DIMACS Technical Report 2000–32
September 2000

Minimal Forbidden Graphs for L(2,1)-Colorings

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DIMACS is a partnership of Rutgers University, Princeton University, AT&T Labs-Research, Bell Labs, NEC Research Institute and Telcordia Technologies (formerly Bellcore).

DIMACS is an NSF Science and Technology Center, funded under contract STC–91–19999; and also receives support from the New Jersey Commission on Science and Technology.

ABSTRACT

The span $\lambda(G)$ of a graph G is the smallest k for which G's vertices can be $L(2,1)$-colored, i.e., colored with integers in $\{0,1,\ldots,k\}$ so that adjacent vertices' colors differ by at least two, and colors of vertices at distance two differ. We study minimal forbidden graphs, graphs with the property that every proper subgraph has smaller span. Fixing the maximum degree, we observe that the first nontrivial case involves minimal forbidden graphs of maximum degree 3 and span 5. We present examples to illustrate the variety of graphs with this property and in particular provide several infinite families of such graphs.

1 INTRODUCTION

$L(2,1)$-colorings assign integers to vertices of a graph G so that the colors assigned to adjacent vertices differ by at least 2 and colors assigned to vertices at distance 2 in the graph differ by at least 1. $L(2,1)$-colorings were first investigated extensively by Yeh [11] and Griggs and Yeh [5] as a generalization of T-colorings for the channel assignment problem that were introduced by Hale [6]: see also Roberts [8]. The ***span*** $\lambda(G)$ of a graph G is the smallest $k \geq 2$ for which G has an $L(2,1)$-coloring using colors in $\{0, 1, \ldots, k\}$. In this paper, we study ***minimal forbidden graphs***, graphs with the property that every proper subgraph has a smaller span. Fixing the maximum degree, we observe that the first nontrivial case involves graphs of maximum degree 3 and span 5. Complete characterizations of the minimal forbidden graphs in this 3-5 case is difficult if not intractable. We observe that the class of such graphs is very complex and provide several infinite families of minimal forbidden graphs and minimal forbidden trees in the 3-5 case.

In the rest of this section, we introduce our notation, note a few fundamental results we shall use throughout, and outline the rest of the paper.

All graphs in this paper will be finite, simple graphs. P_n, C_n, and K_n denote the n-vertex path, cycle, and complete graph, respectively. If $G = (V, E), x \in V, e \in E$, then $G - x$ is the subgraph of G induced by vertices different from x and $G - e$ the subgraph obtained by omitting edge e but not its end-vertices.

We use the notation $[0, k]$ for the set $\{0, 1, \ldots, k\}$. If f is an $L(2,1)$-coloring into $[0, \lambda(G)]$, then f will be called a ***span coloring***. An $L(2,1)$-coloring f into $[0, \lambda(G)]$ has a natural dual g defined by $g(x) = \lambda - f(x)$. We confine attention to the class $\mathcal{G}$ of connected graphs and will base much of our analysis on $\Delta = \Delta(G)$, the *maximum degree* of a vertex in G. Accordingly, let

$$\begin{aligned} \mathcal{G}_\Delta &= \{G \in \mathcal{G} : \ G \text{ has maximum degree } \Delta\} \\ \mathcal{G}_\Delta(\lambda) &= \{G \in \mathcal{G}_\Delta : \ G \text{ has span } \lambda\} \text{ for } \lambda \geq \Delta + 1 \ , \end{aligned}$$

where $\lambda \geq \Delta + 1$ is presumed because $\mathcal{G}_\Delta(\lambda)$ is empty if $\lambda \leq \Delta$. Also let $\mathcal{T}_\Delta$ and $\mathcal{T}_\Delta(\lambda)$ be the subsets of trees in $\mathcal{G}_\Delta$ and $\mathcal{G}_\Delta(\lambda)$, respectively.

The key notion in this paper is that of a ***λ-minimal forbidden graph*** or ***λ-mfg***, a graph G with the property that $\lambda(H) < \lambda(G)$ for all proper subgraphs H of G, where "proper" means that we delete either vertices or edges. A ***λ-minimal forbidden tree*** or ***λ-mft*** is a λ-minimal forbidden graph that is a tree. Let $F(\mathcal{G}_\Delta(\lambda))$ and $F(\mathcal{T}_\Delta(\lambda))$ denote the set of λ-mfg's of maximum degree Δ and the set of λ-mft's of maximum degree Δ, respectively. In Section 2, we shall give descriptions of $F(\mathcal{G}_\Delta(\lambda))$ and $F(\mathcal{T}_\Delta(\lambda))$ for $\Delta = 2$, arbitrary λ, and for $\Delta = 3, \lambda = 4$. The rest of the paper will be devoted to the first nontrivial case, $\Delta = 3, \lambda = 5$. Section 3 will give a variety of examples of graphs in the classes $F(\mathcal{G}_3(5))$ and $F(\mathcal{T}_3(5))$, while Section 4 will provide some infinite classes of trees in the latter cl! ass and Section 5 some infinite classes of graphs in the former class.

We close this section with two observations that will be useful throughout.

Theorem 1.1 *(Griggs and Yeh ([5]). If $T \in \mathcal{T}_\Delta$ then $\lambda(T) \in \{\Delta + 1, \Delta + 2\}$.*

Thus, every tree with maximum degree Δ has span $\Delta + 1$ or $\Delta + 2$. Chang and Kuo [1] present a polynomial time algorithm that determines $\lambda(T)$ for $T \in \mathcal{T}$. On the other hand, Griggs and Yeh [5] prove that determination of $\lambda(G)$ for $G \in \mathcal{G}$ is NP-complete. It is also known [1] that $\lambda(G) \leq \Delta^2 + \Delta$ for $G \in \mathcal{G}_\Delta$, but the exact value of $\max\{\lambda(G) : G \in \mathcal{G}_\Delta\}$ is unknown for most Δ. Tighter bounds on λ and related results for special families of graphs in $\mathcal{G}_\Delta$ are established in [1, 2, 3, 4, 5, 7, 9, 10, 11].

Proposition 1.1 *If $G \in \mathcal{G}_\Delta$, degree of u is Δ, and f is an $L(2,1)$-coloring into $[0, \Delta + 1]$, then $f(u) \in \{0, \Delta + 1\}$ and u has neighbors with all colors except $f(u)$ and either $f(u) + 1$ or $f(u) - 1$.*

Proof. This is straightforward. It also follows from the observation of Griggs and Yeh [5] that $f(u) \in \{0, \Delta + 1\}$ if f is a span coloring for a graph in $\mathcal{G}_\Delta(\Delta + 1)$. Since $\lambda \geq \Delta + 1$, if there is an $L(2,1)$-coloring using colors in $[0, \Delta + 1]$, this must be a span coloring and the graph must be in $\mathcal{G}_\Delta(\Delta + 1)$. The last part is trivial. ∎

Corollary 1.1 *If $G \in \mathcal{G}_\Delta(\Delta+1)$, then a vertex of degree Δ cannot be adjacent to two other vertices of degree Δ.*

2 The Mft's and Mfg's When $\lambda = 2$ and When $\lambda = 3, \Delta = 4$

Theorem 2.1 (Griggs and Yeh ([5]). $\lambda(P_2) = 2; \lambda(P_3) = \lambda(P_4) = 3; \lambda(P_n) = 4$ *for* $n \geq 5; \lambda(C_n) = 4$ *for all* n.

If $\Delta = 2$ and G is a connected graph, then it is a path or cycle. Theorem 2.1 shows that $F(\mathcal{G}_2(3)) = F(\mathcal{T}_2(3)) = \{P_3\}$, $F(\mathcal{G}_2(4)) = \{P_5\} \cup \{C_3, C_4\}$, $F(\mathcal{T}_2(4)) = \{P_5\}$, and $F(\mathcal{G}_2(\lambda)) = \emptyset$ for $\lambda \geq 5$.

Suppose that $\Delta = 3$. Then $\lambda \geq \Delta + 1 = 4$. If G is a connected graph with at least five vertices, then we can remove one vertex and still have $\Delta = 3$ for the others. Hence every member of $F(\mathcal{G}_3(4))$ and $F(\mathcal{T}_3(4))$ must have exactly four vertices. It is easily seen that the only graph in these two sets is the star with three degree-1 vertices. By contrast, $F(\mathcal{G}_3(5))$ and $F(\mathcal{T}_3(5))$, which we examine henceforth, are infinite sets.

3 Some Useful Lemmas and Many Examples

In this section, we state a number of lemmas that will be useful in proving that graphs belong to the classes $F(\mathcal{G}_3(5))$ and $F(\mathcal{T}_3(5))$. We then provide a variety of examples of graphs in these classes.

We shall use the shorthand ***4L-coloring*** to mean an $L(2,1)$-coloring using only colors from $[0, 4]$. It is useful to consider all possible $4L$-colorings of paths in G that start with 0 or 4. In the next lemma, we list these for paths of three to seven vertices, since we will need these results explicitly in what follows. Verification of the lemma is straightforward.

Lemma 3.1 *The following are all possible 4L-colorings of a path of n vertices starting with color 4. Those starting with color 0 are obtained from these by duality.*
(a) If $n = 3$: 402, 403, 413, 420;
(b) If $n = 4$: 4024, 4031, 4130, 4203, 4204;
(c) If $n = 5$: 40240, 40241, 40314, 41302, 41304, 42031, 42041, 42042;
(d) If $n = 6$: 402402, 402403, 402413, 403140, 403142, 413024, 413041, 413042, 420314, 420413, 420420;
(e) If $n = 7$: 4024024, 4024031, 4024130, 4031402, 4031403, 4031420, 4130240, 4130241, 4130413, 4130420, 4203140, 4203142, 4204130, 4204203, 4204204.

Lemma 3.2 *In any 4L-coloring f of the graph of Figure 1A, $f(x) = 2$.*

Proof. Proposition 1.1 and Lemma 3.1(a). ∎

Figure 1 about here

Lemma 3.3 *In any 4L-coloring f of the graph of Figure 1B, either $f(a) \neq 2$ or $f(e) \neq 2$.*

Proof. Proposition 1.1 and Lemma 3.1(e). ∎

Lemma 3.4 *In any 4L-coloring f of the graph of Figure 1C, neither $f(a)$ nor $f(c)$ can be 2.*

Proof. Proposition 1.1 and Lemma 3.1(c). ∎

Lemma 3.5 *In any 4L-coloring f of the graph of Figure 1D:*
(a). If $f(y_1) \in \{1, 3\}$, then $f(y_2) \in \{1, 3\}$.
(b). If $f(y_1) = 2$, then $f(y_2) = 2$.

Proof. By Proposition 1.1, $f(x_1), f(x_2), f(b_2)$ are all in {0,4}. Thus, $f(z_2) \notin \{0, 4\}$. In part (a), we use Lemma 3.1(e) to conclude that the path from x_1 to x_2 can only be colored 4130420 or 0314024. Hence, $f(z_2) = 2$ and $f(y_2) \in \{1, 3\}$. In part (b), the possible colorings for the path are 4204130 or 0240314. Thus, $f(z_2) \in \{1, 3\}$ and $f(y_2) = 2$. ∎

Lemma 3.6 *In any 4L-coloring f of the graph of Figure 1E, $f(y_1) = 2$.*

Proof. By Proposition 1.1, $f(x_0), f(x_1), f(b_1)$ are all in $\{0, 4\}$. Thus, $f(z_1) \notin \{0, 4\}$. By Lemma 3.1(c), the only colorings of the path from x_0 to x_1 are 40314 and 04130. Hence, $f(z_1) \in \{1, 3\}$ and $f(y_1) = 2$. ∎

The next three lemmas are straightforward.

Lemma 3.7 *The only 4L-colorings of C_4 use the colors 0,4,1,3 around the cycle in a clockwise or counterclockwise order.*

Lemma 3.8 *If G consists of C_4 together with a vertex x adjacent to one vertex of C_4, then in any 4L-coloring f of G, $f(x) = 2$.*

Lemma 3.9 *A tree T has $\lambda(H) \leq 4$ for all proper subgraphs H of T if and only if $\lambda(T-x) \leq 4$ for all vertices x of degree 1. In particular, if $T \in \mathcal{T}_3(5)$, then $T \in F(\mathcal{T}_3(5))$ if and only if $\lambda(T-x) \leq 4$ for all leaves x.*

It is also apparent that a graph G has $\lambda(H) \leq 4$ for all proper subgraphs H of G if and only if $\lambda(G-e) \leq 4$ for every edge e of G.

The next two propositions present a variety of examples of graphs in the classes $F(\mathcal{G}_3(5))$ and $F(\mathcal{T}_3(5))$. The proof that each has span greater than 4 follows in most (but not all) cases from the lemmas we have presented. The proof that it has span at most 5 follows from Theorem 1.1 for trees and is straightforward for the non-trees. The proof that every proper subgraph has span at most 4 is straightforward but tedious. We shall present a few of the arguments for illustrative purposes, concentrating mostly on arguments showing that the graphs in question do not have $4L$-colorings.

Proposition 3.1 *All graphs of Figure 2 are in $F(\mathcal{G}_3(5))$.*

Figure 2 about here

To present trees in $F(\mathcal{T}_3(5))$, we observe that every leaf of such a tree must be adjacent to a degree-3 vertex. We use the shorthand of suppressing leaves and vertices of degree 2, thus showing only degree-3 vertices. We put along an edge a number that tells us the number of edges on the path between the two degree-3 vertices. Thus, for example, the trees T_1, T_2, T_{10}, T_{16} of Figure 3 are the trees shown in full in Figure 4.

Proposition 3.2 *All graphs of Figure 3 are in $F(\mathcal{T}_3(5))$.*

Figure 3 about here

That the graph G_1 of Figure 2 has no $4L$-coloring follows by Corollary 1.1. We conclude that $G_1 \in F(\mathcal{G}_3(5))$ if we can show that removal of any edge leads to a graph with $\lambda \leq 4$. This is easy to show.

By Proposition 1.1 and Lemma 3.1(a), the degree-2 vertices of graph G_2 of Figure 2 are each labeled 2 in any $4L$-coloring, which cannot be. The same argument applies to graphs G_3 and G_8.

Consider graph G_7 of Figure 2. By Proposition 1.1, the two degree-3 vertices get colored 0 or 4 in any $4L$-coloring. Then Lemma 3.1(c) shows that there is no way to label the vertices in the path of length 5 between these two vertices.

Next, consider graph G_{10} of Figure 2. By Proposition 1.1, the two degree-3 vertices get colored 0 or 4 in any $4L$-coloring and, by Lemma 3.1(a), the vertex between them gets colored 2. It follows from Proposition 1.1 that on the cycle, one of the neighbors of the degree-3 vertex gets colored 1 or 3. Lemma 3.1(c) shows that there is no possible coloring of the vertices on the cycle.

We turn next to the trees of Figure 4. By Corollary 1.1, there is no $4L$-coloring of tree T_1.

Figure 4 about here

By Lemma 3.1(a), if f is a $4L$-coloring of tree T_2, then the two neighbors of the middle vertex both get color 2, which is impossible.

Consider tree T_{10}. By Lemma 3.4, if f is a $4L$-coloring, then $f(u) \neq 2$ and $f(v) \neq 2$. By Proposition 1.1, the vertices of degree 3 get colors in $\{0, 4\}$. Thus, we know that two of the neighbors of x and of y use colors other than 2 and Proposition 1.1 implies that $f(b) = f(c) = 2$. This contradicts Lemma 3.3.

We give a complete proof that tree T_{16} is in $F(\mathcal{T}_3(5))$. Suppose that f is a $4L$-coloring. By Lemma 3.1(a), $f(b_2) = f(y_0) = 2$. Thus, $f(y_1) \neq 2$ and $f(z_2) \neq 2$. By Proposition 1.1 and Lemma 3.1(b), the path x_1, y_1, z_2, x_2 must get colored 4130 or 0314. By Proposition 1.1, $f(y_2) \in \{0, 4\}$. It now follows by Proposition 1.1 and Lemma 3.1(d) that the path from x_2 to x_3 must get colored 403140 or 041304, so that $f(z_3) \in \{0, 4\}$. However, either $f(x_3) = 4$ or $f(x_4) = 4$, which contradicts $f(z_3) \in \{0, 4\}$. We conclude that there is no such f. We complete the proof by showing that for all leaves x, $T - x$ has a $4L$-coloring. It suffices to give the colors for the vertices other than leaves. There is always a color left over for a leaf. For each leaf x, we shall describe how to color $T - x$ by saying what the colors on the path P from x_0 to x_4 will be and then giving $f(b_2)$ and $f(a_2)$.

Removing a: P: 024130413042; $f(b_2) = 2, f(a_2) = 4$.

Removing b: P: 024130420420; $f(b_2) = 2, f(a_2) = 4$.

Removing c: P: 024130240240; $f(b_2) = 4, f(a_2) = 2$.

Removing d: P: 041304130240; $f(b_2) = 2, f(a_2) = 0$.

Removing e: P: 130240314204; $f(b_2) = 2, f(a_2) = 4$.

The examples of Figures 2 and 3 hint at a vast array of minimal forbidden graphs and trees for the case $\Delta = 3, \lambda = 5$. Although their variety is unknown and perhaps unspecifiable except by algorithm, we do know that they are infinite in number. The demonstration of this fact, both for trees and for arbitrary connected graphs, is the subject of the next two sections.

4 Infinite Families in $F(\mathcal{T}_3(5))$

Theorem 4.1 *The family of trees T_m, $m \geq 1$, defined in Figure 5A is an infinite family in $F(\mathcal{T}_3(5))$.*

Proof. Assume that f is a $4L$-coloring of T_m. Lemma 3.1(a) implies that $f(y_0) = 2, f(y_m) = 2$. By Proposition 1.1, $f(y_1) \in \{1, 3\}$. Lemma 3.5 implies that $f(y_2) \in \{1, 3\}$. Repeating the argument, we conclude that $f(y_m) \in \{1, 3\}$. We have reached a contradiction.

By Lemma 3.9, it remains to show that there is a $4L$-coloring for $T_m - x$ whenever x is a leaf of T_m. By symmetry, it suffices to consider the cases where $x = a_k, 1 < k \leq m + 1$. In describing the $4L$-colorings in these cases, it suffices to give the colors for each non-leaf, as there will always be a color left over for each leaf. If $x = a_{m+1}$, we may use the coloring 420 on the path x_0, y_0, x_1. Then use 0314024 on the path $x_1, y_1, \ldots, z_2, x_2$, 4130420 on the path $x_2, y_2, \ldots, z_3, x_3$, and continue alternating with this pattern until coloring x_m. We now color y_m, x_{m+1} with 13 or 31. We complete the coloring by taking $f(b_i) = 4 - f(x_i)$.

Next suppose that $x = a_k$, $1 < k < m + 1$. Use the above pattern up until the path $x_{k-1}, y_{k-1}, \ldots, z_k, x_k$ is colored using 0314024 or 4130420, without loss of generality the former (by duality). If $k = m$, replace the last 0314024 by 0314204 and let $f(b_m) = 1$, $f(y_m) = 2$, $f(x_{m+1}) = 0$. Define $f(b_i) = 4 - f(x_i)$ for $i < m$. If $k < m$, take $f(b_k) = 1, f(y_k) = 0$, and color the path $x_k, y_k, \ldots, z_{k+1}, x_{k+1}$ using the colors 4024130. Then color the path $x_{k+1}, y_{k+1}, \ldots, z_{k+2}, x_{k+2}$ with 0240314, the path $x_{k+2}, y_{k+2}, \ldots, z_{k+3}, x_{k+3}$ with 4204130, and continue alternating until coloring x_m. The coloring is concluded by taking $f(y_m) = 2, f(x_{m+1}) = 4 - f(x_m)$, and $f(b_i) = 4 - f(x_i)$ for $i \neq k$. ∎

Figure 5 about here

Theorem 4.2 *The family of trees S_m, $m \geq 1$, defined in Figure 5B is an infinite family in $F(\mathcal{T}_3(5))$.*

Proof. Assume that f is a $4L$-coloring of S_m. By Proposition 1.1, $f(z_1) \notin \{0, 4\}$. Using Lemma 3.1(c), we conclude that the path x_0, y_0, w_0, z_1, x_1 is colored 40314 or 04130. Thus, $f(z_1) \in \{1, 3\}$. A symmetric argument using the path $x_{m+1}, z_{m+1}, w_m, y_m, x_m$ shows that $f(y_m) \in \{1, 3\}$. By Proposition 1.1, $f(y_1) = 2$. Lemma 3.5 implies that $f(y_2) = 2$. Repeating the argument, we conclude that $f(y_m) = 2$, which is a contradiction.

We now show that there is a $4L$-coloring for $S_m - x$ whenever x is a leaf of S_m. By symmetry, it suffices to consider the cases where $x = a_k, 1 < k \leq m + 1$. If $x = a_{m+1}$, use the coloring 40314 on the path from x_0 to x_1. Then use 4204130 on the path from x_1 to x_2, 0240314 on the path from x_2 to x_3, and continue alternating these patterns until coloring x_m. Color the path from x_m to x_{m+1} using the colors 42042 or 02402. Finally, take $f(b_i) = 4 - f(x_i)$ for all i.

Suppose that $x = a_k$, $1 < k < m+1$. Use the coloring above up through x_{k-1}. Thus, the path $x_{k-1}, y_{k-1}, \ldots, z_k, x_k$ is colored using 4204130 or 0240314, without loss of generality the former (by duality). Replace these colors by 4203140. If $k = m$, then color the path from

x_m to x_{m+1} by 03140 and let $f(b_i) = 4 - f(x_i)$ for $i \neq k, f(b_k) = 2$. If $k < m$, color the path from x_k to x_{k+1} using colors 0314024, the path from x_{k+1} to x_{k+2} using colors 4130420, and keep alternating these patterns until coloring x_m. Color the path from x_m to x_{m+1} using colors 03140 or colors 41304. Finally, take $f(b_i) = 4 - f(x_i), i \neq k, f(b_k) = 2$. ∎

Theorem 4.3 *The family of trees R_m, $m \geq 1$, defined in Figure 5C is an infinite family in $F(\mathcal{T}_3(5))$.*

Proof. Suppose f is a $4L$-coloring of R_m. As in the proof of Theorem 4.2, we have $f(y_m) = 2$. By Lemma 3.4, $f(w_1) \neq 2$ and $f(w_2) \neq 2$. By Proposition 1.1, $f(z_{m+1}) = 2$. This violates Lemma 3.3.

We now show that $R_m - x$ has a $4L$-coloring for all leaves x. We consider $a_k, 0 \leq k \leq m$ and c_1.

If we remove c_1, then consider the coloring of $S_m - a_{m+1}$ in the proof of Theorem 4.2. Use that coloring left to right for R_m starting from x_0 until hitting x_m. Then color the path from x_m to x_{m+1} using the colors 4204130 or 0240314. Color the path from x_{m+1} to d_1 using the colors 42042 or 02402 and the path from x_{m+1} to d_2 using the colors 40314 or 04130. Let $f(b_i) = 4 - f(x_i)$, all i.

If we remove a_0, color the path from x_0 to x_1 with 20420, the path from x_1 to x_2 with 0314024, the path from x_2 to x_3 with 4130420, and alternate these latter two patterns until coloring x_{m+1}. Now color the path from x_{m+1} to d_1 with 04204 or 40240 and the path from x_{m+1} to d_2 with 03140 or 41304. Let $f(b_i) = 4 - f(x_i)$ for all i.

If we remove $a_k, 1 \leq k \leq m$, color the path from x_0 to x_1 with 04130, the path from x_1 to x_2 with 0241304, the path from x_2 to x_3 with 4203140, alternating these two patterns until coloring x_k. Then on the path from x_k to x_{k+1}, use the colors 0314024 or 4130420, alternating these patterns until coloring x_{m+1}. Color the path from x_{m+1} to d_1 with 04204 or 40240 and the path from x_{m+1} to d_2 with 03140 or 41304. Finally, let $f(b_i) = 4 - f(x_i)$ for $i \neq k$ and $f(b_k) = 2$. ∎

5 Infinite Families in $F(\mathcal{G}_3(5))$

Relatively simple variants of the examples of the previous section give examples of non-trees in $F(\mathcal{G}_3(5))$.

Theorem 5.1 *The family of graphs G_m, $m \geq 1$, defined in Figure 6 is an infinite family in $F(\mathcal{G}_3(5))$.*

Proof. Suppose f is a $4L$-coloring of G_m. Lemma 3.8 implies that $f(y_0) = f(z_{m+1}) = 2$. By Proposition 1.1, $f(z_1) \notin \{0, 4\}$. Thus, Lemma 3.1(e) implies that the path from x_0 to x_1 gets colored 4204130 or 0240314. Therefore, $f(z_1) \in \{1, 3\}$ and we have $f(y_1) = 2$. Lemma 3.5 implies that $f(y_2) = 2$ and, repeating the argument, that $f(y_m) = 2$. But this and $f(z_{m+1}) = 2$ contradict Lemma 3.3. Removal of any edge leads to a $4L$-coloring, which completes the proof. The details are left to the reader. ∎

Figure 6 about here

Theorem 5.2 *The family of graphs H_m, $m \geq 1$, defined in Figure 6 is an infinite family in $F(\mathcal{G}_3(5))$.*

Proof. Suppose f is a $4L$-coloring of H_m. Lemma 3.8 implies that $f(z_{m+1}) = 2$. By Proposition 1.1, $f(z_1) \notin \{0, 4\}$. By Lemma 3.1(c), the path from x_0 to x_1 must get the colors 40314 or, by duality, 04130. Thus, $f(z_1) \in \{1, 3\}$ and we have $f(y_1) = 2$. Repeated uses of Lemma 3.5 give us the conclusion that $f(y_m) = 2$, which contradicts Lemma 3.3. Removal of any edge leads to a $4L$-coloring, which completes the proof. The details are left to the reader. ∎

6 Closing Remarks

In this paper, we have illustrated the wide variety of graphs in the classes $F(\mathcal{G}_3(5))$ and $F(\mathcal{T}_3(5))$. One is left with the problem of characterizing these classes, a problem which might be difficult to solve except possibly algorithmically. We leave it to future research to investigate the classes $F(\mathcal{G}_\Delta(\lambda))$ and $F(\mathcal{T}_\Delta(\lambda))$ for other values of $\lambda \geq 3$ and $\Delta \geq 5$.

Acknowledgment

We are indebted to Jan Kratochvil and Aleksandar Pekeč for discussions on $L(2,1)$-colorings. Fred Roberts thanks the National Science Foundation for its support under grant NSF-SBR-9709134 to Rutgers University.

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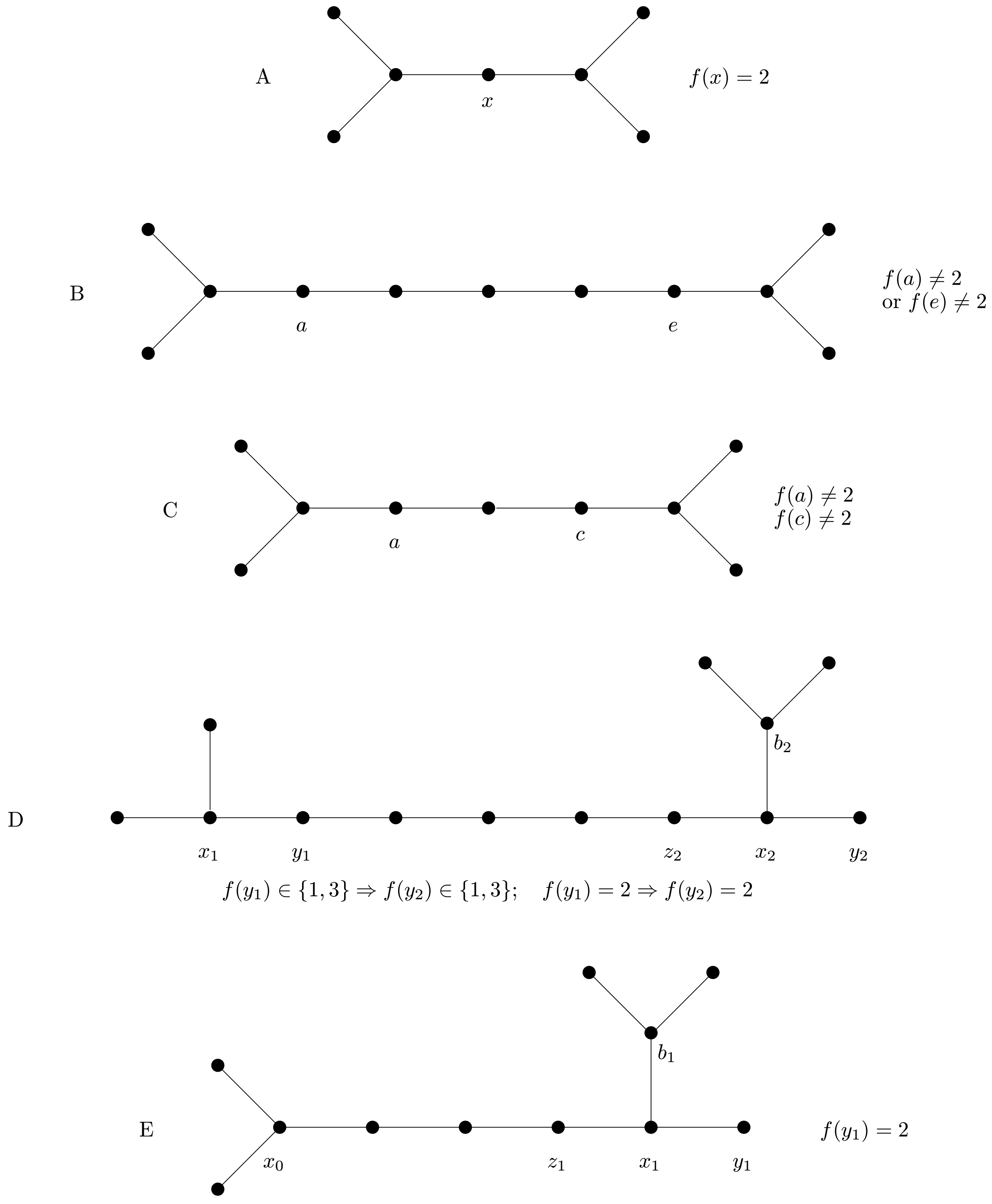

Figure 1. Necessary colors and implications for any $4L$-coloring f.

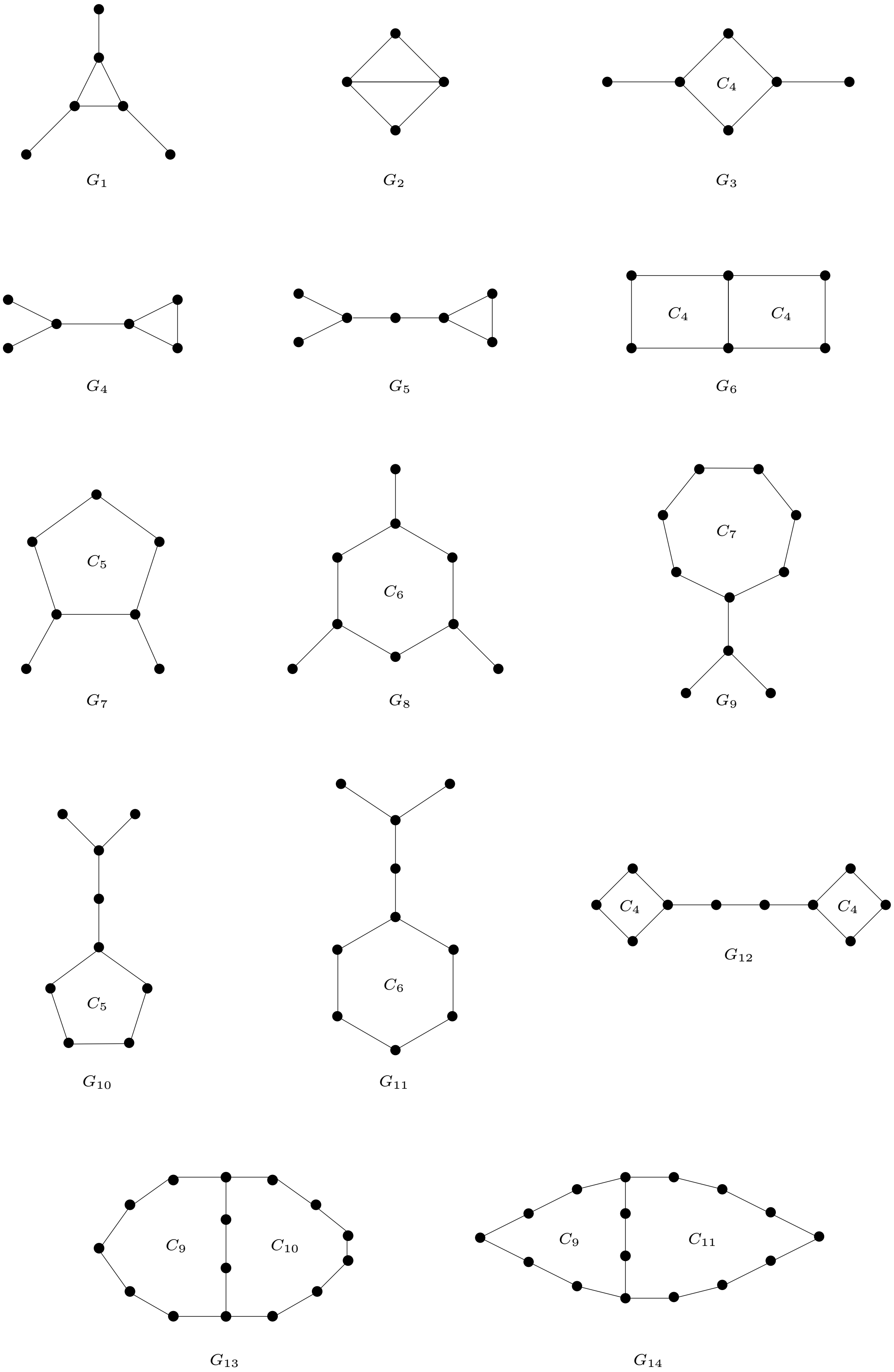

Figure 2. Graphs in $F(\mathcal{G}_3(5))$.

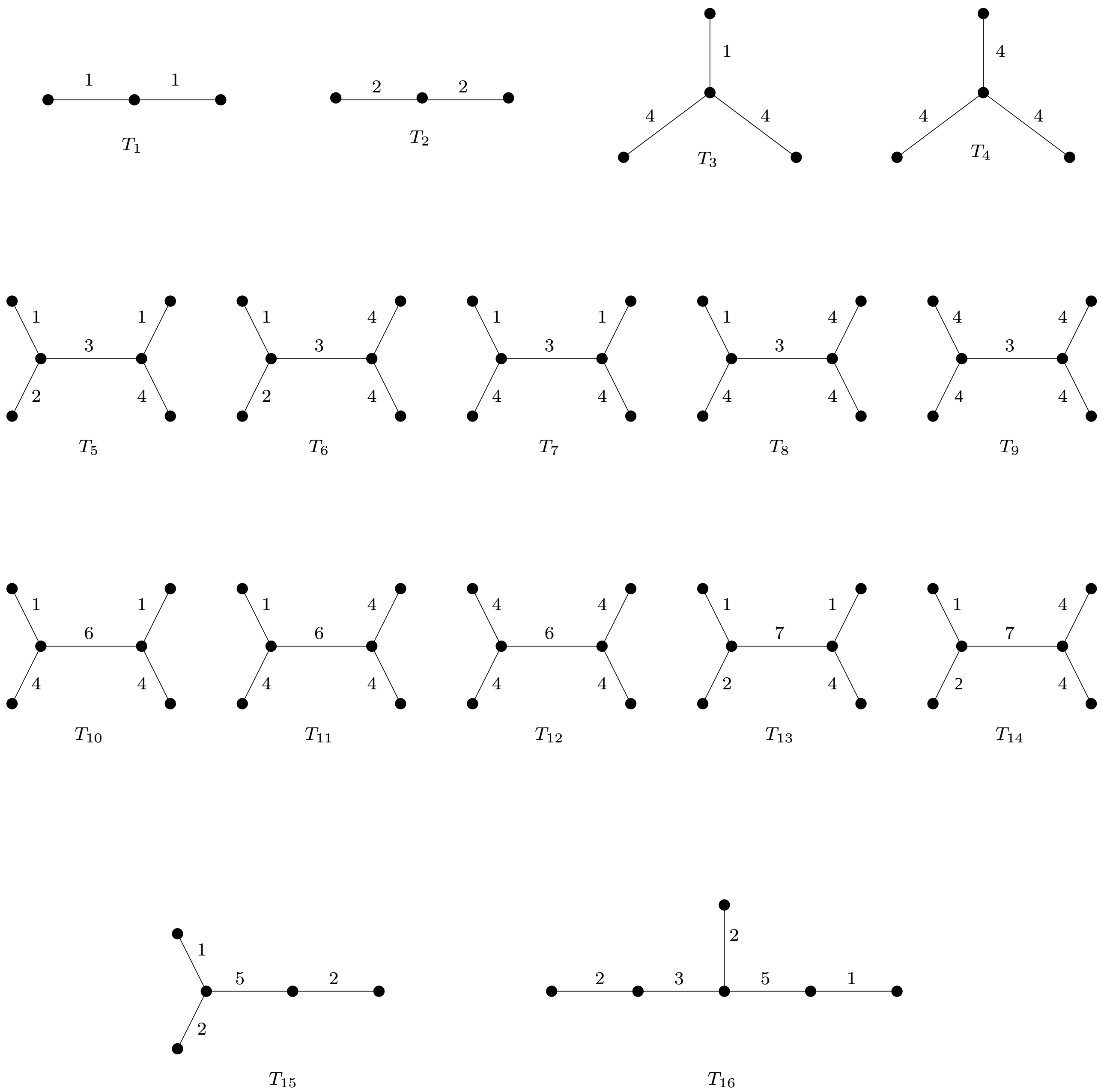

Figure 3. Trees in $F(\mathcal{T}_3(5))$. Edge number is length of path between degree-3 vertices. Leaves are not shown.

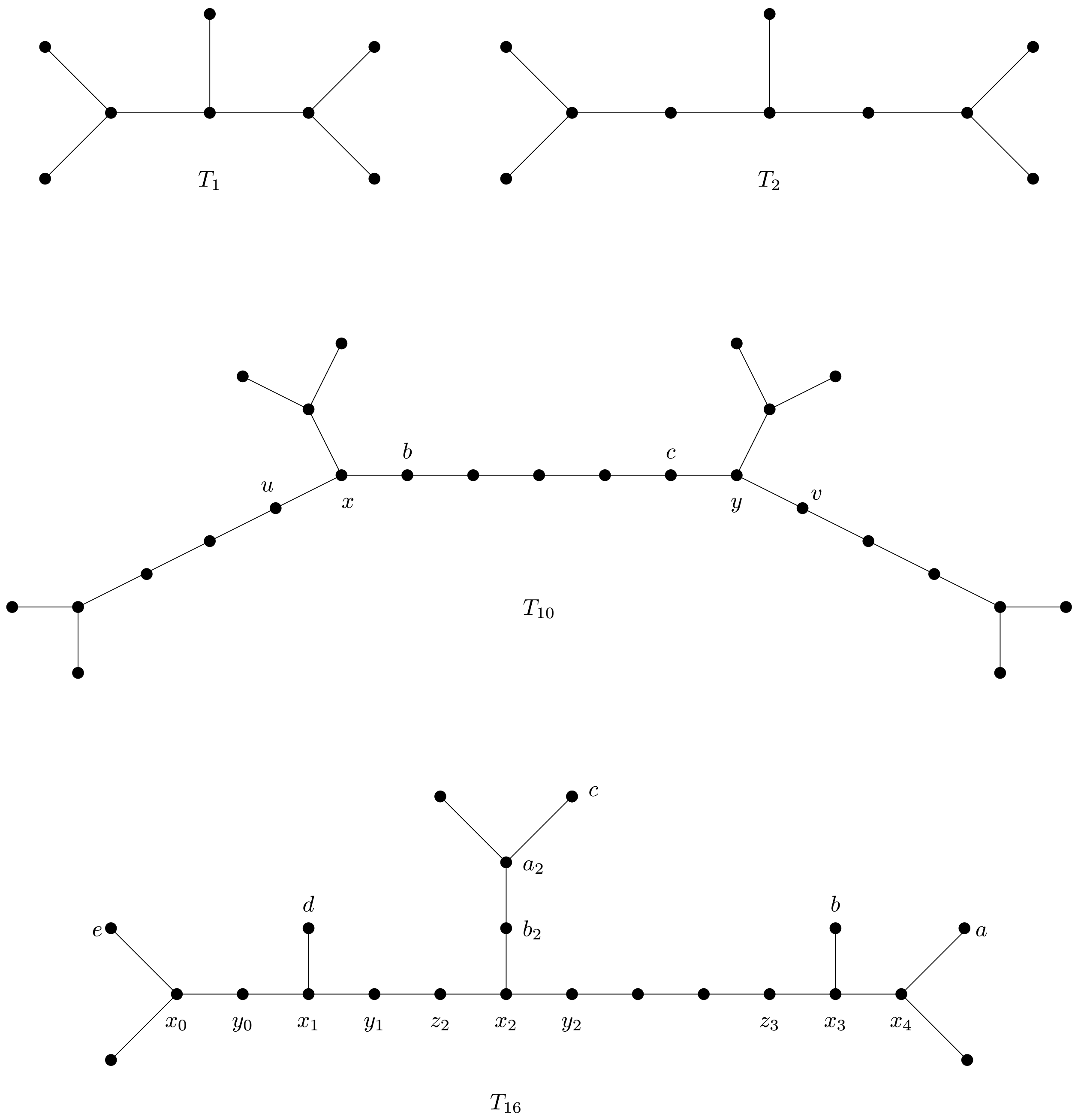

Figure 4. Full versions of some trees of Figure 3.

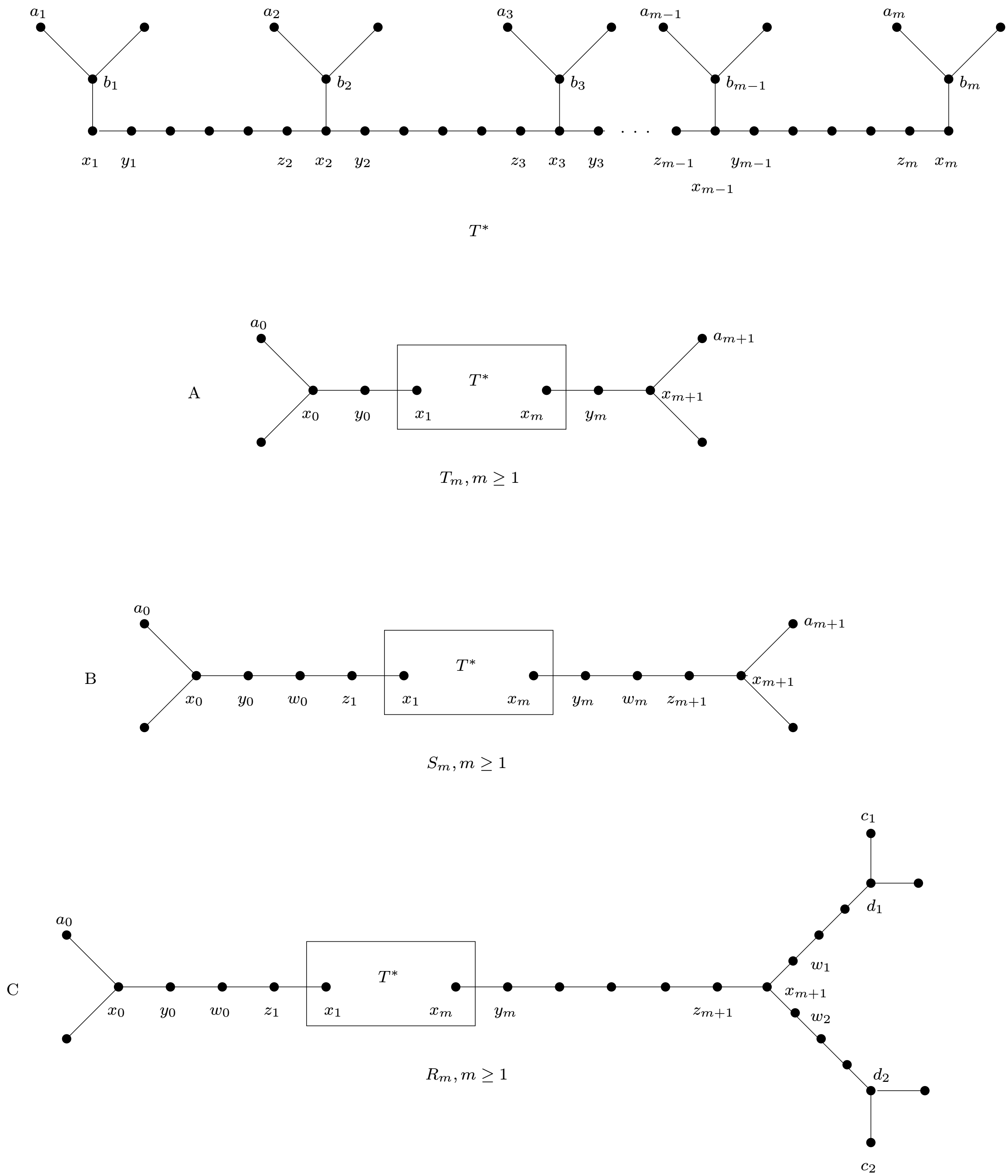

Figure 5. Every R_m, S_m, T_m is in $F(\mathcal{T}_3(5))$.

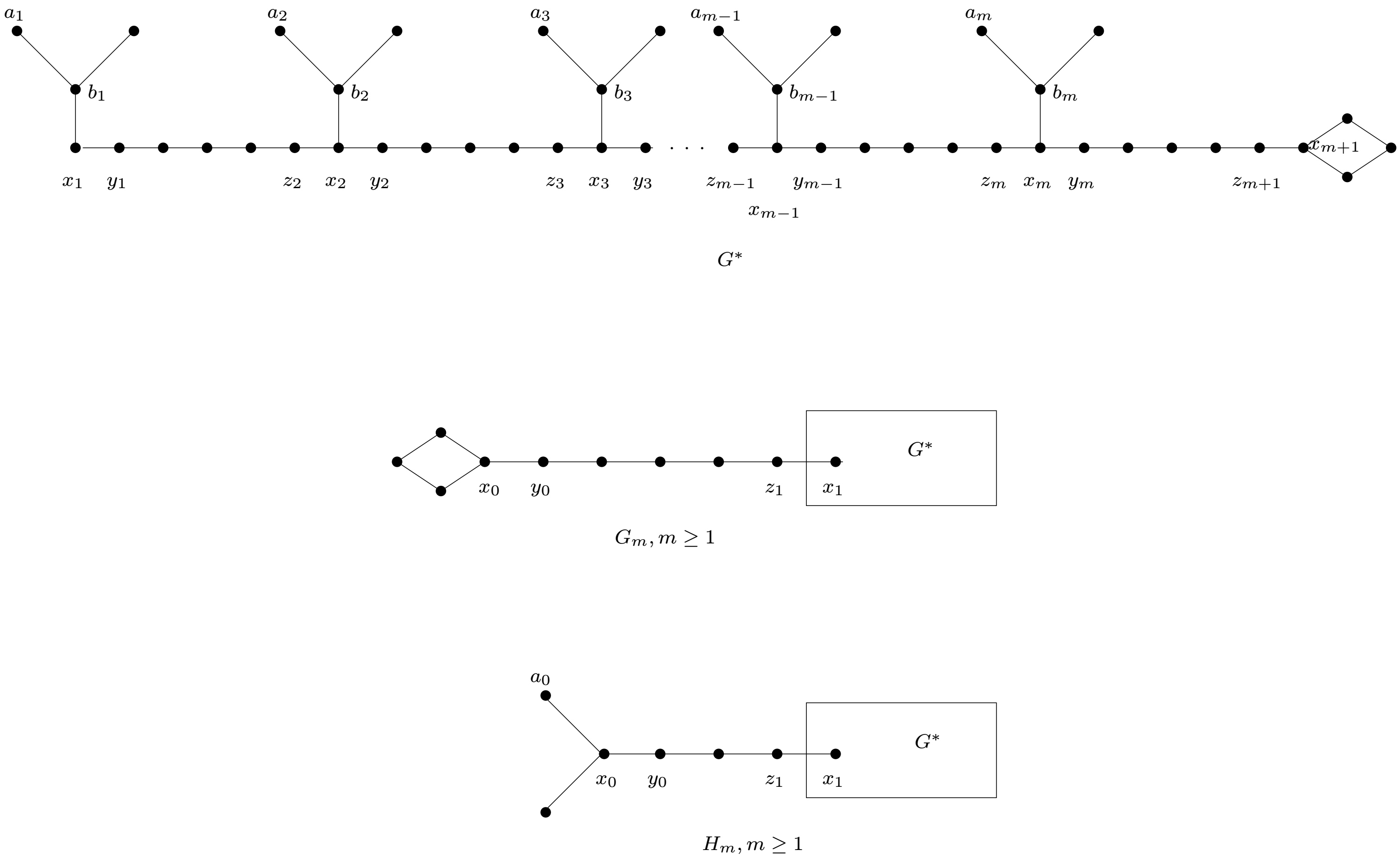

Figure 6. Every G_m and H_m is in $F(\mathcal{G}_3(5))$.