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HOW MANY DISJOINT 2–EDGE PATHS MUST A CUBIC GRAPH HAVE ?

by

Alexander Kelmans [1]
RUTCOR, Rutgers University,
New Brunswick, New Jersey 08903
and
Department of Mathematics, University of Puerto Rico
San Juan, PR 00931,
Email: `kelmans@rutcor.rutgers.edu`

Dhruv Mubayi [2]
Georgia Institute of Technology,
Atlanta, GA 30332-0160, USA.
Email: `mubayi@math.gatech.edu`

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ABSTRACT

In this paper we show that every simple cubic graph on n vertices has a list of at least $\lceil n/4 \rceil$ disjoint 2–edge paths and that this bound is sharp. Our proof provides a polynomial time algorithm for finding such a list in a simple cubic graph.

Keywords: packing, 2–edge path, cubic graph, polynomial time algorithm.

1 Introduction

Contents

We consider simple undirected graphs. All notions on graphs that are not defined here can be found in [2].

Given a graph G and a set $\mathcal{F}$ of subgraphs of G, an *$\mathcal{F}$–packing* of G is a subgraph of G whose components are members of $\mathcal{F}$. The *$\mathcal{F}$–packing problem* is that of finding an $\mathcal{F}$–packing having the maximum number of vertices.

Various $\mathcal{F}$–packing problems for a graph have extensively been studied by many authors for different families $\mathcal{F}$ (e.g. [1, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13]). It is not surprising that $\mathcal{F}$–packing turns out to be NP–hard for most of the families $\mathcal{F}$. Surprisingly the problem can be solved in polynomial time for some non–trivial families. For example, Edmonds (see [13]) showed that a classical matching problem can be solved in polynomial time. It is also known that the problem of packing stars of at least one and at most k edges is polynomially solvable even if the stars we pack are required to be induced [7]. On the other hand, the problem of finding in G the maximum number of disjoint subgraphs, isomorphic to a given connected graph H of at least three vertices, is NP–hard [6].

In this paper we consider a special case of the above mentioned problem when $H = \Lambda$, a path of two edges (or simply, a *2–path*). Although the Λ–packing problem is NP–hard, i.e. possibly intractable in general, this problem turns out to be tractable for some natural classes of graphs. Here are some examples of such results.

Let $v(G)$ and $\lambda(G)$ denote the number of vertices and maximum number of disjoint 2–edge paths in a graph G. Obviously $\lambda(G) \le \lfloor v(G)/3 \rfloor$.

A graph is called *claw–free* if it contains no induced subgraph isomorphic to $K_{1,3}$ (which is called a *claw*). A block of a connected graph is called an *end–block* if it has at most one vertex in common with any other block of the graph.

1.1 [5] *Suppose that G is a connected claw–free graph having at most two end–blocks (in particular, a 2–connected claw–free graph). Then $\lambda(G) = \lfloor v(G)/3 \rfloor$.*

Let G^{Δ} be the graph obtained from a cubic graph by "replacing" every vertex by a triangle. From **1.1** we have, in particular, for cubic graphs:

1.2 *Suppose that G is a connected cubic graph having at most two end–blocks (in particular, a 2–connected graph). Then $\lambda(G^{\Delta}) = \lfloor v(G^{\Delta})/3 \rfloor$.*

Let $eb(G)$ denote the number of end–blocks of G.

1.3 [5] *Suppose that G is a simple connected claw–free graph and $eb(G) \geq 2$. Then $\lambda(G) \geq \lfloor (v(G) - eb(G) + 2)/3$, and this lower bound is sharp.*

1.4 [9] *Let T be a tree on t vertices. Let G_n be a d_n-regular graph on n vertices. Suppose that $d_n \to \infty$ when $n \to \infty$. Then G_n contains at least $(1 - o(1))n/t$ (and, obviously, at most n/t) disjoint trees isomorphic to T.*

In particular, $\lambda(G_n) = (1 - o(1))n/3$.

Given **1.1** and **1.4**, it seems natural to consider regular sparse graphs that may have induced claws. In this paper we consider the first such nontrivial case, that of cubic graphs. Our main question on Λ–packings is:

How many disjoint 2–edge paths must a cubic graph have?

Let $\mathcal{C}_n$ denote the set of simple cubic graphs on n vertices and

$$\lambda[\mathcal{C}_n] = \min\{\lambda(G) : G \in \mathcal{C}_n\}.$$

Obviously every graph in $\mathcal{C}_n$ contains at most $\lfloor n/3 \rfloor$ disjoint 2-paths.

Our main result is

1.5 *Let $n > 2$ be an even positive integer. Then $\lambda[\mathcal{C}_n] \geq \lceil n/4 \rceil$.*

The following examples show that the lower bound in **1.5** is sharp.

(E1) Obviously $\lambda(K_4) = 1 = \lceil v(K_4)/4 \rceil$ and $\lambda(K_{3,3}) = 2 = \lceil v(K_{3,3})/4 \rceil$. If H is a simple cubic graph on 8 vertices then clearly $\lambda(H) = 2 = \lceil v(H)/4 \rceil$.

Note that in all these examples $\lambda(G) = \lceil v(G)/4 \rceil = \lfloor v(G)/3 \rfloor$, and so the lower bound in **1.5** and the trivial upper bound mentioned above are the same. Below we give examples of graphs for which the lower bound in **1.5** is attained and is different from the upper bound.

(E2) Let K' denote the graph obtained from K_4 by subdividing one edge. Let H_1, H_2, H_3 be disjoint copies of K' and h_i be a (unique) vertex of degree two in H_i. Let Y be the graph obtained from $H_1 \cup H_2 \cup H_3$ by adding a new vertex x and three edges xh_1, xh_2, and xh_3 (see Figure 1).

It is easy to see that $v(Y) = 16$ and $\lambda(Y) = 4 = \lceil v(Y)/4 \rceil$.

(E3) Let F be the graph obtained from $K_{3,3}$ by removing one edge and let f_1 and f_2 be the two vertices in F of degree two. Let Y' be the graph obtained from disjoint graphs F and $Y - xh_1$ by adding two edges f_1h_1 and f_2x (see Figure 1). It is easy to see that $v(Y') = 22$ and $\lambda(Y') = 6 = \lceil v(Y')/4 \rceil$.

Obviously K_4, H, Y, and Y' are simple cubic graphs. Thus $\lambda(G) = \lceil v(G)/4 \rceil$ for every graph G such that each component of G is isomorphic to one of the graphs K_4, $K_{3,3}$, H, Y, and Y'. Therefore we have, in particular:

1.6 *Let $n > 2$ be an even positive integer. Then $\lambda[\mathcal{C}_n] = \lceil n/4 \rceil$.*

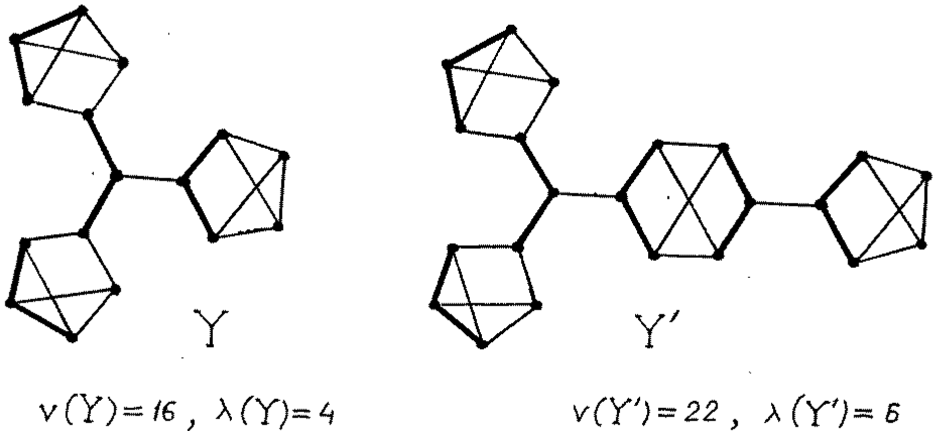

Figure 1: Extremal graphs Y and Y'

It is easy to show (see **4.3** below) that $\lambda[\mathcal{C}_n] \ge c\, v(G)$ for some $c \in (0, 1/3)$. So the main problem is to show that the above inequality holds for $c = 1/4$.

Our proof provides a polynomial time algorithm for finding in a cubic graph G a list $\mathcal{P}$ of at least $v(G)/4$ disjoint 2–edge paths. Therefore this algorithm is a polynomial approximation algorithm that guarantees for cubic graphs a Λ–packing such that $|\mathcal{P}| \ge \frac{3}{4}\lambda(G)$.

2 Main notions and notation

Let G be a simple graph. We use the following notation and notions:

$V = V(G)$ and $E = E(G)$ are the sets of vertices and edges of a graph G, respectively,

$v(G) = |V(G)|$ and $e(G) = |E(G)|$,

if X is a subgraph of G, then $N(X) = N(X, G)$ is the set of vertices of $G - X$ adjacent to vertices in X,

if $x \in V(G)$, then $d(x, G) = d(x) = |N(X, G)|$,

if X and Y are subgraphs of G, then $[X, Y]$ is the set of edges connecting a vertex in X with a vertex in Y,

$|X, Y| = |[X, Y]|$,

$\delta(G) = \min\{d(x, G) : x \in V(G)\}$,

$\Delta(G) = \max\{d(x, G) : x \in V(G)\}$,

$\lambda(G)$ is the maximum number of vertex disjoint 2–paths in G,

if $P = xcy$ is a 3–vertex path then $c(P)$ is the center vertex of P, i.e. $c(P) = c$.

Disjoint subgraphs X and Y of G are *adjacent* if $[X, Y] \ne \emptyset$.

A *diamond* is the graph obtained from K_4 by removing one edge.

A *net* is the graph obtained from a triangle, say xyz, by adding three new vertices x', y', z' (the *feet of the net*) and three new edges xx', yy', zz' (the *legs of the net*)(see Figure 6(d3)).

K' denotes the graph obtained from K_4 by subdividing one edge (see Figure 6(d1)).

A set of disjoint 2–edge paths in G is called a Λ–*packing of* G. Let $\mathcal{P}$ be a Λ–packing of G. We use the following notation related with $\mathcal{P}$:

$A = A(\mathcal{P})$ is the subgraph of G induced by $\cup\{P : P \in \mathcal{P}\}$,

$B = B(\mathcal{P}) = G - A(\mathcal{P})$, and

if $P \in \mathcal{P}$ then $\bar{P} = \bar{P}(\mathcal{P})$ is the subgraph of G induced by $P \cup (N(P) \cap B(\mathcal{P}))$.

We will consider the following list of parameters of a Λ–packing $\mathcal{P}$ of G:

$|\mathcal{P}|$ is the number of 2–edge paths in $\mathcal{P}$,

$\tau(\mathcal{P})$ is the number of 2–edge paths in $\mathcal{P}$ that induce a triangle in G,

$\beta(\mathcal{P})$ is the number of edges in B, and

$\sigma(\mathcal{P})$ is the number of edges in G connecting centers of paths in $\mathcal{P}$.

In our proof we consider a Λ–packing $\mathcal{P}$ of G that is lexicographically optimal with respect to the above (ordered) list of parameters:

$\mathcal{P}$ is *maximum* or 1–*optimal* if $\mathcal{P}$ has the maximum number of members,

$\mathcal{P}$ is 2–*optimal* if $\mathcal{P}$ is 1–optimal and $\tau(\mathcal{P}) = max\{\tau(\mathcal{L}) : \mathcal{L}$ is 1–optimal$\}$,

$\mathcal{P}$ is 3–*optimal* if $\mathcal{P}$ is 2–optimal and $\beta(\mathcal{P}) = max\{\beta(\mathcal{L}) : \mathcal{L}$ is 2–optimal$\}$,

$\mathcal{P}$ is 4–*optimal* if $\mathcal{P}$ is 3–optimal and $\sigma(\mathcal{P}) = max\{\sigma(\mathcal{L}) : \mathcal{L}$ is 3–optimal$\}$.

Clearly if $\mathcal{P}$ is k–optimal, $1 \le k \le 4$, then $\mathcal{P}$ is s–optimal for every $1 \le s \le k$.

3 Some reductions

Our strategy in proving **1.5** is to exhibit various properties of a minimum counterexample, and finally conclude that it cannot exist. At various stages in this process, we will need some operations that reduce a minimum counterexample to a smaller one providing a contradiction. In this section we present such operations.

3.1 *Let H_1, H_2, and Q be disjoint graphs such that H_1 and H_2 are isomorphic to K' and Q is a cubic graph. Let h_i be the vertex of degree two in H_i and let Q' be obtained from Q by subdividing an edge q_1q_2 by a vertex q. Suppose that $\lambda(Q) \ge v(Q)/4$ and G can be obtained from $H_1 \cup H_2 \cup Q$ by adding a new vertex x and three new edges xh_1, xh_2, and xq* (see Figure 2). *Then $\lambda(G) \ge v(G)/4$.*

Proof Clearly $v(Q) = v(G) - 12$. Since $\lambda(Q) \ge v(Q)/4$, Q has a Λ–packing $\mathcal{P}'$ such that $|\mathcal{P}'| = \lambda(Q) \ge v(Q)/4 = v(G)/4 - 3$. It is easy to see that $G' = G - Q' = H_1 \cup H_2 \cup \{x, xh_1, xh_2$, and so G' has three disjoint 2–paths, say L_1, L_2, and L_3. Then

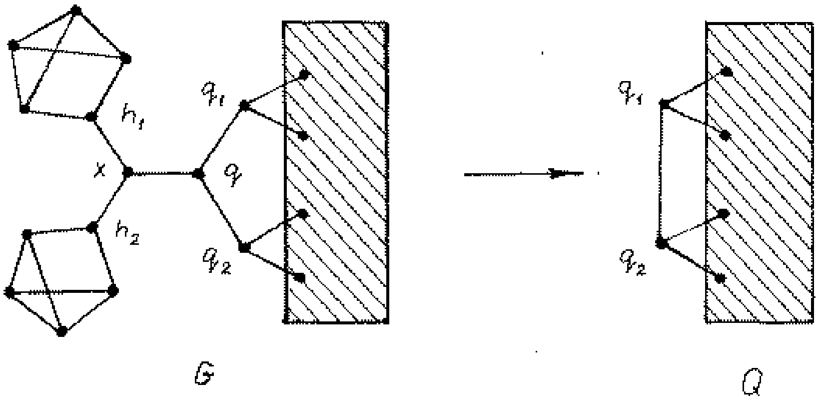

Figure 2:

$\mathcal{R} = \mathcal{P}' \cup \{L_1, L_2, L_3\}$ is a Λ–packing of G such that $|\mathcal{R}| = |\mathcal{P}'| + 3 \geq \lambda(G_2') + 3 \geq v(G_2')/4 + 3 = v(G)/4$. □

3.2 *Suppose that G is a cubic graph and N is a net in G with the triangle xyz and dangling edges xx', yy', and zz'. Let G' be obtained from $G - \{x, y, z\} \cup x'z'$ by replacing the 2-path with center y' by an edge* (see Figure 3). *Suppose that $\lambda(G') \geq v(G')/4$. Then $\lambda(G) \geq v(G)/4$.*

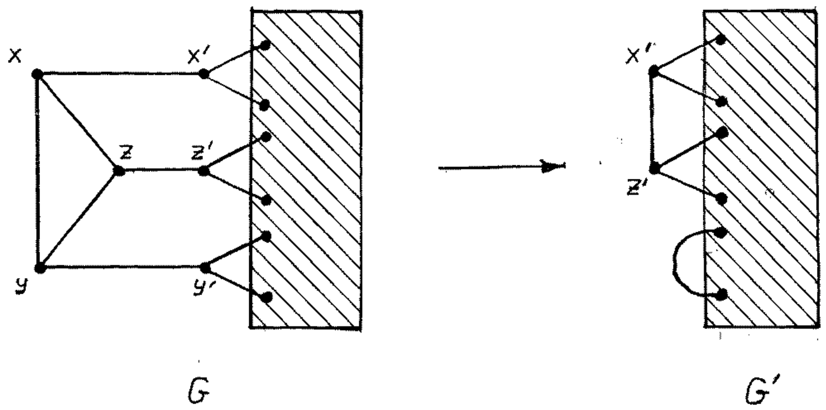

Figure 3:

Proof Clearly $v(G') = v(G) - 4$. Let $\mathcal{P}'$ be a 1–optimal Λ–packing in G', and so $|\mathcal{P}'| = \lambda(G')$.

Suppose $x'z'$ belongs to no 2–path in $\mathcal{P}'$. Consider a 2–path $L = xyz$ in G. Let $\mathcal{P} = \mathcal{P}' \cup L$.

Now suppose that $x'z'$ belongs to a 2–path $L' = x'z's$ in $\mathcal{P}'$. Consider 2–paths $L_1 = sz'z$ and $L_2 = x'xy$ in G. Let $\mathcal{P} = \mathcal{P}' \setminus L' \cup \{L_1, L_2\}$.

In both cases $\mathcal{P}$ is a Λ–packing in G and $|\mathcal{P}| = |\mathcal{P}'|+1$. Since $|\mathcal{P}'| = \lambda(G') \geq v(G')/4$ and $v(G') = v(G) - 4$, we have $\lambda(G) \geq |\mathcal{P}| \geq v(G')/4 + 1 = v(G)/4$. □

3.3 *Suppose that G is a cubic graph, D is an induced diamond of G, x and y are the two vertices of degree two in D, $xx', yy' \in E(G)$ where $x', y' \in V(G-D)$ and $x' \neq y'$. Let $G' = G - D \cup x'y'$* (see Figure 4). *Suppose that $\lambda(G') \geq v(G')/4$. Then $\lambda(G) \geq v(G)/4$.*

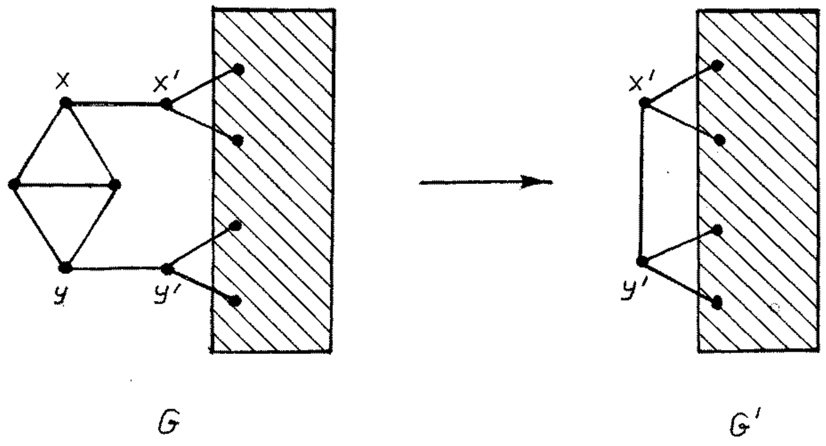

Figure 4:

Proof Clearly $v(G') = v(G) - 4$. Let $\mathcal{P}'$ be a 1–optimal Λ–packing in G', and so $|\mathcal{P}'| = \lambda(G')$.

Suppose $x'y'$ belongs to no 2–path in $\mathcal{P}'$. Let L be a 2–path in D. Let $\mathcal{P} = \mathcal{P}' \cup L$.

Now suppose that $x'y'$ belongs to a 2–path $L' = x'y's$ in $\mathcal{P}'$. Consider in G the 2–paths $L_1 = sy'y$ and L_2 where L_2 induces the triangle in $D - y$. Let $\mathcal{P} = \mathcal{P}' \setminus L' \cup \{L_1, L_2\}$.

In both cases $\mathcal{P}$ is a Λ–packing in G and $|\mathcal{P}| = |\mathcal{P}'|+1$. Since $|\mathcal{P}'| = \lambda(G') \geq v(G')/4$ and $v(G') = v(G) - 4$, we have $\lambda(G) \geq |\mathcal{P}| \geq v(G')/4 + 1 = v(G)/4$. □

3.4 *Suppose that G is a cubic graph, H is an induced subgraph of G isomorphic to $K_{2,3}$. Let G' be the graph obtained from $G - E(H)$ by identifying the vertices of H with a new vertex h'* (see Figure 5). *Suppose that $\lambda(G') \geq v(G')/4$. Then $\lambda(G) \geq v(G)/4$.*

Proof Clearly $v(G) = v(G') = 4$. Since $\lambda(G') \geq v(G')/4$, there is a Λ–packing $\mathcal{P}'$ of G' such that $|\mathcal{P}'| \geq v(G')/4 = v(G)/4 - 1$.

Suppose that h' belongs to no 2–path in $\mathcal{P}'$. Let L be a 2–path in H. Then $\mathcal{P} = \mathcal{P}' \cup L$ is a Λ–packing of G and $\lambda(G) \geq |\mathcal{P}| = |\mathcal{P}'| + 1 \geq v(G')/4 + 1 = v(G)/4$.

Therefore let h' belong to a (unique) 2–path $P' = a'c'b'$ in $\mathcal{P}'$.

Let x, y, and z be the three vertices of degree two in H, $xx', yy', zz' \in E(G)$ where x', y', z' are distinct vertices in $V(G - H)$. Let p, q be the two vertices of degree three in H (as well as in G).

If h' is an end vertex of P', say $c' = x'$, then $c' \in \{x', y', z'\}$, and let $L_1 = xx'b'$ and $L_2 = ypz$.

If h' is the center vertex c' of P', i.e. $h = c'$, then $\{a', b'\} \subset \{x', y', z'\}$, say $a' = x'$ and $b' = y'$, and let $L_1 = qxx'$ and $L_2 = ypz$.

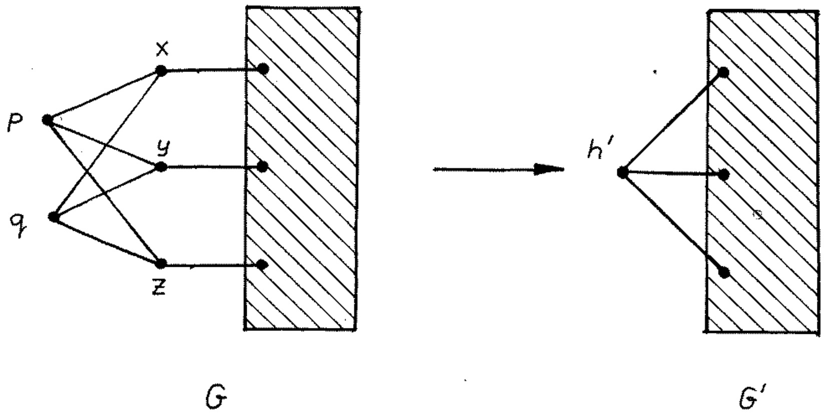

Figure 5:

In both cases $\mathcal{P} = \mathcal{P}' \setminus P' \cup \{L_1, L_2\}$ is a Λ–packing of G and $\lambda(G) \geq |\mathcal{P}| = |\mathcal{P}'| + 1 \geq v(G')/4 + 1 = v(G)/4$. □

4 Properties of 4–optimal Λ–packings

In this section, we establish a number of structural properties of 4-optimal Λ–packings. These, together with the reductions in Section 3 will be used in Section 5 to prove **1.5**.

Recall that for a Λ–packing $\mathcal{P}$, $A = A(\mathcal{P})$ is the subgraph of G induced by $\cup\{P : P \in \mathcal{P}\}$, and $B = B(\mathcal{P}) = G - A(\mathcal{P})$. As above we write V and E for $V(G)$ and $E(G)$, respectively.

We begin with the following easy observation.

4.1 *Let G be a simple cubic graph and $\mathcal{P}$ be a 1-optimal Λ–packing in G. Then every vertex in $B(\mathcal{P})$ is of degree at most one, i.e. every component of B is either an isolated vertex or is of one edge.*

4.2 *Suppose that $\delta(G) \geq 3$, $\mathcal{P}$ is a 1-optimal Λ–packing of G, and $|A(\mathcal{P}), B(\mathcal{P})| \leq \alpha|\mathcal{P}|$. Then $\lambda(G) \geq 2v(G)/(\alpha + 6)$.*

Proof (uses **4.1**). Let β denote the number of edges of $B = B(\mathcal{P})$ and γ the number of isolated vertices of B. Let $\lambda(G) = \lambda$ and $v(G) = n$. Since by **4.1**, every vertex of B is of degree at most one $|A(\mathcal{P}), B(\mathcal{P})| \geq 4\beta + 3\gamma$. Since $\mathcal{P}$ is a 1-optimal Λ–packing, $|\mathcal{P}| = \lambda$. Therefore by the inequality in our assumption, $4\beta + 3\gamma \leq \alpha\lambda$. Obviously, $n = \gamma + 2\beta + 3\lambda$, and so $4\beta = 2n - 2\gamma - 6\lambda$. Now we have from the above inequality: $\lambda \geq (2n + \gamma)/(\alpha + 6)$. Since $\gamma \geq 0$, we have $\lambda \geq 2n/(\alpha + 6)$. □

From **4.2** we have:

4.3 *Let G be a simple cubic graph. Then $\lambda(G) \geq 2v(G)/11$.*

Proof (uses **4.2**). Let $\mathcal{P}$ be a 1-optimal Λ–packing in G. Since G is cubic, $|P, B(\mathcal{P})| \leq 5$ for every $P \in \mathcal{P}$. Therefore $|A(\mathcal{P}), B(\mathcal{P})| \leq 5|\mathcal{P}|$, and our claim follows from **4.2**. □

4.4 *Let $\mathcal{P}$ be a 1–optimal Λ–packing in G and $L = xcy \in \mathcal{P}$. Suppose that $\{b_x, b\} \subseteq V(B)$, $b \neq b_x$, $bb_x \notin E$, and $\{xb_x, zb\} \subseteq E$ where $z \in \{c, y\}$. Then b_x is an isolated vertex in B.*

Proof Suppose, on the contrary, that b_x is not an isolated vertex in B, i.e. there is a vertex a adjacent to b_x. Clearly $L_2 = xb_xa$ is a 2–path and the subgraph of G, induced by $\{b, c, y\}$, contains a 2–path, say L_1. Let $\mathcal{P}' = \mathcal{P} \setminus L \cup \{L_1, L_2\}$. Obviously $\mathcal{P}'$ is a Λ–packing in G and $|\mathcal{P}'| = |\mathcal{P}| + 1$. Therefore $\mathcal{P}$ is not 1–optimal, a contradiction. □

4.5 *Let $\mathcal{P}$ be a 3–optimal Λ–packing in G and $L = xcy \in \mathcal{P}$. Suppose that $xy \notin E$, $\{b_x, b\} \subseteq V(B)$, $b_x \neq b$, $bb_x \notin E$, and $\{xb_x, zb\} \subseteq E$ where $z \in \{c, y\}$. Then b is not an isolated vertex in B and $yb \notin E$.*

Proof (uses **4.4**). Suppose, on the contrary, b is an isolated vertex in B. By **4.4**, b_x is an isolated vertex in B. Clearly the subgraph of G, induced by $\{b, c, y\}$, contains a 2–path, say L_1. Let $\mathcal{P}' = \mathcal{P} \setminus L \cup L_1$. Obviously $\mathcal{P}'$ is a Λ–packing in G and $|\mathcal{P}'| = |\mathcal{P}|$. Since $xy \notin E$, $\tau(\mathcal{P}') \geq \tau(\mathcal{P})$. Also $xb_x \notin E(B(\mathcal{P}))$, and $E(B(\mathcal{P}')) = E(B(\mathcal{P})) \cup xb_x$. Therefore $\beta(\mathcal{P}') = \beta(\mathcal{P}) + 1$, and so $\mathcal{P}$ is not 3–optimal, a contradiction. Now by **4.4**, $yb \notin E(G)$. □

Recall that if $L \in \mathcal{P}$ then $\bar{L}$ denotes the subgraph in G induced by $L \cup (N(L) \cap B)$, and so $V(\bar{L} - L)$ is the set of vertices in B adjacent to L.

4.6 *Suppose that $\Delta(G) \leq 3$. Let $\mathcal{P}$ be a 3–optimal Λ–packing in G and $L = xcy \in \mathcal{P}$. Then $|L, B| \leq 3$.*

Proof (uses **4.5**). Let us assume, on the contrary, that $|L, B| \geq 4$. Since every vertex in G is of degree at most three, we have: $xy \notin E$ and there are two vertices $b_x, b_y \in V(\bar{L} \cap B)$ such that $xb_x, yb_y \in E$. By **4.5**, $b_xb_y \in E$.

Suppose that no vertex in $B - \{b_x, b_y\}$ is adjacent to L. Since $|L, B| \geq 4$, clearly c is adjacent to $\{b_x, b_y\}$, say $cb_x \in E$. Let $L' = cxb_x$ and $\mathcal{P}' = \mathcal{P} \setminus L \cup L'$. Obviously $\mathcal{P}'$ is a Λ–packing in G and $|\mathcal{P}'| = |\mathcal{P}|$. Since L does not induce a triangle (because $xy \notin E$) and L' does, $\tau(\mathcal{P}') = \tau(\mathcal{P}) + 1$. Therefore $\mathcal{P}$ is not a 2–optimal Λ–packing in G, a contradiction.

Now suppose that a vertex b from $B - \{b_x, b_y\}$ is adjacent to L. Then the subgraph of G, induced by $L \cup \{b, b_x, b_y\}$, contains two disjoint 2–paths, say L_1 and L_2. Let $\mathcal{P}' = \mathcal{P} \setminus L \cup \{L_1, L_2\}$. Obviously $\mathcal{P}'$ is a Λ–packing in G and $|\mathcal{P}'| = |\mathcal{P}| + 1$. Therefore $\mathcal{P}$ is not 1–optimal, a contradiction. □

From **4.6** it follows that the assumption of **4.2** holds for $\alpha = 3$. Therefore we have from **4.2** the following intermediate result:

4.7 *Let G be a simple cubic graph. Then $\lambda(G) \geq 2v(G)/9$.*

Our main goal now is to replace in the last inequality 2/9 by 1/4, namely, to show that the assumption of **4.2** is satisfied for $\alpha = 2$.

4.8 *Suppose that $\Delta(G) \le 3$ and $\mathcal{P}$ is a 3–optimal Λ–packing in G. Then for every $L \in \mathcal{P}$*
$(a1)$ $|V(\bar{L} - L)| \le 3$ (i.e. there are at most three vertices in B adjacent to L), and
$(a2)$ if $|V(\bar{L} - L)| = 3$ then $\bar{L}$ is an induced net in G, L induces the triangle of the net, and each foot of the net is an isolated vertex in B.

Proof (uses **4.5**). Suppose that there are three distinct vertices b, b_1, and b_2 in G adjacent to some $L = xcy \in \mathcal{P}$. We can assume that $xb_1 \in E$.

(**p1**) Suppose that b_1 is adjacent to $\{b, b_2\}$, say $b_1 b_2 \in E$.

Suppose that $yb_2 \in E$. Since b is adjacent to L, the subgraph induced by $L \cup \{b, b_1, b_2\}$ contains two disjoint 2–paths, say L_1 and L_2.

Now suppose that an end vertex, say y, is adjacent to no vertex in $\{b, b_1, b_2\}$. Since $\Delta(G) \le 3$, we can assume that $xb_2, cb \in E$. Let $L_1 = b_1 x b_2$ and $L_2 = bcy$.

In both cases let $\mathcal{P}' = \mathcal{P} \setminus L \cup \{L_1, L_2\}$. Obviously $\mathcal{P}'$ is a Λ–packing in G and $|\mathcal{P}'| = |\mathcal{P}| + 1$. Therefore $\mathcal{P}$ is not 1–optimal, a contradiction.

(**p2**) Suppose that b_1 is not adjacent to $\{b, b_2\}$ and $xy \notin E$. Since $d(x) \le 3$, clearly $b'z \in E$ for some $b' \in \{b, b_2\}$ and $z \in \{c, y\}$. Then by **4.5**, $yb' \notin E$ (and so $cb' \in E$). Since $d(c) \le 3$, exactly one of b, b_2 is adjacent to c, say $b' = b_2$. Then $bx \in E$. Let $L_1 = bxb_1$, $L_2 = b_2 cy$, and $\mathcal{P}' = \mathcal{P} \setminus L \cup \{L_1, L_2\}$. Obviously $\mathcal{P}'$ is a Λ–packing in G and $|\mathcal{P}'| = |\mathcal{P}| + 1$. Therefore $\mathcal{P}$ is not 1–optimal, a contradiction.

(**p3**) Now suppose that b_1 is not adjacent to $\{b, b_2\}$ and $xy \in E$. Since vertices x, y, and c are of degree at most three in G, we can assume that $yb_2, cb \in E$.

Suppose that $bb_2 \in E$. Let $L_1 = cxb_1$ and $L_2 = ybb_2$. Then $\mathcal{P}' = \mathcal{P} \setminus L \cup \{L_1, L_2\}$ is a Λ–packing in G and $|\mathcal{P}'| = |\mathcal{P}| + 1$. Therefore $\mathcal{P}$ is not 1–optimal, a contradiction.

Thus $L \cup \{b, b_1, b_2\}$ induces a net in G, L induces the triangle of the net, and $V(\bar{L} - L) = \{b, b_1, b_2\}$. Hence $|V(\bar{L} - L)| = 3$, and so $(a1)$ holds.

Suppose that one of the vertices in $\{b, b_1, b_2\}$ is not an isolated vertex in B. By symmetry, we can assume that $bb' \in E$ for some $b' \in V(B - \{b_1, b_2\})$. Let $L_1 = b'bc$, $L_2 = b_1 xy$, and $\mathcal{P}' = \mathcal{P} \setminus L \cup \{L_1, L_2\}$. Obviously $\mathcal{P}'$ is a Λ–packing in G and $|\mathcal{P}'| = |\mathcal{P}| + 1$. Therefore $\mathcal{P}$ is not 1–optimal, a contradiction.

Thus each vertex in $\{b, b_1, b_2\}$ is an isolated vertex in B, and so $(a2)$ holds. □

4.9 *Suppose that $\Delta(G) \le 3$, $\mathcal{P}$ is a 3–optimal Λ–packing in G, and $L = xcy \in \mathcal{P}$. Suppose also that there are exactly two vertices, say b_1 and b_2, in B adjacent to L, and $b_1 b_2 \notin E$. Then $|L, B| \le 2$.*

Proof (uses **4.4** and **4.5**). Suppose, on the contrary, that $|L, B| \ge 3$. Since $\Delta(G) \le 3$, each vertex in L is of degree at most three in G.

(**p1**) Suppose that $xy \in E$. Then L induces a triangle T in G. Since x, y, and c are of degree at most three in G and $|L, B| \ge 3$, we have: $|L, B| = 3$, each vertex of T is adjacent to exactly one vertex in $\{b_1, b_2\}$, and $|b', T| = 2$, $|b'', T| = 1$ where $\{b', b''\} = \{b_1, b_2\}$. Then b' is adjacent to an end vertex, say x, of L, and b'' is adjacent to a vertex t in $L - x$. By **4.4**, b' is an isolated vertex in B. Let L' be a 2-path with the center in b' and with the two edges connecting b' with T. Let $\mathcal{P}' = \mathcal{P} \setminus L \cup L'$. Then $\mathcal{P}'$ is a Λ–packing in G and $|\mathcal{P}'| = |\mathcal{P}|$. Since L' induces a triangle in G, $\tau(\mathcal{P}') = \tau(\mathcal{P})$. Since b' is an isolated vertex in B, $E(B(\mathcal{P}')) = E(B(\mathcal{P})) \cup b''t$, and so $\beta(\mathcal{P}') = \beta(\mathcal{P}) + 1$. Therefore $\mathcal{P}$ is not 3–optimal, a contradiction.

(**p2**) Now suppose that $xy \notin E$. We also know that $b_1 b_2 \notin E$. If $\bar{L}$ contains a triangle T, then replacing L in $\mathcal{P}$ by any 2-edge path in T yields $\mathcal{P}'$ with $\tau(\mathcal{P}') = \tau(\mathcal{P}) + 1$, contradicting the 2–optimality of $\mathcal{P}$. Hence we may assume that $\bar{L}$ has no triangle. Then $|L, B| \geq 3$ implies that $b'x, b'y \in E$ for some $b' \in \{b_1, b_2\}$. Let $\{b', b''\} = \{b_1, b_2\}$. Then $|L, B| \geq 3$ implies that $b''c \in E$. By **4.5**, $b''x, b''y \notin E$ and there is a vertex b in $V(B - \{b_1, b_2\})$ adjacent to b''. Let $L_1 = xb'y$ and $L_2 = bb''c$. Then L_1 and L_2 are disjoint 2–paths in G. Let $\mathcal{P}' = \mathcal{P} \setminus L \cup \{L_1, L_2\}$. Obviously $\mathcal{P}'$ is a Λ–packing in G and $|\mathcal{P}'| = |\mathcal{P}| + 1$. Therefore $\mathcal{P}$ is not 1–optimal, a contradiction. □

4.10 *Suppose that $\Delta(G) \leq 3$, $\mathcal{P}$ be a 3–optimal Λ–packing in G, $L = xcy \in \mathcal{P}$, and $|L, B| = 3$. Then precisely one of the following conditions holds:*

($d0$) *$\bar{L} = K_4 = G$,*

($d1$) *$\bar{L}$ is isomorphic to K', L belongs to one of the two triangles of $\bar{L}$, $\bar{L} - L$ is a one edge component, say pp', in B, and this component contains a (unique) vertex, say p, of degree two in $\bar{L}$* (Figure 6 (d1)),

($d2$) *$\bar{L}$ is isomorphic to $K_{2,3}$ and c is a vertex of degree three and $\bar{L} - L$ is a one edge component, say pp', in B, and this component contains a vertex, say p, of degree two in $\bar{L}$* (Figure 6 (d2)), *and*

($d3$) *$\bar{L}$ is an induced net in G, L induces the triangle of the net, and each foot of the net is an isolated vertex in B* (Figure 6 (d3)).

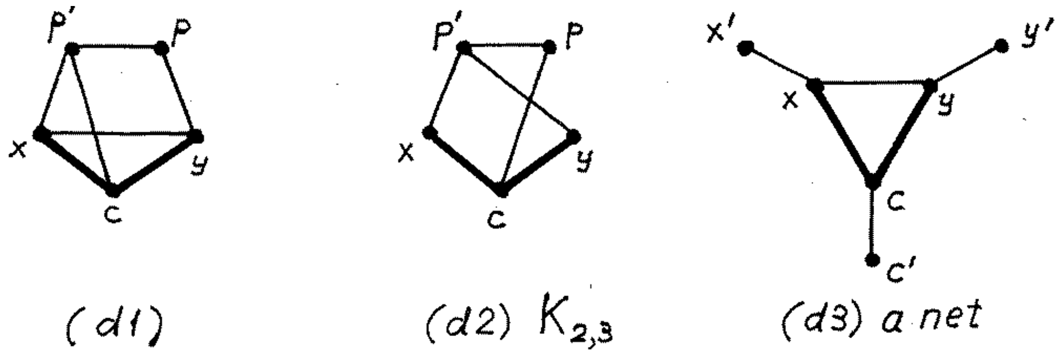

Figure 6: Conditions ($d1$), ($d2$), and ($d3$) in **4.10**

Proof (uses **4.8** and **4.9**). Let us assume that ($d3$) does not hold. Then by **4.8**, $|V(\bar{L} \cap B)| \leq 2$. Since $|L, B| \leq 3$ and $\Delta(G) \leq 3$, clearly $|V(\bar{L} \cap B)| \geq 1$.

(**p1**) Suppose $|V(\bar{L} \cap B)| = 1$, say $V(\bar{L} \cap B) = \{b\}$. Since $|L, B| = 3$, we have $bx, bc, by \in E$. If $xy \in E$ then $\bar{L} = K_4$. Since $\Delta(G) \leq 3$ and G is connected, clearly $G = \bar{L} = K_4$, and so ($d0$) holds. Therefore we can assume that $xy \notin E$. Let $L' = bcx$ (and so L' is a 2–path that induces a triangle in G). Let $\mathcal{P}' = \mathcal{P} \setminus L \cup L'$. Obviously $\mathcal{P}'$

is a Λ–packing in G and $|\mathcal{P}'| = |\mathcal{P}|$. Since $xy \notin E$ and $xb \in E$, clearly $\tau(\mathcal{P}') = \tau(\mathcal{P})+1$. Therefore $\mathcal{P}$ is not 2–optimal, a contradiction.

(**p2**) Now suppose $|V(\bar{L} \cap B)| = 2$, say $V(\bar{L} \cap B) = \{b_1, b_2\}$. If $b_1b_2 \notin E$ then by **4.9**, $|L, B| \leq 2$, a contradiction. Therefore let $b_1b_2 \in E$.

(**p2.1**) Suppose that one of the end vertices of L, say x, is adjacent to both b_1 and b_2, and so $\{b_1, b_2, c\} \subseteq N(x)$. Since $|N(x)| \leq 3$, clearly $xy \notin E$. Let $L' = b_1xb_2$ (and so L' is a 2–path that induces a triangle in G). Let $\mathcal{P}' = \mathcal{P} \setminus L \cup L'$. Obviously $\mathcal{P}'$ is a Λ–packing in G and $|\mathcal{P}'| = |\mathcal{P}|$. Since $xy \notin E$ and $xb \in E$, clearly $\tau(\mathcal{P}') = \tau(\mathcal{P}) + 1$. Therefore $\mathcal{P}$ is not 2–optimal, a contradiction.

(**p2.2**) Now suppose that no end vertex of L is adjacent to both b_1 and b_2. Since $|L, B| = 3$, we can assume that $xb_1 \in E$.

Suppose that $yb_2 \in E$. Since $|L, B| = 3$, we may assume by symmetry that $cb_2 \in E$. If $xy \in E$ then $(d1)$ holds. So let $xy \notin E$. Let $L' = cb_2y$ (and so L' is a 2–path that induces a triangle in G). Let $\mathcal{P}' = \mathcal{P} \setminus L \cup L'$. Obviously $\mathcal{P}'$ is a Λ–packing in G and $|\mathcal{P}'| = |\mathcal{P}|$. Since $xy \notin E$ and $xb \in E$, clearly $\tau(\mathcal{P}') = \tau(\mathcal{P}) + 1$. Therefore $\mathcal{P}$ is not 2–optimal, a contradiction.

Now suppose that $yb_2 \notin E$. Since $|L, B| = 3$, we have $yb_1, cb_2 \in E$. Then $(d2)$ holds. $\square$

Let $\mathcal{P}_3$ denote the set of paths L in $\mathcal{P}$ such that $|L, B| = 3$. Let $\mathcal{P}_3^i$ denote the set of 2–paths P in $\mathcal{P}_3$ such that $\bar{P}$ satisfies condition (di) in **4.10**, $i = 0, 1, 2, 3$.

Consider $P \in \mathcal{P}_3^1$. Then by $(d1)$ in **4.10**, $\bar{P}$ has exactly one vertex p of degree two and exactly one edge pp' from B. Since $d(p) = 3$, there exists exactly one vertex $z(P)$ in $G - \bar{P}$ adjacent to p. Since $pp' \in E(B)$, by **4.1**, $z(P) \in V(A)$. Let $z(P) = z$ and $g(P)$ denote the 2–path in $\mathcal{P}$ containing $z(P)$. Put $g(P) = P_1 = x_1c_1y_1$ (see Figure 7).

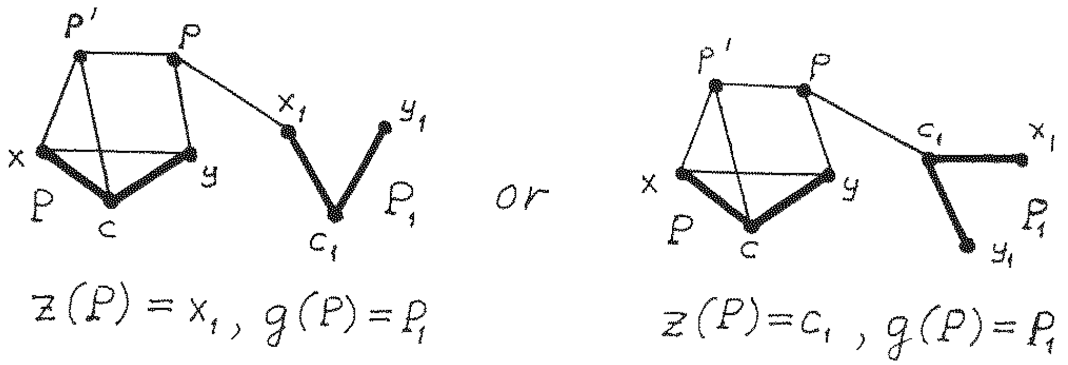

Figure 7:

4.11 *Suppose that* $\Delta(G) \leq 3$. *Let* $\mathcal{P}$ *be a 4–optimal* Λ*–packing in* G *and* $P \in \mathcal{P}_3^1$. *Then*

$(a1)$ $|P_1, B| \leq 2$ *and*

($a2$) *if* $|P_1, B| = 2$, *then* (see Figure 8) ($a2.0$) z *is an end vertex of* P_1, $bz \in E$ *for some* $b \in V(B - \bar{P})$, *and* $[P_1, B - \bar{P}] = \{bz\}$,

($a2.1$) P_1 *is an induced 2–path in* G (*i.e.* $x_1y_1 \notin E$), *and*

($a2.2$) *the center* c_1 *of the 2–path* P_1 *is adjacent to the center of another path in* $\mathcal{P}$.

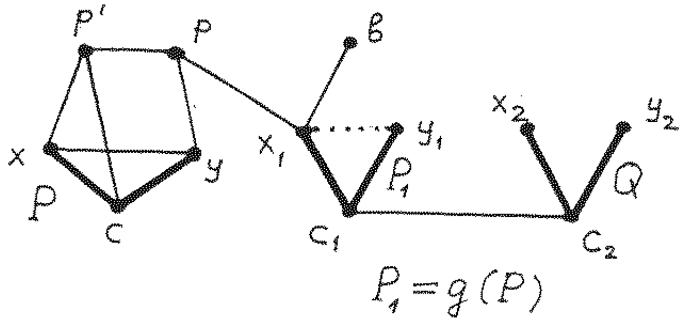

Figure 8:

Proof (uses **4.5**). Since $P \in \mathcal{P}_3^1$, P induces a triangle in G. We can assume that the center c of P is adjacent to p in $\bar{P}$, i.e. $cp \in E$. Let $z(P) = z$ and $L = zpp'$ where pp' is a (unique) edge in $\bar{P} - P$.

(**p1**) Suppose that z is an end vertex of P_1, say $z = x_1$.

Suppose that $bu \in E$ for some $b \in V(B)$ and $u \in \{c_1, y_1\}$. Then the subgraph, induced by $\{b, c_1, y_1\}$, contains a 2–path, say R. Let $\mathcal{P}' = (\mathcal{P} \setminus P_1) \cup \{L, R\}$. Clearly $\mathcal{P}'$ is a Λ–packing and $|\mathcal{P}'| = |\mathcal{P}| + 1$. Therefore $\mathcal{P}'$ is not 1–optimal, a contradiction. Hence $|P_1, B| \le 2$.

Now assume that $|P_1, B| = 2$. Then $bz \in E$ for some $b \in V(B-\bar{P})$, and $[P_1, B-\bar{P}] = \{bz\}$, and so ($a1$) and ($a2.0$) hold.

Since $d(x_1) = 3$, we have $N(x_1) = \{b, c_1, p\}$. Therefore $x_1y_1 \notin E$, and so ($a2.1$) holds.

Suppose that ($a2.2$) (see Figure 8) is not true, i.e. c_1 is not adjacent to the center of another path in $\mathcal{P}$. Let $\mathcal{P}' = (\mathcal{P} \setminus P_1) \cup L$. Clearly $\mathcal{P}'$ is a Λ–packing and $|\mathcal{P}'| = |\mathcal{P}|$. Since by ($a2.1$), $x_1y_1 \notin E$, we have $\tau(\mathcal{P}') = \tau(\mathcal{P})$. Now $E(B(\mathcal{P}')) = E(B(\mathcal{P})) - pp' \cup c_1y_1$, and so $\beta(\mathcal{P}') = \beta(\mathcal{P})$. Hence $\mathcal{P}'$ is 3–optimal. Since p is the center of the 2–path L and $cp \in E$, clearly $\sigma(\mathcal{P}') = \sigma(\mathcal{P}) + 1$. Therefore $\mathcal{P}$ is not 4–optimal, a contradiction.

(**p2**) Now suppose that z is not an end vertex of P_1 (and so $z = c_1$).

Since $d(c_1) = 3$, we have $N(c_1) = \{p, x_1, y_1\}$. Therefore c_1 is not adjacent to the center of another path in $\mathcal{P}$ or a vertex in $B - p$. We want to show that $\{x_1, y_1\}$ is adjacent to no vertex in B, which implies $|P_1, B| \le 1$.

Suppose, on the contrary, that there is a vertex b in B adjacent to $\{x_1, y_1\}$, say $bx_1 \in E$.

Suppose that $x_1y_1 \in E$. Let $R = bx_1y_1$. Then $\mathcal{P}' = (\mathcal{P} \setminus P_1) \cup \{L, R\}$ is a Λ–packing and $|\mathcal{P}'| = |\mathcal{P}| + 1$. Therefore $\mathcal{P}'$ is not 1–optimal, a contradiction.

Now suppose that $x_1y_1 \notin E$. Let $\mathcal{P}' = (\mathcal{P} \setminus P_1) \cup L$. Clearly $\mathcal{P}'$ is a Λ–packing and $|\mathcal{P}'| = |\mathcal{P}|$. Since $x_1y_1 \notin E$, we have $\tau(\mathcal{P}') = \tau(\mathcal{P})$. Since $E(B(\mathcal{P}')) = E(B(\mathcal{P})) - pp' \cup$

bx_1, we have $\beta(\mathcal{P}') = \beta(\mathcal{P})$. Therefore $\mathcal{P}'$ is 3–optimal. Now $cp \in E$, p is the center of the 2–path L, and c_1 is not adjacent to the center of another path in $\mathcal{P}$. Hence $\sigma(\mathcal{P}') = \sigma(\mathcal{P}) + 1$. Therefore $\mathcal{P}$ is not 4–optimal, a contradiction. □

Consider $P \in \mathcal{P}_3^2$ such that $|N(\bar{P})| = 2$. Then by $(d2)$ in **4.10**, $\bar{P}$ has exactly three vertices of degree two each being adjacent to exactly one vertex in $G - \bar{P}$ and moreover, $N(\bar{P}) \subseteq V(A - \bar{P})$. Hence $|\bar{P}, G - \bar{P}| = |\bar{P}, A - \bar{P}| = 3$. Since $|N(\bar{P})| = 2$, there is a unique vertex $z(P)$ in $N(\bar{P})$ which is adjacent to exactly two vertices in $\bar{P}$. Clearly $z(P) \in V(A - \bar{P})$. As above, let $z(P) = z$ and let $g(P)$ denote the 2–path in $\mathcal{P}$ containing $z(P)$. Put $g(P) = P_1 = x_1c_1y_1$ (see Figure 9).

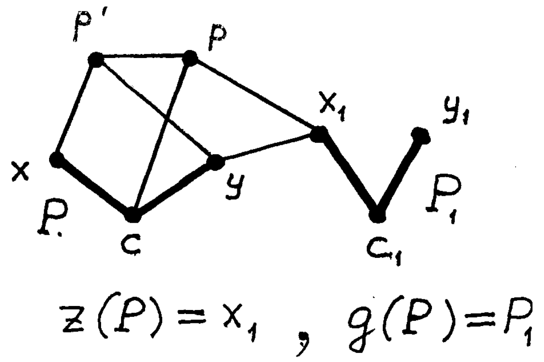

Figure 9:

4.12 *Suppose that $\mathcal{P}$ is a 1–optimal Λ–packing in G and $P \in \mathcal{P}_3^2$. Then $[P_1, B - \bar{P}] = \emptyset$, and so $|P_1, B| \leq 1$.*

Proof Since $P \in \mathcal{P}_3^2$, we have $z(P)$ is adjacent to exactly two vertices, say p_1 and p_2, in $\bar{P}$, and $\{p_1, p_2\} \cap V(B) \leq 1$. Since $z = z(\bar{P})$ is of degree three in G, clearly z is an end vertex of 2–path P_1, say $z = x_1$.

Suppose, on the contrary, that there is a vertex b in $B - \{p_1, p_2\}$ adjacent to a vertex in $g(P)$. Then $b \in B - \bar{P}$ and since $d(x) = 3$ and $N(x_1) = \{c_1, p_1, p_2\}$, clearly b is not adjacent to x_1. Therefore b is adjacent to a vertex in $\{c_1, y_1\}$. We know that $\bar{P} \cup \{x_1p_1\}$ contains two disjoint 2-paths, say P' and P''. Clearly the subgraph induced by $\{b, c_1, y_1\}$ contains a 2–path, say Q. Let $\mathcal{P}' = \mathcal{P} \setminus \{P, g(P)\} \cup \{P', P'', Q\}$. Then $\mathcal{P}'$ is a Λ–packing of G and $|\mathcal{P}'| = |\mathcal{P}| + 1$. Therefore $\mathcal{P}$ is not 1–optimal, a contradiction. Hence $|g(P), B| \leq 1$. □

Now consider $P \in \mathcal{P}_3^3$, and so $\bar{P}$ is an induced net in G, P induces the triangle, say T, of the net and each foot of the net is an isolated vertex in B. Let tt' be a leg and t' the corresponding foot of $\bar{P}$. Let $N(t') = \{r, s, t\}$. Since t' is an isolated vertex in B, there are $R, S \in \mathcal{P}$ such that $r \in R$, $s \in S$, and $t' \notin R \cup S$ (possibly, $R = S$).

4.13 *Suppose that G is a cubic graph, $\lambda(G) < v(G)/4$, and if H is a cubic graph with $v(H) < v(G)$ then $\lambda(H) \geq v(G)/4$. Let $\mathcal{P}$ be a 3–optimal Λ–packing in G, and*

$P \in \mathcal{P}_3^3$. Suppose that t' belongs to a triangle $\{r, s, t'\}$. Then $R \neq S$, $N(R) \cap V(B) = N(S) \cap V(B) = \{t'\}$, and so $|R, B| = |S, B| = 1$.

Proof (uses **3.3** and **4.5**).

(**p1**) Suppose, on the contrary, that $R = S$. Then $r, s \in R$. Let L be the 2–path $rt's$ and $\mathcal{P}' = \mathcal{P} \setminus R \cup L$. Clearly, $\mathcal{P}'$ is a Λ–packing of G and $|\mathcal{P}'| = |\mathcal{P}|$.

(**p1.1**) Suppose that R does not induce a triangle in G. Then $rs \in E(R)$. Since R does not induce a triangle in G and L does, clearly $\tau(\mathcal{P}') = \tau(\mathcal{P}) + 1$. Therefore $\mathcal{P}$ is not 2–optimal, a contradiction.

(**p1.2**) Now suppose that R induces a triangle in G. Then $R \cup t'$ induces a diamond D in $G - T$ and t' is a vertex of degree two in D. Let d' be the other vertex of degree two in D and $N(d') \setminus \{r, s\} = \{d\}$. Let $G' = G - D \cup dt$. Clearly G' is a cubic graph and $v(G') < v(G)$. Since G is cubic, clearly $P \cap D = \emptyset$, $d \neq t$ and $dt \notin E$. Since every foot of $\bar{P}$ is a vertex in B and $d' \in V(A)$, clearly d' is not a foot of $\bar{P}$. Therefore G' has no parallel edges. By minimality of G, $\lambda(G') \geq v(G')/4$. Now by **3.3**, $\lambda(G) \geq v(G)/4$, a contradiction.

(**p2**) Now suppose that $R \neq S$. Suppose, on the contrary, that $|R, B| \geq 2$. Since both r and s are of degree at most three in G, clearly r is an end vertex of R, say $R = rr'r''$, and s is an end vertex of S, say $S = ss's''$. Since $|R, B| \geq 2$, there exists $bz \in E$ such that $b \in V(B)$ and $z \in V(R)$. Since $\Delta(G) \leq 3$, $N(t') = \{r, s, t\}$, $N(r) = \{r', s, t'\}$, and $N(s) = \{s'r, t'\}$, we have $b \neq t'$, $z \neq r$, $bt', rr'' \notin E$, and S does not induce a triangle in G. By **4.5**, we have: $br'' \notin E$ and $br' \in E$ (and so $z = r'$). Let $L' = br'r''$ and $\mathcal{P}' = \mathcal{P} - \{R, S\} \cup \{L, L'\}$. Clearly, $\mathcal{P}'$ is a Λ–packing of G and $|\mathcal{P}'| = |\mathcal{P}|$. Since L induces a triangle and both R and S do not, we have $\tau(\mathcal{P}') > \tau(\mathcal{P})$. Therefore $\mathcal{P}$ is not 2–optimal, a contradiction. □

4.14 *Suppose that $\mathcal{P}$ is a 3-optimal Λ–packing in G, $P \in \mathcal{P}_3^1$, $g(P) = P_1 = x_1c_1y_1$, and $|P_1, B| = 2$. Suppose that $P_2 = x_2c_2y_2 \in \mathcal{P} \setminus \{P, P_1\}$ and $c_1c_2 \in E$* (see Figure 8). *Then $|P_2, B| \leq 1$.*

Proof (uses **4.5**). Since $d(c_1) = 3$, we have $N(c_1) = \{c_2, x_1, y_1\}$. Therefore c_1 is adjacent to no vertex in B, and so $z = z(P)$ is an end vertex of P_1, say $z = x_1$. Since $d(c_2) = 3$, we have $N(c_2) = \{c_1, x_2, y_2\}$. Therefore c_2 is adjacent to no vertex in B.

Suppose that there is a 2–path R in G such that $V(R) \cap V(A - \{x_2, y_2\}) = \emptyset$. Then $R \cap Q = \emptyset$ for every $Q \in \mathcal{P} \setminus P_2$ and $R \cap \{x_2, y_2\} \neq \emptyset$. Let $\mathcal{P}' = (\mathcal{P} \setminus \{P_1), P_2\} \cup \{R, c_2c_1y_1, x_1pp'\}$. Clearly $\mathcal{P}'$ is a Λ–packing and $|\mathcal{P}'| = |\mathcal{P}| + 1$. Therefore $\mathcal{P}$ is not 1–optimal, a contradiction.

Now suppose that there is no such R. Then $|P_2, B| \geq 2$ implies $x_2y_2 \notin E$ and $x_2x', y_2y' \in E$ for some pair of isolated vertices x', y' in B. This contradicts **4.5**. □

5 Main result

Now we are ready to prove our main result.

5.1 *Let G be a cubic graph. Then $\lambda(G) \geq \lceil v(G)/4 \rceil$.*

Proof (uses **3.1**, **3.2**, **4.2**, **4.5**, **4.10** - **4.14**).

We can assume without loss of generality that G is a connected graph. If $v(G) = 4$ then $G = K_4$ and our claim is obviously true. Let us assume that, on the contrary, our claim is not true. Let G be a counterexample with the minimum number of vertices, and so $\lambda(G) < v(G)/4$. Clearly $v(G) \geq 6$.

Let $\mathcal{P}$ be a 4–optimal Λ–packing of G, and so $\mathcal{P}$ is s–optimal for every $s = 1, 2, 3, 4$.

If $|L, B| \leq 2$ for every $L \in \mathcal{P}$ then $|A, B| \leq \sum\{L, B| : L \in \mathcal{P}\} \leq 2|\mathcal{P}| = 2\lambda(G)$. Now we are done by **4.2**. Therefore we can assume that there exists $P \in \mathcal{P}$ such that $|P, B| \geq 3$. By **4.10**, $|P, B| \leq 3$. Therefore $|P, B| = 3$, i.e. $P \in \mathcal{P}_3$.

Also by **4.10**, $\bar{P}$ satisfies precisely one of conditions $(d0)$, $(d1)$, $(d2)$, and $(d3)$.

Suppose that $\bar{P}$ satisfies $(d0)$. Since G is connected and cubic, $\bar{P} = K_4$ implies $G = K_4$, contradicting $v(G) \geq 6$.

Therefore $\bar{P}$ satisfies precisely one of conditions $(d1)$, $(d2)$, $(d3)$ in **4.10**. Let, as above, $\mathcal{P}_3^i$ denote the set of 2–paths P in $\mathcal{P}$ such that $\bar{P}$ satisfies (di), $i = 1, 2, 3$.

(**p1**) Suppose that $P, Q \in \mathcal{P}_3$, $P \neq Q$, $g(P)$ and $g(Q)$ are defined, and $g(P) = g(Q) = P_1 = x_1c_1y_1$. Then by **4.11** and **4.12**, $P, Q \in \mathcal{P}_3^1$ and $z(Q) = z(P) = z$. Since $p, q \in N(z) \setminus V(P_1)$ and $d(z) = 3$, we have $z \in \{x_1, y_1\}$, say $z = x_1$.

Let $N(c_1) = \{x_1, y_1, y_2\}$. By **4.11** (a2.2), c_1 is adjacent to the center c_2 of another 2-path $P_2 \in \mathcal{P}$, and so $c_2 \in \{y_1, y_2\}$. Since $d(c_2) = 3$, we have $y_1y_2 \notin E$. Since G is cubic, $G - x_1c_1$ consists of exactly two components G_1 and G_2 where G_1 is the subgraph of G induced by $\bar{P} \cup \bar{Q} \cup z$ (i.e. G_1 is obtained from $\bar{P} \cup \bar{Q}$ by adding a vertex z and the two edges pz and qz). Clearly $y_1, y_2 \in G_2$. Let G_2' be the graph obtained from $G_2 - c_1$ by adding a new edge $e' = y_1y_2$. Obviously G_2' is cubic. Also G_2' is simple because $y_1y_2 \notin E$.

Clearly $v(G_2') = v(G) - 12 < v(G)$. Therefore by minimality of G, G_2' has a Λ–packing $\mathcal{P}'$ such that $|\mathcal{P}'| = \lambda(G_2') \geq v(G_2')/4$. Then by **3.1**, $\lambda(G) \geq v(G)/4$, a contradiction.

(**p2**) Suppose that $P \in \mathcal{P}_3^2$ and $|N(\bar{P})| \neq 2$. Clearly $|N(\bar{P})| \in \{1, 2, 3\}$, and by **4.10**, $\bar{P}$ is isomorphic to $K_{2,3}$.

(**p2.1**) Suppose that $|N(\bar{P})| = 1$, say $N(\bar{P}) = \{x\}$. Then $\bar{P} \cup x$ induces G, $v(G) = 6$, and $\lambda(G) = 2$, and so $\lambda(G) \geq v(G)/4$, a contradiction.

(**p2.2**) Now suppose that $|N(\bar{P})| = 3$, say $N(\bar{P}) = \{x', y', z'\}$. Let G' be obtained from $G - E(\bar{P})$ by identifying the vertices of $\bar{P}$ with a new vertex p'. Then G' is a cubic loopless graph. Since the vertices x', y', and z' are distinct, G' has no parallel edges. Thus G' is a simple graph. Therefore by minimality of G, $\lambda(G') \geq v(G')/4$. Then by **3.4**, $\lambda(G) \geq v(G)/4$, a contradiction.

(**p3**) Suppose that $P \in \mathcal{P}_3^3$ (and so $\bar{P}$ is an induced net in G) and at least one of the feet $\{x', y', z'\}$ of the net $\bar{P}$, say z', does not belong to a triangle in G.

Let G' be obtained from $G - P \cup x'y'$ by replacing the 2-path with center z' by an edge. Clearly G' is a cubic graph with no loops and $v(G') < v(G)$. Since z belongs to no triangle and $\bar{P}$ is an induced net in G, it follows that G' has no parallel edges. Thus G' is a simple cubic graph. By minimality of G, $\lambda(G') \geq v(G')/4$. Now by **3.2**, $\lambda(G) \geq v(G)/4$, a contradiction.

(**p4**) Now suppose that every P in $\mathcal{P}_3$ satisfies the assumptions of neither (**p1**) nor

(**p2**) nor (**p3**). Namely, we suppose that
($s1$) if $P, Q \in \mathcal{P}_3 \setminus \mathcal{P}_3^3$ then $g(P) \neq g(Q)$,
($s2$) if $P \in \mathcal{P}_3^2$ then $|N(\bar{P})| = 2$, and
($s3$) if $P \in \mathcal{P}_3^3$ then each foot of the net $\bar{P}$ belongs to a triangle.

By **4.13**, ($s3$) is equivalent to
($s3'$) if $P \in \mathcal{P}_3^3$ then each foot of the net $\bar{P}$ belongs to a triangle which is not a triangle of a diamond.

Note that, because of assumption ($s2$), $z(P)$ and $g(P)$ are defined not only for every $P \in \mathcal{P}_3^1$ but also for $P \in \mathcal{P}_3^2$. Let, as above, $g(P) = P_1 = x_1c_1y_1$.

Let us define a directed graph Γ as follows:
$V(\Gamma) = \mathcal{P}$ and $PQ \in E(\Gamma)$ if $P, Q \in \mathcal{P}$, $P \neq Q$, and exactly one of the following holds:
($a1$) $P \in \mathcal{P}_3^1$ and either $|g(P), B| = 2$ and $Q = g(P)$ or $|g(P), B| = 2$ and $Q \in \mathcal{P} \setminus \{P, g(P)\}$ where the centers of $g(P)$ and Q are adjacent (i.e. $c(g(P)), c(Q) \in E$),
($a2$) $P \in \mathcal{P}_3^2$ and $Q = g(P)$, and
($a3$) $P \in \mathcal{P}_3^3$ and a foot of the net $\bar{P}$ is adjacent to a vertex in Q.

Clearly if $P \in \mathcal{P}_3$ then P is not an isolated vertex in Γ and if $PQ \in E(\Gamma)$ then $P \in \mathcal{P}_3$.

In case ($a1$), by **4.11**, $|Q, B| \leq 2$. In case ($a2$), by **4.12**, $|Q, B| \leq 1$. In case ($a3$), by **4.13**, $|Q, B| \leq 1$.

Therefore every edge in Γ starts in $\mathcal{P}_3$ and ends in $\mathcal{P} \setminus \mathcal{P}_3$.

(**p4.1**) We claim that
each component of Γ contains exactly one vertex from $\mathcal{P}_3$, and therefore each component of Γ is a star centered in a vertex in $\mathcal{P}_3$.

Suppose, on the contrary, that $P, L \in \mathcal{P}_3$, $P \neq L$, and P, L belong to the same component of Γ. By **4.13**, $P, L \in \mathcal{P}_3 \setminus \mathcal{P}_3^3$. By assumption ($s1$), $g(P) \neq g(L)$. Therefore by ($s1$), we can assume that $P \in \mathcal{P}_3^1$, $g(P) = P_1 = x_1c_1y_1$, $g(L) = P_2 = x_2c_2y_2$, and $z(L) = x_2$ where $P_2 \in \mathcal{P} \setminus \{P, P_1\}$ and $c_1c_2 \in E$. Let $ll' = E(\bar{L} - L)$, $lx_2 \in E$, , and, as above, $pp' = E(\bar{P} - P)$, $px_1 \in E$. Let $\mathcal{P}' = \mathcal{P} \setminus \{P_1, P_2\} \cup \{l'lx_2, c_1c_2y_2, p'px_1\}$. Clearly $\mathcal{P}'$ is a Λ–packing and $|\mathcal{P}'| = |\mathcal{P}| + 1$. Therefore $\mathcal{P}$ is not 1–optimal, a contradiction.

(**p4.2**) Let C be a component of Γ and put $|C, B| = \sum\{|L, B| : L \in C\}$. Let $P \in C$.

If $C = \{P\}$ then $|P, B| \leq 2$. Therefore $|C, B| = |P, B| \leq 2 = 2|V(C)|$.

If $P \in \mathcal{P}_3^1$ and $|g(P), B| \leq 1$ then by (**p4.1**) and ($a1$) in the definition of $E(\Gamma)$, $C = \{P, g(P)\}$. Therefore $|C, B| = |P, B| + |g(P), B| \leq 3 + 1 = 2|V(C)|$.

If $P \in \mathcal{P}_3^1$ and $g(P) \in \mathcal{P}_2$ then by ($a1$), $\{P, g(P), Q\} \subseteq C$, where the centers of $g(P)$ and Q are adjacent. Since the centers of $g(P)$ and Q are of degree three in G, there is no $L \in \mathcal{P} \setminus \{g(P), Q\}$ whose center is adjacent to the center of either $g(P)$ or Q. Hence by (**p4.1**), $C = \{P, g(P), Q\}$ (see Figure 8). By **4.14**, $|Q, B| \leq 1$. Therefore $|C, B| = |P, B| + |g(P), B| + |Q, B| \leq 3 + 2 + 1 = 2|V(C)|$.

If $P \in \mathcal{P}_3^2$ then by (**p4.1**) and ($a2$) in the definition of $E(\Gamma)$, $C = \{P, g(P)\}$. By **4.12**, $|g(P), B| \leq 1$. Therefore $|C, B| = |P, B| + |g(P), B| \leq 3 + 1 = 2|V(C)|$.

For $P \in \mathcal{P}_3^3$, let $\mathcal{P}(P)$ denote the set of 2–paths in $\mathcal{P} \setminus P$ adjacent to a foot of the net $\bar{P}$ in G.

If $P \in \mathcal{P}_3^3$ then by (**p4.1**) and $(a2)$ in the definition of $E(\Gamma)$, $C = P \cup \mathcal{P}(P)$. By $(s3')$, every foot of the net $\bar{P}$ belongs to a triangle which is not a triangle of an induced diamond. Hence by **4.13**, $|\mathcal{P}(P)| \geq 6$ and $|L, B| \leq 1$ for every $L \in \mathcal{P}(P)$. Therefore
$|C, B| \leq 3 + |\mathcal{P}(P)|$. Since $|\mathcal{P}(P)| \geq 6$, we have $|C, B| < 2|V(C)|$.

Let $\mathcal{C}(\Gamma)\}$ denote the set of components of Γ. From the above inequalities we have:

$$\sum_{L \in \mathcal{P}} |L, B| = \sum_{C \in \mathcal{C}(\Gamma)} \sum_{L \in C} |L, B| = \sum_{C \in \mathcal{C}(\Gamma)} |C, B| \leq 2 \sum_{C \in \mathcal{C}(\Gamma)} |V(C)| = 2|V(\Gamma)| = 2|\mathcal{P}|.$$

Now by **4.2**, $\lambda(G) \geq v(G)/4$, a contradiction. $\square$

The approach, we developed in this paper to analyze Λ–packings in cubic graphs, allowed us to obtain similar results for regular graphs of higher degree. These results will be presented in another paper.

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