

where the threshold $\bar{X}$ is the unique solution to

$$1 = \sum_i [1 - \frac{\bar{X}}{a_i}]. \tag{6}$$

The threshold $\bar{X}$ is by definition such that the set of users opting out have $a_i < y_i + \sum y_{j \neq i} = \bar{X}$. Users for whom $a_i < \bar{X}$ set $y_i = 0$; here if $a_i = a$ then $\bar{X} = a\frac{m-1}{m}$. The optimal decision rule in a symmetric network thus is to participate in the system:

$$y_i = \frac{a(m-1)}{m^2 P}. \tag{7}$$

The game defined by (2) exemplifies the externality problem under congestion: a decentralized Nash-equilibrium need not be Pareto-optimal.

Definition 2.3. *An externality is present whenever the payoff of a user (player) is directly affected by the actions of the other users (players).*

2.3 Congestion Control Under Certainty

The question for the optimal congestion control is if there is a pricing rule such that individual optimization according to a concave utility function leads to the Pareto-optimal resource allocation. In what follows, let $y = u'^{-1}(P)$ be the demand for data rate of a representative user; like in [11], y is the demand as defined in microeconomics [12].

The nonnegativity conditions for the total utility (TU) in a system with pricing is given by

$$TU = nu'(y)\mathbf{y} - n(m-1)\mathbf{y} \geq \mathbf{0}. \tag{8}$$

Letting $\mathbf{F}$ denote the matrix of interference (externality) coefficients [13], this can be rewritten as

$$\frac{1}{u'(y)}\mathbf{Fx} \leq \mathbf{Gx}. \tag{9}$$

The optimal signal-to-noise ratio α is characterized by (cf. Zander (1992)):

$$\mathbf{Gx}^* \geq \alpha\mathbf{Fx}^* \tag{10}$$

where $\alpha = \frac{1}{m-1}$. Comparing (9) and (10) suggests that optimally $u'(y) = P = m - 1$. Optimal congestion price forces each user to pay the externality (interference) cost caused to the other users, thus ruling out transmit power warfare [8].

Remark 2.1. *The optimal price rule is to charge a congestion based price even under resource costs; here there is a separation between the optimal assignment of users to bases and optimal pricing. The reason is that interference from all other users (from all cells) is considered as an externality cost (like in [6]).*

The congestion price is a sufficient mechanism to achieve the Pareto-optimal resource allocation under a distributed regime with utility functions (1) [8]. In what follows it is argued that a positive price is sufficient for an efficient equilibrium under a distributed regime with the utility functions as specified in (2) above.

3 Stability of the Network Game

The goal of this section is to study the convergence of a distributed network under pricing. From (6) the equilibrium threshold for user i is

$$\bar{X} = \frac{m-1}{\sum \frac{1}{a_i}} \tag{11}$$

and so user i participates in the system if

$$\Pi_i = a_i - \bar{X} \geq 0 \tag{12}$$

and opts out if $\Pi_i < 0$. The threshold (11) is an increasing function in the number of users m; thus, following [2] the payoff functions $\{\Pi_i\}_{i=1}^m$ define an externality game.

3.1 Non-atomic Games

In this paper a special class of externality games is considered:

Definition 3.1. *A non-atomic game is a game with a continuum of users, in which no single player can affect the other players by changing only her own action. A nonatomic externality game (NAEG) is a nonatomic game with submodular payoff functions [3].*

The key restriction on a representative user i's payoff u_i is that it only depends on her own strategy y_i (resource usage) and the externality vector $\{y_j\}_{j\neq i}$:

$$u_i(y_i, f(\mathbf{y})) = \frac{a_i y_i}{f(\mathbf{y})} = \frac{a_i y_i}{\sum_j y_j} \tag{13}$$

Here by definition u_i is a submodular function i.e. function with decreasing differences: for all y, y' in the strategy space and $y > y'$, (for $P = 0$) $u_i(y, f) - u_i(y', f)$ is nonincreasing in f. Adding the pricing linear penalty function to evaluate the total utility of a user does not change this. A congested communication network is thus another example of a nonatomic externality game. Hence, learnability in a congested network can be established by learnability conditions for an $NAEG$ [2].

3.2 Learnability in a Decentralized System

Following [2] consider a discrete-time environment with M possible strategies. Let r_i^t denote the payoff (capacity) for playing strategy i at time t, where as above, $0 \leq r_i^t \leq 1$. The state of an automaton is a probability vector such that at time t the automaton picks strategy i with probability p_i^t at random. A standard parameterized (by β) updating rule (ibid.) is

$$p_i^{t+1} = p_i^t + \beta r_i^t (1 - p_i^t) \tag{14}$$

$$\forall j \neq i: \quad p_j^{t+1} = p_j^t (1 - \beta r_i^t) \tag{15}$$

Assuming the r_i^t are chosen from some stationary probability distribution so that $P(r_i^t \leq x)$ is given by $F_i(x)$ independent of t. The LAs are ϵ-optimal in that for any $\epsilon > 0$ there exists some β such that

$$\lim_{t \to \infty} E(\sum_{i=1}^{M} p_i^t r_i^t) > \max(E(r_i^s)) - \epsilon \tag{16}$$

The learning automaton defined by (14)-(15) is ϵ-optimal (ibid.): the automaton can achieve an asymptotic expected payoff arbitrarily close to the optimal payoff. Here consider $M = 2$ possible strategies: accept A or opt out R. Then the play converges to a single strategy: accept or opt out; in effect, the expected value of m is being learned. The payoff function for player i is given by $\Pi_i = a_i - \bar{X}$, where a_i is the value of participation for user i. In what follows the learning of optimal strategy is studied from the point of view of objective functions of type (2).

3.3 An Implementation Example

In Figure 1 the 5 users learn an optimal strategy regarding demand for received power. Payoff of user i r_i is given by (2). Let users choose received powers from $S_i = S = \{\frac{1}{m}, \frac{1}{m-1}, .., 1\}$. Under pricing with $P = 1$, demand is finite, ruling out a transmit power warfare that would occur if the price was zero: recall that (2) is maximized letting $y_i \to \infty$ in the absence of a price. Also, in the absence of a price penalty, further simulations (not demonstrated here) show that all the users in the system prefer the highest power level, as opposed to the case under pricing as in the figure where all users learn to demand the lowest power level (corresponding to the Nash equilibrium defined above). Pricing thus encourages energy efficiency which is a Pareto-improvement only *if* there are real resource costs related to transmit power energy. Here however, the resource costs are not accounted for; instead, *the price mechanism here works to rule out excessive congestion* like in [8] and unlike in [5]: the price prevents the user from increasing the demand indefinitely to steal capacity (SNR) from other users (thus enforcing incentive-compatibility [12]).

4 Conclusion

A fundamental problem in distributed network optimization is to deal with various externality costs, by definition not accounted for in a system based on individual optimization.

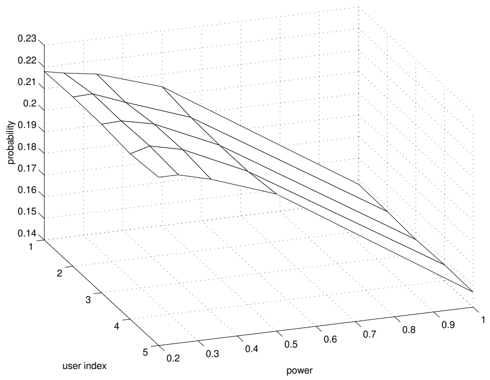

Figure 1: Probabilities for Received Powers for $m = 5$

A simple linear pricing rule is applicable for the distributed control of a communication network. Convergence to an equilibrium point in a distributed communication network can be established from learnability point of view. Preliminary numerical analysis suggests that a greedy user optimizing an individual objective function can behave optimally from the point of view of the whole system under pricing.

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