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# ASYMPTOTICALLY OPTIMAL TREE–PACINGS IN REGULAR GRAPHS

by

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**ABSTRACT**

Let $T$ be a tree with $t$ vertices. Clearly, an $n$ vertex graph contains at most $n/t$ vertex disjoint trees isomorphic to $T$. In this paper we show that for every $\epsilon > 0$, there exists a $D(\epsilon, t) > 0$ such that, if $d > D(\epsilon, t)$ and $G$ is a simple $d$-regular graph on $n$ vertices, then $G$ contains at least $(1 - \epsilon)n/t$ vertex disjoint trees isomorphic to $T$.

**Keywords:** graph, hypergraphs, packing, trees, matchings

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# 1 Introduction

We consider simple undirected graphs. Given a graph $G$ and a family $\mathcal{F}$ of graphs, an *$\mathcal{F}$-packing* of $G$ is a subgraph of $G$ each of whose components is isomorphic to a member of $\mathcal{F}$. The *$\mathcal{F}$-packing problem* is to find an $\mathcal{F}$–packing of the maximum number of vertices. There are various results on the $\mathcal{F}$–packing problem (see e.g. [3, 10, 12, 13, 14]).

When $\mathcal{F}$ consists of a single graph $F$, we abuse notation by writing *$F$–packing*. The very special case of the $F$–packing problem when $F = K_2$, a single edge, is simply that of finding a maximum matching. This problem is well-studied, and can be solved in polynomial time (see, for example, [14]). However, if $F$ is a connected graph with at least three vertices then the $F$-packing problem is known to be NP-hard [12]. There are various directions for studying this generally intractable problem.

Here are some results for the $F$–packing problem where $F$ is the 3-vertex path $\Lambda$.

**Theorem 1** [9] *Suppose that $G$ is a claw–free graph (i.e. $G$ contains no induced subgraph isomorphic to $K_{1,3}$). Suppose also that $G$ is connected and has at most two end–blocks (in particular, vertex 2–connected). Then $G$ contains at least $v(G)/3$ vertex disjoint 3-vertex paths.*

**Theorem 2** [11] *Suppose that $G$ is a 3-regular graph. Then $G$ contains at least $v(G)/4$ vertex disjoint 3-vertex paths that can be found in polynomial time (and so for 3-regular graphs there is a polynomial approximation algorithm that guarantees at least a 3/4–optimal solution for the $\Lambda$–packing problem).*

On the other hand, there is a series of asymptotic packing results, beginning with the following deep theorem of Hajnal and Szemerédi.

**Theorem 3** [8] *If $G$ has $n$ vertices and minimum degree at least $(1 - 1/r)n$, then $G$ contains $\lfloor n/r \rfloor$ vertex-disjoint copies of $K_r$.*

Theorem 3 has been generalized by Alon and Yuster for graphs other than $K_r$.

**Theorem 4** [2] *For every $\gamma > 0$ and for every positive integer $h$, there exists an $n_0 = n_0(\gamma, h)$ such that for every graph $H$ with $h$ vertices and for every $n > n_0$, any graph $G$ with $hn$ vertices and with minimum degree $\delta(G) \geq (1-1/\chi(H)+\gamma)hn$ contains $n$ vertex-disjoint copies of $H$.*

In this paper we consider an asymptotic version of the $F$–packing problem, where $F$ is a tree. Our main result is the following.

**Theorem 5** *Let $T$ be a tree on $t$ vertices and let $\epsilon > 0$. Suppose that $G$ is a $d$-regular graph on $n$ vertices and $d \geq \frac{128t^3}{\epsilon^2} \ln(\frac{128t^3}{\epsilon^2})$. Then $G$ contains $(1-\epsilon)n/t$ vertex disjoint copies of $T$.*

Both Theorem 3 and Theorem 4 require $G$ to have $\Omega(n^2)$ edges. Theorem 5 differs from these results in that our graphs are not required to be dense. Indeed, $d$ above is only a function of $\epsilon$ and the size of the tree and does not depend on $n$. Consequently, Theorem 5 cannot possibly be extended to graphs other than trees, since the Turán number of a cycle of length $2t$ is known to be at least $\Omega(n^{(2t+1)/2t})$ [4], and there exist essentially regular graphs with about this many edges that contain no copy of $C_{2t}$.

In this paper, we present two approaches for obtaining tree-packing results in regular graphs. First, in Section 2 we give a short proof of an asymptotic version of Theorem 5. This proof relies on powerful hypergraph packing results of Frankl and Rödl [7] and Pippenger and Spencer [16]. Next, in Section 3 we present a proof of Theorem 5, based on a probabilistic approach. It uses another powerful result called the Lovász Local Lemma (see e.g., [1]). In addition it provides an explicit dependence of the degree on $t$ and $\epsilon$. Section 4 contains some concluding remarks and an open question.

# 2 $T$-packings from matchings in hypergraphs

In this section we present the proof of the following asymptotic version of Theorem 5.

**Theorem 6** *Let $T$ be a tree on $t$ vertices. Let $G_n$ be a $d_n$-regular graph on $n$ vertices. Suppose that $d_n \to \infty$ when $n \to \infty$. Then $G_n$ contains at least $(1-o(1))n/t$ (and, obviously, at most $n/t$) disjoint trees isomorphic to $T$.*

The proof of this theorem is based on a hypergraph packing result of Pippenger and Spencer [16]. The main idea behind this proof came from a result of Rödl [17] that solved an old packing conjecture of Erdős and Hanani [5]. Rödl's idea, now known as his "nibble", was used by Frankl-Rödl [7] to prove that under certain regularity and local density conditions, a hypergraph has a large matching. Pippenger and Spencer used probabilistic methods to extend and generalize the result in [7].

First we introduce some notions about hypergraphs. All hypergraphs we consider are allowed to have multiple edges. Given a hypergraph $\mathcal{H} = (V, E)$, the *degree* $d(v)$ of a vertex $v \in V$ is the number of edges containing $v$. For vertices $v, w$, the *codegree* $cod(v, w)$ of $v$ and $w$ is the number of edges containing both $v$ and $w$. Let

$$\Delta(\mathcal{H}) = \max_{v \in V} d(v), \qquad \delta(\mathcal{H}) = \min_{v \in V} d(v), \qquad C(G) = \max_{u,v \in V, u \neq v} cod(u, v).$$

A *matching* in $\mathcal{H}$ is a set of pairwise disjoint edges of $\mathcal{H}$. Let $\mu(\mathcal{H})$ be the size of the largest matching in $\mathcal{H}$. A matching $M$ is *perfect* if every vertex of $\mathcal{H}$ is in exactly one edge of $M$. A hypergraph $\mathcal{H}$ is $t$-uniform if each of its edges consists of exactly $t$ elements.

**Theorem 7** [16] *For every $t \geq 2$ and $\varepsilon > 0$, there exist $\varepsilon' > 0$ and $n_0$ such that if $\mathcal{H}$ is a $t$-uniform hypergraph on $n(\mathcal{H}) \geq n_0$ vertices with $\delta(\mathcal{H}) \geq (1-\varepsilon')\Delta(\mathcal{H})$, and $C(\mathcal{H}) \leq \varepsilon'\Delta(\mathcal{H})$, then*

$$\mu(\mathcal{H}) \geq (1-\varepsilon)n/t.$$

We rephrase Theorem 7 in more convenient asymptotic notation.

**Theorem 7'** *Let $\mathcal{H}_1, \mathcal{H}_2, \ldots$ be sequence of hypergraphs, with $|V(\mathcal{H}_k)| \to \infty$. If $\delta(\mathcal{H}_k) \sim \Delta(\mathcal{H}_k)$, and $C(\mathcal{H}_k) = o(\Delta(\mathcal{H}_k))$, then $\mu(\mathcal{H}_k) \sim |V(\mathcal{H}_k)|/t$.*

The above result says that under certain regularity and local density conditions on $\mathcal{H}$, one can find an almost perfect matching $M$ in $\mathcal{H}$, i.e., the number of vertices in no edge of $M$ is negligible. In fact, [16] proves something much stronger, namely that one can decompose almost all the edges of $\mathcal{H}$ into almost perfect matchings, but we need only the weaker statement.

Next we show how Theorem 7 can be applied to provide asymptotically optimal tree-packings of regular graphs. For convenience, we omit the subscript $k$ and the use of integer parts in what follows. Our goal is to produce a large $T$-packing in $G$. By a copy of $T$ we mean a subgraph isomorphic to $T$.

Given $u, v \in V(G)$ let $c(v)$ and $c(u,v)$ denote the number of copies of $T$ in $G$ containing $v$ and $\{u,v\}$, respectively (note that different copies may have the same vertex set). The following lemma provides necessary estimates for the numbers $c(v)$ and $c(u,v)$.

**Lemma 8** *Let $T$ be a tree with $t$ vertices. Suppose that $G$ is a $d$-regular graph on $n$ vertices. Then*
*(s1) $c(v) = (1+o(1))c_T d^{t-1}$ ($d \to \infty$) for every $v \in V(G)$, where $c_T$ depends only on $T$ and does not depend on the choice of $v$, and*
*(s2) $c(a,b) = O(d^{t-2})$ for every pair $a, b \in V(G)$, $a \neq b$.*

**Proof.**

**(p1)** We first estimate $c(v)$. Let us consider the rooted tree $R$ obtained from $T$ by specifying a vertex $r$ of $T$ as a root. Let $c_r(v)$ denote the number of copies of $R$ in $G$ such that $r$ is located in $v \in V(G)$.

It suffices to show that $c_r(v) = (1+o(1))d^{t-1}$ for all $r \in V(T)$ and $v \in V(G)$. Indeed, this would imply that $c(v) = \sum\{c_r(v)/g : r \in V(T)\} = (1+o(1))(t/g)d^{t-1} = (1+o(1))c_T d^{t-1}$, where $g$ is the size of the automorphism group of $T$.

Let $x_1$ be a leaf of $R$ distinct from $r$, $R_1 = R - x_1$, and $y_1$ be the vertex in $R_1$ adjacent to $x_1$. If $(x_i, y_i, R_i)$ is already defined, let $x_{i+1}$ be a leaf of $R_i$ distinct from $r$, $R_{i+1} = R_i - x_{i+1}$, and $y_{i+1}$ be the vertex in $R_{i+1}$ adjacent to $x_{i+1}$. Clearly $r = y_{t-1} = R_{t-1}$. Now we estimate $c_r(v)$ as follows. There is only one way to allocate $r$ in $G$, namely, to allocate $r$ in $v$. Since $v$ is of degree $d$ in $G$ and $G$ is simple, there are $d$ ways to allocate $x_{t-1}$ in $G$. Suppose that $R_i$, $1 \leq i < t-1$, is already allocated in $G$, and $y_i$ is allocated in a vertex $v_i$ in $G$. Since $v_i$ is of degree $d$ in $G$ and $G$ is simple, there are at most $d$ and at least $d - t + i$ ways to allocate $x_i$ in $G$. Therefore

$$(d-t)^{t-1} < c_r(v) < d^{t-1}. \qquad (*)$$

Since $d \to \infty$, we have: $c_r(v) = (1 + o(1))d^{t-1}$ for all $r \in V(T)$ and $v \in V(G)$.

**(p2)** Now we will estimate $c(a, b)$, the number of copies of $T$ in $G$ containing both $a$ and $b$ where $a \neq b$. For $x, y \in V(T)$, let $c_{x,y}(a, b)$ denote the number of copies of $T$ containing $a, b$, with $a$ playing the role of $x$ and $b$ playing the role of $y$. Clearly

$$c(a, b) \leq \binom{t}{2} \max_{x,y \in V(T)} c_{x,y}(a, b),$$

because $a, b$ play the role of some pair $x, y$ in each copy of $T$ containing them. Hence it suffices to show that $c_{x,y}(a, b) \leq d^{t-2}$.

Split $T$ in two nontrivial trees $X$ and $Y$ where $X$ is rooted at $x$ and $Y$ is rooted at $y$, $V(X) \cap V(Y) = \emptyset$, and $V(X) \cup V(Y) = V(T)$. This can be done by deleting any edge from the unique path between $x$ and $y$. By (*), there are at most $d^{|V(X)|-1}$ copies of $X$ in $G$ with $a$ playing the role of $x$, and at most $d^{|V(Y)|-1}$ copies of $Y$ in $G$ with $b$ playing the role of $y$. Thus $c_{x,y}(a, b) \leq d^{|V(X)|-1} d^{|V(Y)|-1} = d^{t-2}$. □

**Proof of Theorem 6** Given $G$, we must find a $T$-packing of size at least $(1-o(1))n/t$. From $G$ construct the hypergraph $\mathcal{H} = (V, E)$ with $V = V(G)$ and $E$ consisting of vertex sets of copies of $T$ in $G$ (note that $\mathcal{H}$ can have multiple edges). Then Lemma 8 part (s1) implies that $\delta(\mathcal{H}) = \Delta(\mathcal{H}) \sim c_T d^{t-1}$, and Lemma 8 part (s2) implies that $C(\mathcal{H}) = O(d^{t-2}) = o(d^{t-1}) = o(\Delta(\mathcal{H}))$. Theorem 7 now implies that $\mu(\mathcal{H}) \sim |V(H)|/t = n/t$. This clearly yields a $T$-packing in $G$ of the required size. □

# 3 $T$-packings from the Lovász Local Lemma

This section contains a proof of Theorem 5 based on a probabilistic approach and the so called Lovász Local Lemma. We use the following symmetric version of the Lovász Local Lemma.

**Theorem 9** [1] *Let $A_1, \ldots, A_n$ be events in a probability space. Suppose that each event $A_i$ is mutually independent of a set of all the other events $A_j$ but at most $d$, and that $Prob[A_i] \leq p$ for all $i$. If $ep(d + 1) \leq 1$, then $Prob[\wedge \overline{A_i}] > 0$.*

Here we make no attempt to optimize our absolute constants. First we need the following lemma. Given a partition $V_1, \ldots, V_t$ of the vertex set of a graph $G$, let $d_i(v)$ denote the number of neighbors of a vertex $v$ of $G$ in $V_i$.

**Lemma 10** *Let $t$ be an integer and let $G$ be a $d$-regular graph satisfying $d \geq 4t^3$. Then there exists a partition of $V(G)$ into $t$ subsets $V_1, \ldots, V_t$ such that*
$\frac{d}{t} - 4\sqrt{\frac{d}{t} \ln d} \leq d_i(v) \leq \frac{d}{t} + 4\sqrt{\frac{d}{t} \ln d}$
*for every $v \in V$ and $1 \leq i \leq t$.*

**Proof.** Partition the set of vertices $V$ into $t$ subsets $V_1, V_2, \ldots, V_t$ by choosing for each vertex randomly and independently an index $i$ in $\{1, \ldots, t\}$ and placing it into $V_i$. Let $A_{i,v}$ denote the event that $d_i(v)$ is either greater than $\frac{d}{t} + 4\sqrt{\frac{d}{t} \ln d}$ or less than $\frac{d}{t} - 4\sqrt{\frac{d}{t} \ln d}$. Observe that if none of the events $A_{i,v}$ holds, then our partition satisfies

the assertion of the lemma. Hence it suffices to show that with positive probability no event $A_{i,v}$ occurs. We prove this by applying Theorem 9.

Since the number of neighbors of any vertex $v$ in $V_i, i = 1, 2, \ldots, t$, is a binomially distributed random variable with parameters $d$ and $1/t$, it follows by the standard estimates for Binomial distributions (cf. , e.g., [15], Theorem 2.3) that for every $v \in V$

$$Pr\left(|d_i(v) - \frac{d}{t}| > a\frac{d}{t}\right) \leq 2e^{-\frac{a^2(d/t)}{2(1+a/3)}}.$$

By substituting $a$ to be $4\sqrt{(t/d)\ln d}$, we obtain that the probability of the event $A_{i,v}$ is at most $2e^{-4\ln d} = 2d^{-4}$. Clearly each event $A_{i,v}$ is independent of all but at most $td(d-1)$ others, as it is independent of all events $A_{j,u}$ corresponding to vertices $u$ whose distance from $v$ is larger than 2. Since $e \cdot 2d^{-4} \cdot (td(d-1)+1) < e \cdot 2d^{-4} \cdot td^2 < 1$, we conclude, by Theorem 9, that with positive probability no event $A_{i,v}$ holds. This completes the proof of the lemma. □

Next we prove the following tree-packing result for nearly-regular, $t$-partite graphs, which is interesting in its own right.

**Proposition 11** *Let $T$ be a fixed tree with the vertex set $u_1, \ldots, u_t$ and let $H$ be a $t$-partite graph with parts $V_1, \ldots, V_t$ such that $|V_1| = h$ and for every vertex $v \in V(H)$ and every $1 \leq i \leq t$ the number $d_i(v)$ of neighbors of $v$ in $V_i$ satisfies $(1-\delta)k \leq d_i(v) \leq (1+\delta)k$ for some $k > 0$ and $0 \leq \delta < 1$. Then $H$ contains $(1-2(t-1)\delta)h$ vertex disjoint copies of $T$ with the property that $V_i$ contains the vertex of each copy corresponding to $u_i$, $1 \leq i \leq t$.*

**Proof.** We use induction on $t$. For $t = 1$ the assertion is trivially true. Therefore let $t \geq 2$. Without loss of generality, we can assume that $u_t$ is a leaf adjacent to the vertex $u_{t-1}$. Let $T' = T - u_t$ and $H' = H - V_t$. Then by the induction hypothesis, we can find at least $(1-2(t-2)\delta)h$ vertex disjoint copies of $T'$ in $H'$ such that in all these copies the vertices, corresponding to $u_{t-1}$, belong to $V_{t-1}$. Denote the set of these vertices by $S$. Consider all the edges between $S$ and $V_t$. In the resulting bipartite graph $B$ each vertex is of degree at least $(1-\delta)k$ and at most $(1+\delta)k$. Therefore the edges of $B$ can be covered by $(1+\delta)k$ disjoint matchings. Since the number of edges in $B$ is at least $(1-\delta)k|S|$, we conclude that $B$ contains a matching of size at least

$$\frac{(1-\delta)k|S|}{(1+\delta)k} = \frac{1-\delta}{1+\delta}|S| \geq (1-2\delta)|S|.$$

By adding the edges of this matching to the appropriate copies of $T'$, we obtain at least $(1-2\delta)|S| = (1-2\delta)(1-2(t-2)\delta)k \geq (1-2(t-1)\delta)k$ vertex disjoint copies of $T$. This completes the proof of the statement. □

Having finished all necessary preparations, we are now ready to complete the proof of Theorem 5.

**Proof of Theorem 5.** Let $G$ be a $d$-regular graph on $n$ vertices with $d \geq \frac{128t^3}{\epsilon^2}\ln(\frac{128t^3}{\epsilon^2})$ and let $T$ be a tree with $t$ vertices. By Lemma 10, we can partition vertices of $G$ into

$t$ parts $V_1, \ldots, V_t$ such that $|V_1| \geq n/t$ (pick $V_1$ to be the largest part) and for every vertex the number of its neighbors in $V_i$, $1 \leq i \leq t$, is bounded by $(1 \pm \delta)d/t$, where $\delta = 4\sqrt{(t/d)\ln d} \leq \epsilon/2t$. Thus by Proposition 11, $G$ contains at least $(1 - 2(t-1)\delta)|V_1| \geq (1-\epsilon)n/t$ vertex disjoint copies of $T$. □

# 4 Concluding remarks

- The regularity requirement in Theorem 5 cannot be weakened to a minimum degree requirement. To see this, let $G_d$ be the complete bipartite graph with parts $X, Y$ of sizes $d$ and $d^2$, respectively. The minimum degree of $G_d$ is $d \to \infty$, but clearly the largest $T$-packing has size at most $d = o(|V(G_d)|)$. On the other hand, it is easy to see that the proof of Theorem 5 remains valid for nearly-regular graphs. More precisely one can show the following.

  **Proposition 12** *Let $T$ be a tree on $t$ vertices. For all $t$ and $\epsilon > 0$, there exist two positive numbers $\gamma = \gamma(t, \epsilon)$ and $D(t, \epsilon)$ such that the following holds: if $d > D(t, \epsilon)$ and $G$ is a graph on $n$ vertices with $(1-\gamma)d \leq \delta(G) \leq \Delta(G) \leq (1+\gamma)d$, then $G$ contains $(1-\epsilon)n/t$ vertex disjoint copies of $T$.*

  It is also easy to see that the above results can be extended to $d$-regular multigraphs provided all multiplicities are bounded.

- The dependency of the degree of the graph on both $t$ and $\epsilon$ is needed in the statement of Theorem 5. To see this, let $G$ be a regular graph consisting of $\lceil \epsilon n/t \rceil$ disjoint cliques of size $k$, where $k = \Theta(t/\epsilon)$ is an integer such that $k \equiv t-1 \ (mod\ t)$. Clearly any packing of $G$ by a tree on $t$ vertices misses at least $t-1$ vertices in each clique. Therefore altogether it will miss at least $(t-1)(\epsilon n/t) = \Omega(\epsilon|V(G)|)$ vertices. This shows that in the statement of Theorem 5 the degree of the graph should be at least $\Omega(t/\epsilon)$. Thus there is a big gap between the upper and lower bounds and it is an interesting

  **Question.** What is the correct dependency of the degree of the graph $G$ on $t$ and $\epsilon$ to guarantee $(1-\epsilon)n/t$ vertex disjoint copies of $T$ in $G$?

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