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Difference Graphs

by

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ABSTRACT

Intersection and measured intersection graphs are quite common in the literature. In this paper we introduce the analogous concept of measured difference graphs: Given an arbitrary hypergraph $\mathcal{H} = \{H_1, ..., H_n\}$, let us associate to it a graph on vertex set $[n] = \{1, 2, ..., n\}$ in which (i, j) is an edge iff the corresponding sets H_i and H_j are "sufficiently different". More precisely, given an integer threshold k, we consider three definitions, according to which (i, j) is an edge iff (1) $|H_i \setminus H_j| + |H_j \setminus H_i| \geq 2k$, (2) $max\{|H_i \setminus H_j|, |H_j \setminus H_i|\} \geq k$, and (3) $min\{|H_i \setminus H_j|, |H_j \setminus H_i|\} \geq k$. It is not difficult to see that each of the above define hereditary graph classes, which are monotone with respect to k. We show that for every graph G there exists a large enough k such that G arises with any of the definitions above. We prove that with the first two definitions one may need $k = \Omega(\log n)$ in any such realizations of certain graphs on n vertices. However, we do not know a graph G which could not be realized by the last definition with $k = 2$.

Key Words: Graph realization, intersection graphs, difference graphs

1 Introduction

The intersection and measured intersection graphs are quite often considered in the literature; see e.g. the surveys [2, chapter 4], [5, 6]. Here we introduce an analogous concept of measured difference graphs.

Let us introduce the notation $[n] = \{1, 2, ..., n\}$. For a set X and integer l let us denote by $\binom{X}{l}$ the family of all subsets of X of size l.

Let $\mathcal{H} \subseteq 2^X$ be an arbitrary hypergraph over some finite base set X. Let furthermore ρ be a symmetric real function in two variables, $\rho : \mathbb{R}^2 \mapsto \mathbb{R}$, and let $k \in \mathbb{Z}_+$ be an integer threshold. We shall associate to the triplet $(\mathcal{H}, \rho, k)$ a graph $G^{\rho,k}_{\mathcal{H}}$ defined on the vertex set $V(G^{\rho,k}_{\mathcal{H}}) = \mathcal{H}$, and in which $(H, H') \in E(G^{\rho,k}_{\mathcal{H}})$ iff

$$\rho(|H \setminus H'|, |H' \setminus H|) \geq k, \tag{1}$$

where $H \setminus H' = \{x \in X | x \in H, x \notin H'\}$. Let us further say that a simple graph $G = (V, E)$ is (ρ, k)-*realizable* if $G = G^{\rho,k}_{\mathcal{H}}$ for some hypergraph $\mathcal{H}$ (let us note that we do not assume anything about the base set X of such a hypergraph.) Let us finally denote by $\mathcal{G}^{\rho,k}$ the family of (ρ, k)-realizable graphs.

We shall also assume that the symmetric real function ρ is monotone increasing, satisfying the inequality

$$\rho(a + 1, b + 1) \geq \rho(a, b). \tag{2}$$

In particular, we shall consider in this paper the cases $\rho \in \{\text{avg}, \max, \min\}$, where

$$\text{avg}(a, b) = \frac{a + b}{2}, \tag{3}$$

$$\max(a, b) = \max\{a, b\}, \qquad \text{and} \tag{4}$$

$$\min(a, b) = \min\{a, b\}. \tag{5}$$

For instance, if $\mathcal{H} = \binom{[5]}{2}$ is the family of all ten distinct subsets of cardinality 2 of a given set of cardinality 5 then $G^{\text{avg},2}_{\mathcal{H}} = G^{\max,2}_{\mathcal{H}} = G^{\min,2}_{\mathcal{H}}$ are all realizations of the Petersen graph, see Figure 1.

It is easy to see that the above properties (classes of graphs) are hereditary. Indeed, if $G = G^{\rho,k}_{\mathcal{H}}$, and G' is an induced sub-graph of G, then $G' = G^{\rho,k}_{\mathcal{H}'}$, where $\mathcal{H}'$ is the sub-hypergraph corresponding to the vertices $V(G') \subset \mathcal{H} = V(G)$.

It is also easy to see that these properties (classes of graphs) are monotone with respect to k, that is $\mathcal{G}^{\rho,k} \subseteq \mathcal{G}^{\rho,k+1}$. Indeed, given a graph $G = G^{\rho,k}_{\mathcal{H}}$, we can define $\mathcal{H}' = \{H \cup \{H\} | H \in \mathcal{H}\}$ yielding $G = G^{\rho,k+1}_{\mathcal{H}'}$, due to (2).

For a given graph $G = (V, E)$ let $k^\rho(G)$ denote the smallest integer k for which G is (ρ, k)-realizable. First, we show that every simple graph is realizable for $\rho \in \{\text{avg}, \max, \min\}$ with some finite threshold k.

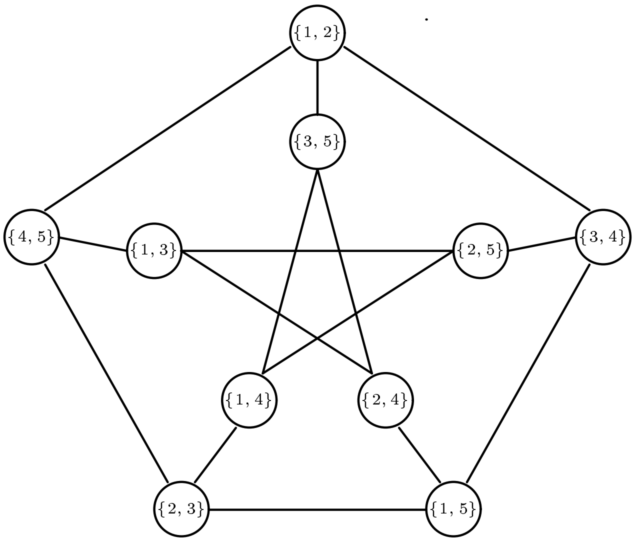

Figure 1: Realizations of the Petersen graph.

Theorem 1 *For every graph $G = (V, E)$ and $\rho \in \{avg, \max, \min\}$ we have $k^\rho(G) \leq |V| - 1$.*

For $\rho \in \{\text{avg}, \max, \min\}$, let us denote by $k^\rho(n)$ the maximum of $k^\rho(G)$, where the maximum is taken over all simple graphs on n vertices. We will demonstrate that in cases of $\rho \in \{\text{avg}, \max\}$

$$k^\rho(n) = \Omega(\log n)$$

.

More precisely, let $\mathcal{H} = 2^X$ be a family of all the distinct subsets of a given set X of cardinality $|X| = 2k$. It is not difficult to see that $G^{\text{avg},k} = 2^{2k-1} K_2$ is a matching consisting of 2^{2k} vertices and 2^{2k-1} disjoint edges. Though this construction seems to be an extremal one, we can only show a slightly weaker statement.

Theorem 2 *Consider the graph $G = t\, K_2$ consisting of t pairwise disjoint edges on $n = 2t$ vertices. If this graph G is (avg, k)-realizable, then*

$$t = O\left(k^2 2^{2k}\right), \tag{6}$$

implying that $k^{avg}(n) \geq k^{avg}(G) = \Omega(\log n)$.

Analogously, for the case of $\rho = \max$ let us consider the hypergraph $\mathcal{H} = \binom{X}{k}$, i.e. the family of all the subsets of cardinality k of the base set X of cardinality $2k$. It is not difficult to see that $G^{\max,k}_{\mathcal{H}} = \frac{1}{2}\binom{2k}{k} K_2$ is a matching consisting of $\binom{2k}{k}$ vertices and $\frac{1}{2}\binom{2k}{k}$ pairwise disjoint edges. Another possible way to realize the same matching is by a hypergraph $\mathcal{H}$ consisting of all the subsets of cardinality k or $k-1$ from a base set of cardinality $2k-1$.

Though these constructions look also extremal, we can show only the following slightly weaker statement.

Theorem 3 *Let $G = t\,K_2$ be the graph consisting of t pairwise disjoint edges on $n = 2t$ vertices. If G is $(\max, k)$-realizable, then*

$$t = O\left(k^2 \binom{3k}{k}\right), \tag{7}$$

implying that $k^{\max}(n) \geq k^{\max}(G) = \Omega(\log n)$.

Interestingly, for $k^{\min}(n)$ we cannot show a similarly fast growing lower bound. Clearly, $\min\{|H \setminus H'|, |H' \setminus H|\} < 1$ iff H is a subset of H' or vice versa. Thus, the graphs in $\mathcal{G}^{\min,1}$ are exactly the co-comparability graphs, see [2]. Since not all graphs belong to this family, we immediately obtain

$$k^{\min}(n) \geq 2.$$

In fact, several families of graphs can be shown to belong to $\mathcal{G}^{\min,2}$, including trees, cycles, line graphs and the complements of the above, P_4-free graphs, etc., see [3]. No general construction technique arose so far, nor did we manage to find a non $(\min, 2)$-realizable graph, leaving open the questions: "Is $k^{\min}(n)$ bounded by a constant?", or even more "Does $\mathcal{G}^{\min,2}$ contain all simple graphs?"

2 Proofs of the Theorems

For the proof of Theorem 1, let us define a *stable non-edge covering* of a given simple graph $G = (V, E)$ as a collection $\mathcal{S} = \{S_1, ..., S_m\}$ of stable sets of G such that for every pair of vertices $u, v \in V$ forming a non-edge $(u, v) \notin E$ there exists a stable set $S \in \mathcal{S}$ containing both vertices u and v.

Proof of Theorem 1. Let us consider a stable non-edge covering $\mathcal{S}$ of the given graph $G = (V, E)$, let $d_{\mathcal{S}}(v)$ for $v \in V$ denote the number of stable sets $S \in \mathcal{S}$ containing vertex v, and let $k = \max_{v \in V} d_{\mathcal{S}}(v)$. Let us then define

$$H_v = \{S \in \mathcal{S} | S \ni v\} \cup B_v \text{ for } v \in V,$$

where the sets V and B_v, $v \in V$ are pairwise disjoint and $|B_v| = k - d_{\mathcal{S}}(v)$ for $v \in V$. Then, we have $|H_v| = k$ for all $v \in V$, and $H_v \cap H_u = \emptyset$ if and only if $(u, v) \in E$. Hence, $G = G_{\mathcal{H}}^{\rho,k}$ for $\rho \in \{\text{avg}, \max, \min\}$, and for the hypergraph $\mathcal{H} = \{H_v | v \in V\}$.

Since the family of non-edges $E(\overline{G})$ form a trivial stable non-edge cover, with $d_{E(\overline{G})}(v) \leq |V| - 1$ for all $v \in V$, the claim follows. □

Let us remark that by choosing the right stable non-edge covering, one may get substantially smaller value for k than $|V| - 1$. For instance, for the Petersen graph on Figure 1 one can cover the non-edges by 5 stable sets, each of size 4, such that every vertex belongs to exactly 2 of these.

For the proofs of the next two Theorems, we shall need to recall the following result of Füredi [4], generalizing an earlier result of Bollobás [1].

Theorem 4 ([4]) *Let a, b and c positive integers, and $\{A_1, \ldots, A_l\}$ and $\{B_1, \ldots, B_l\}$ be two families of sets such that $|A_i| = a$, $|B_i| = b$ and $|A_i \cap B_i| \leq c$ for any $1 \leq i \leq l$, and $|A_i \cap B_j| > c$ for any $1 \leq i \neq j \leq l$. Then,*

$$l \leq \binom{a+b-2c}{a-c}.$$

Proof of Theorem 2. Suppose that $tK_2 = G_{\mathcal{H}}^{\mathrm{avg},k}$, where $\mathcal{H} = \{A_1, \ldots, A_t, B_1, \ldots, B_t\}$ and where we assumed that the pairs (A_i, B_i), $i = 1, ..., t$ are forming the edges of the matching. By the definition of (avg, k)-realization, this implies that

$$\begin{aligned} d(A_i, B_i) &\geq 2k \quad \text{for } i = 1, ..., t, \text{ and} \\ d(A_i, B_j) &< 2k \quad \text{for } 1 \leq i \neq j \leq t, \end{aligned}$$

where $d(A, B) = |A \setminus B| + |B \setminus A|$.

For integers a, b let us define $I_{a,b} = \{1 \leq i \leq t : |A_i| = a, |B_i| = b\}$, and set $c = \lfloor \frac{1}{2}(a + b - 2k) \rfloor$. The above inequalities then imply

$$\begin{aligned} |A_i \cap B_j| &= \frac{1}{2}(|A_i| + |B_j| - d(A_i, B_j)) > c \quad \text{if } i \neq j, \quad \text{and} \\ |A_i \cap B_i| &= \frac{1}{2}(|A_i| + |B_j| - d(A_i, B_j)) \leq c \quad \text{otherwise.} \end{aligned}$$

Hence, we can apply the above cited result of Füredi for the families $\{A_i | i \in I_{a,b}\}$ and $\{B_i | i \in I_{a,b}\}$, and obtain

$$|I_{a,b}| \leq \binom{a+b-2c}{a-c} \leq \binom{2k+1}{a-c} \leq 2^{2k+1}.$$

Let us observe finally that $d(A_i, B_1) < 2k$ and $d(B_i, A_1) < 2k$ for any $i \geq 2$ imply $|B_1| - 2k < |A_i| < |B_1| + 2k$ and $|A_1| - 2k < |B_i| < |A_1| + 2k$ for $i = 2, ..., t$. Therefore $|\{(a, b) \mid I_{a,b} \neq \emptyset\}| \leq 1 + (4k - 1)^2 = O(k^2)$, thus the claimed inequality

$$t \leq \sum_{\{(a,b): I_{a,b} \neq \emptyset\}} |I_{a,b}| = O(k^2 2^{2k})$$

follows. $\square$

Proof of Theorem 3. Let us suppose again that $tK_2 = G_{\mathcal{H}}^{\max,k}$, where $\mathcal{H} = \{A_1, \ldots, A_t, B_1, \ldots, B_t\}$ and where we assumed the pairs (A_i, B_i), $i = 1, ..., t$ to correspond to the edges of the matching such that $|A_i \setminus B_i| \geq k$ for $i = 1, ..., t$, and $|A_i \setminus B_j| < k$ and $|B_j \setminus A_i| < k$ for $1 \leq i \neq j \leq t$.

Analogously to the previous proof, let us define $I_{a,b} = \{1 \leq i \leq t : |A_i| = a, |B_i| = b\}$ for integers a, b, and set $c = a - k$. Then, for any distinct indices $i, j \in I_{a,b}$ we have

$$|A_i \cap B_j| = |A_i| - |A_i \setminus B_j| > a - k = c \quad \text{and} \quad |A_i \cap B_i| \leq c.$$

Thus, we can apply again Theorem (4) for the families $\{A_i | i \in I_{a,b}\}$ and $\{B_i | i \in I_{a,b}\}$, and obtain

$$|I_{a,b}| \leq \binom{a+b-2c}{a-c} = \binom{b-a+2k}{k}. \tag{8}$$

If $I_{a,b}$ contains at least two distinct elements, say i and j, then $b-a = |B_j| - |A_i| \leq |B_j - A_i| < k$, hence (8) implies $|I_{a,b}| \leq \binom{3k}{k}$.

Finally, let us note that since $|A_1 \setminus B_i| < k$ and $|B_1 \setminus A_i| < k$ for any $i \geq 2$, it follows that $|B_1| - 2k < |A_i| < |B_1| + 2k$ and $|A_1| - 2k < |B_i| < |A_1| + 2k$, hence $|\{(a, b) : I_{a,b} \neq \emptyset\}| = O(k^2)$. Thus,

$$t \leq O\left(k^2 \binom{3k}{k}\right)$$

follows. □

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