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THE SHERRINGTON–KIRKPATRICK MODEL WITH SHORT RANGE FERROMAGNETIC INTERACTIONS

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Abstract

We study a model of Ising spins with short range ferromagnetic and long range SK interactions. We generalize the results obtained for the standard SK model, computing in particular the high temperature pressure.

RÉSUMÉ. Nous étudions un modèle de spins binaires qui est soumis à la fois à des interactions de type Ising et de type Sherrington-Kirkpatrick. Nous généralisons des résultats obtenus dans le cas du modèle SK seul, en calculant en particulier la pression à haute température.

1.Version française abrégée

Dans cette Note, nous considérons un modèle d'Ising ferromagnétique perturbé par une interaction de verre de spins de champ moyen sur le réseau carré de dimension d. L'hamiltonien du système est la somme d'une composante deterministe à courte portée

$$H_N^I(\sigma) = -\frac{1}{2} \sum_{i,j \in \Lambda_N} K(i-j)\sigma_i\sigma_j, \tag{1.1}$$

et d'une composante désordonnée à longue portée

$$H_N^{SK}(\sigma) = \frac{1}{\sqrt{2|\Lambda_N|}} \sum_{i,j \in \Lambda_N} J_{i,j}\sigma_i\sigma_j, \tag{1.2}$$

où $\Lambda_N = \{-N, \dots, N\}^d$, $\sigma \in \{-1,+1\}^{\Lambda_N}$, $K(\cdot) \geq 0$, $\sum_i K(i) < \infty$, et $\{J_{i,j}\}_{i,j \in \mathbb{Z}^d}$ est une famille de variables aléatoires indépendantes de loi gaussienne $\mathcal{N}(0,1)$. La fonction de partition du système est donnée par

$$\tilde{Z}_N(\kappa,\beta) = \sum_{\sigma \in \{-1,+1\}^{\Lambda_N}} \exp\left\{-\kappa H_N^I(\sigma) - \beta H_N^{SK}(\sigma)\right\}, \tag{1.3}$$

avec $\beta > 0$ and $\kappa > 0$. Le cas $\beta = 0$ correspond au modèle d'Ising habituel à temperature $1/\kappa$, tandis que pour $\kappa = 0$ on retrouve le modèle SK introduit par Kirkpatrick et Sherrington [6]. Le modèle SK a été étudié rigoureusement lorsque $\beta < 1$, d'abord par des techniques de moments et d'expansions [1], puis par calcul stochastique [2] et par des méthodes de concentration [11]. Cette dernière réference traite également le cas bien plus complexe avec champ extérieur. Notre modèle est motivé par les systèmes de verres de spins sujets à la fois à des interactions ferromagnétiques à courte portée homogènes en espace, et des interactions à longue portée fortement oscillantes représentées ici par les couplages aléatoires $J_{i,j}$. Quand la portée de ces dernières tend vers l'infini (*asymptotique de Kac)*, le système est alors décrit par le modèle (1.3), comme cela a été remarqué dans [4,Theorem B] en l'absence d'interaction courte. Notons que les deux composantes de l'hamiltonien sont concurrents: pour des réalisations typiques des $J_{i,j}$ les configurations de minimum d'energie seront des mélanges de spins + et −. Indépendamment des conditions frontières la pression du système est donnée par la limite quand $N \to \infty$ de

$$\tilde{p}_N(\kappa,\beta) = \frac{1}{|\Lambda_N|} \log \tilde{Z}_N(\kappa,\beta). \tag{1.4}$$

dont l'existence, non évidente en présence de l'interaction SK, est partie de nos résultats. Le modèle d'Ising sous-jacent joue un rôle important dans la suite; dans la région d'unicité nous noterons $\mathbf{P} = \mathbf{P}_\kappa$ [resp., $\mathbf{E} = \mathbf{E}_\kappa$] la mesure de Gibbs [5] sur $\{-1,+1\}^{\mathbb{Z}^d}$ avec interaction de paires K et température inverse κ [resp., l'intégrale correspondante], en réservant la notation $\mathbb{P}$ (resp. $\mathbb{E}$) pour les variables J. Définissons $\theta_\kappa > 0$ la quantité

$$\theta_\kappa^2 = \sum_{i\in\mathbb{Z}^d} \left[\mathbf{E}_\kappa(\sigma_0\sigma_i)\right]^2 , \tag{1.5}$$

connue dans la littérature sous le nom de *bubble diagram* [3], ainsi que

$$c_\kappa(i) = c(i) = \mathbf{E}_\kappa(\sigma_0\sigma_i) , \qquad \chi_\kappa = \sum_{i\in\mathbb{Z}^d} c(i), \tag{1.6}$$

et κ_c la plus petite valeur de κ telle que $\chi_\kappa = \infty$; il est bien connu que $c(i) \geq 0$ est une fonction croissante de κ. Soit D le domaine de l'espace des paramètres

$$D = \{(\kappa,\beta);\ \kappa < \kappa_c, \exists\varepsilon > 0\ :\ \sup_N \mathbf{E}_\kappa^{\otimes 2} \exp\frac{\beta^2+\varepsilon}{2} X_N(\sigma,\sigma')^2 < \infty\} \quad ,$$

$$X_N(\sigma,\sigma') = \frac{1}{\sqrt{|\Lambda_N|}} \sum_{i\in\Lambda_N} \sigma_i\sigma_i' \tag{1.7}$$

Sur D la variable $\tilde{Z}_N/\mathbb{E}\tilde{Z}_N$ est bornée dans L^2 uniformément en N. Résumons nos résultats principaux, qui sont énoncés dans le Théorème 2.1 ci-dessous.

i) Le domaine D contient $\{\kappa < \kappa_c, \beta^2\chi_\kappa < 1\}$.

ii) Sur le domaine D, $\mathbf{E}_\kappa[\exp(-\beta H_N^{SK})]/\mathbb{E}\mathbf{E}_\kappa[\exp(-\beta H_N^{SK})]$ converge vers une loi log-normale.

iii) Si $(\kappa,\beta)\in\overline{D}$, la limite $\tilde{p}(\kappa,\beta)$ quand $N\to\infty$ de $\tilde{p}_N(\kappa,\beta)$ existe $\mathbb{P}$–p.s., et elle est donnée par $\tilde{p}(\kappa,\beta) = p^I(\kappa) + \frac{\beta^2}{4}$ où $p^I(\kappa) = \lim_N(|\Lambda_N|)^{-1}\log\tilde{Z}_N(\kappa,0)$ désigne la pression du modèle d'Ising.

Notre preuve suit l'approche par le calcul calcul stochastique introduite dans [2]. Nous utilisons aussi de façon cruciale les propriétés du modèle d'Ising (avec l'interaction ferromagnétique seule); le champ aléatoire $(\sigma_i)_i$ est positivement associé (inégalité FKG), et ses moments sont majorés par ceux du champ gaussien possédant les mêmes covariances, d'après un résultat de C. Newman [7].

2. English Version

2.1 The "Ising + SK"model.

We consider the Ising ferromagnetic d–dimensional model, perturbed by a mean field interaction. A spin configuration is an element σ of $\{-1,+1\}^{\Lambda_N}$, with $\Lambda_N =$

$\{-N, \dots, N\}^d$, $N \in \mathbb{Z}^+$, and the Hamiltonian of the system is given by $H_N^I(\sigma) + H_N^{SK}(\sigma)$ (cf. (1.1) and (1.2)). The partition function is given by (1.3), in which there are two positive parameters β and κ, which play the role of two *independent* inverse temperatures. If $\beta = 0$ the model reduces to the Ising model with temperature $1/\kappa$, and if $\kappa = 0$ we are back to the standard SK model [6]. The SK model has been rigorously studied for $\beta < 1$, first with expansion techniques [1], and then via a stochastic calculus technique [2] and finally by using concentration inequalities [11]. In [11] the (much more complex) model with an external magnetic field is treated. Our model is motivated by spin glass systems in which there are two different types of interactions: one of short range, ferromagnetic and translation invariant, and the other one of long range type with strongly oscillating (random) couplings $J_{i,j}$. When the range of the J–interaction tends to infinity (Kac asymptotics), the system is described by a model with partition function (1.3), as remarked in [4,Theorem B] for the case *without* short range interactions. Observe that the two terms in the Hamiltonian are competing: for typical realizations of $J_{i,j}$, the minimal energy configurations are mixtures of ± 1 spins. Independently of the boundary conditions, the pressure of the system is given by the limit $N \to \infty$ of (1.4), the existence of which is part of our result.

The underlying Ising model plays a crucial role: in the uniqueness region we denote by $\mathbf{P} = \mathbf{P}_\kappa$ [resp., $\mathbf{E} = \mathbf{E}_\kappa$] the Gibbs measure [5] on $\{-1, +1\}^{\mathbb{Z}^d}$ with 2–body interaction K and inverse temperature κ [resp., the corresponding expectation], and we will use the notation $\mathbb{P}$ [resp. $\mathbb{E}$] for the random interactions J. An important role will be played by the so called *bubble diagram* (see e.g. [3]), i.e. the positive quantity $\theta_\kappa > 0$ defined by (1.5), as well as the quantities c_κ (the correlations), χ_κ (the susceptibility), both defined in (1.6), and κ_c, which is the smallest value of κ such that $\chi_\kappa = \infty$. It is well known that $c_\kappa(i)$ is non–negative and increasing in κ. If $(\kappa, \beta) \in D$ (cf. (1.7)) the random variable $\tilde{Z}_N / \mathbb{E}\tilde{Z}_N$ is uniformly bounded in L^2 with respect to N. We have the following:

Theorem 2.1.

i) The domain D contains the parameter values $\{\kappa < \kappa_c, \beta^2 \chi_\kappa < 1\}$.

ii) In the domain D,

$$\lim_{N \to \infty} \frac{\mathbf{E}_\kappa \left[\exp\left(-\beta H_N^{SK}\right)\right]}{\mathbb{E}\mathbf{E}_\kappa \left[\exp\left(-\beta H_N^{SK}\right)\right]} = \exp\left(Y - \frac{1}{2}\phi(\beta^2\theta_\kappa^2)\right) \tag{2.1}$$

in distribution, with $Y \overset{\mathcal{D}}{=} \mathcal{N}(0, \phi(\beta^2\theta_\kappa^2))$ and $\phi(t) = (1/2)\log(1/1-t)$.

iii) If $(\kappa, \beta) \in \overline{D}$, the limit $\tilde{p}(\kappa, \beta)$ of $\tilde{p}_N(\kappa, \beta)$ as $N \to \infty$ exists $\mathbb{P}$–a.s., and it is given by

$$\tilde{p}(\kappa, \beta) = p^I(\kappa) + \frac{\beta^2}{4}, \tag{2.2}$$

where $p^I(\kappa) = \lim_{N\to\infty}(|\Lambda_N|)^{-1}\log \tilde{Z}_N(\kappa, 0)$ *is the pressure of the Ising model.*

Our proof follows the stochastic calculus method introduced in [2]. A crucial element in our proofs are the properties of the Ising model with short range ferromagnetic interactions: besides the FKG inequality, we will use the fact that the moments of such an Ising field are bounded by those of a Gaussian field with the same covariances [7].

We observe also that the concentration method [11] gives the the statement iii), but it does not identify the limit fluctuation behavior in ii). On the other hand the method in [1], succesfully applied to the case $\kappa = 0$ where the underlying measure is of the product form, seems to be hard to adapt to our case. The computations in Section 4 and Section 5 of [2] still hold and, for example, we have convergence in law of the average energy

$$\mathcal{D}\text{–}\lim_{N\to\infty}\left[\frac{\mathbf{E}_\kappa\left(-H_N^{SK}\exp(-\beta H_N^{SK})\right)}{\mathbf{E}_\kappa\left(\exp(-\beta H_N^{SK})\right)} - \frac{\beta}{2}|\Lambda_N|\right] = \mathcal{N}(\beta v_\kappa, v_\kappa)$$

under the conditions stated above,with $v_\kappa = \theta_\kappa^2/2(1-\beta^2\theta_\kappa^2)^2$. We have also a similar result on the entropy of the Gibbs measure with random interactions with respect to $\mathbf{P}_\kappa$, namely the entropy can be written as $|\Lambda_N|\beta^2/4 + U_N$ and U_N converges in law to a (non centered) Gaussian variable.

2.2 Preliminary results.

We discuss in this Section some results about products of independent realizations of Ising spins with $\kappa < \kappa_c$, which are needed to study the random model later on. We fix such a κ from now on, so we will forget the subscripts κ in our notations.

Since $\chi = \sum_i c(i) < \infty$ in D it follows from Newman's Central Limit Theorem for stationary fields of positively associated variables (see Proposition 4 together with Theorem 3 in [8]) that:

Lemma 2.2. *If* η, η', η'' *are independent Gaussian variables with common variance* θ*, we have for finite* χ

$$\mathcal{D}\text{–}\lim_{N\to\infty}\left(X_N(\sigma,\sigma'), X_N(\sigma,\sigma''), X_N(\sigma',\sigma'')\right) = (\eta,\eta',\eta''), \tag{2.3}$$

Recall now the Gaussian inequality for the Ising model with ferromagnetic two–body interaction, proven by C. Newman [7,Th.3]:

Proposition 2.3. *Let $\xi \in \mathbb{R}^{\mathbb{Z}^d}$ be a collection of centered random variables which are jointly Gaussian under the measure $\mathcal{P}$ ($\mathcal{E}$ denotes the expectation), with covariance $\mathcal{E}(\xi_i \xi_j) = c(i-j)$ for all $i, j \in \mathbb{Z}^d$. Then, for any integer $k > 0$ and any $\{i_q\}_{q=1,\dots,2k}$, $i_q \in \mathbb{Z}^d$, we have that*

$$\mathbf{E}\left\{\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_{2k}}\right\} \le \mathcal{E}\left\{\xi_{i_1}\xi_{i_2}\dots\xi_{i_{2k}}\right\}. \tag{2.4}$$

Proof of Theorem 2.1, i): We will prove that $\mathbf{E}^{\otimes 2}\exp(tX_N^2/2)) \le \exp\phi(t\chi)$ for $t\chi < 1$. For any positive integer k we have

$$\begin{aligned}\mathbf{E}^{\otimes 2}\left\{X_N^{2k}\right\} &= \frac{1}{|\Lambda_N|^k}\sum_{i_1\in\Lambda_N}\cdots\sum_{i_{2k}\in\Lambda_N}\mathbf{E}\left\{\sigma_{i_1}\dots\sigma_{i_{2k}}\right\}\mathbf{E}\left\{\sigma'_{i_1}\dots\sigma'_{i_{2k}}\right\}\\ &\le \mathbf{E}\mathcal{E}\left\{\left[\frac{1}{\sqrt{|\Lambda_N|}}\sum_{i\in\Lambda_N}\sigma_i\xi'_i\right]^{2k}\right\}\end{aligned} \tag{2.5}$$

where we have used the first GKS inequality [9] $\mathbf{E}\{\sigma_{i_1}\dots\sigma_{i_{2k}}\} \ge 0$, applied Proposition 2.3 to the σ'–variables on the right–hand side of (2.5) and re-summed the expression. Therefore, expanding the exponential in power series we get, for $t\chi < 1$

$$\begin{aligned}\mathbf{E}^{\otimes 2}\left(\exp\left(\frac{t}{2}X_N^2\right)\right) &\le \mathbf{E}\mathcal{E}\left(\exp(t/2)\left[\frac{1}{\sqrt{|\Lambda_N|}}\sum_{i\in\Lambda_N}\sigma_i\xi'_i\right]^2\right)\\ &= \mathbf{E}\left(\exp\phi\left(t|\Lambda_N|^{-1}\sum_{i,j\in\Lambda_N}c(i-j)\sigma_i\sigma_j\right)\right)\\ &\le \exp\phi(t\chi),\end{aligned} \tag{2.6}$$

In (2.6) we used the fact that, for all fixed σ, $|\Lambda_N|^{-1/2}\sum_{i\in\Lambda_N}\sigma_i\xi'_i$ has a normal distribution with mean 0 and variance $|\Lambda_N|^{-1}\sum_{i,j\in\Lambda_N}c(i-j)\sigma_i\sigma_j$ to get the first equality and then that the maximum value of the variance is achieved when $\sigma_i = 1\ \forall i$ and that $|\Lambda_N|^{-1}\sum_{i,j\in\Lambda_N}c(i-j)\le\chi$. $\square$

2.3 Martingale embedding and convergence.

Following [2] we introduce a collection of independent, standard Brownian motions $\{B_{i,j}(t)\}_{i,j\in\mathbb{Z}^d}$, $t\ge 0$, defined on some probability space $(\Omega, A, \mathbb{P})$, and we consider the following positive, mean 1 martingales

$$e_N(t,\sigma) = \exp\left\{\frac{1}{\sqrt{2|\Lambda_N|}}\sum_{i,j\in\Lambda_N}B_{i,j}(t)\sigma_i\sigma_j - \frac{t}{4}|\Lambda_N|\right\} \quad , \quad Z_N(t) = \mathbf{E}[e_N(t,\sigma)] \ . \tag{2.7}$$

Note that

$$Z_N(\beta^2) \overset{\mathcal{D}}{=} \frac{\mathbf{E}\left[\exp\left(-\beta H_N^{SK}\right)\right]}{\mathbb{E}\mathbf{E}\left[\exp\left(-\beta H_N^{SK}\right)\right]}, \tag{2.8}$$

and observe that by definition we have $\sup_N \mathbb{E} Z_N(t)^2 < \infty$ if $(\kappa, t^{1/2}) \in D$. Set $\mathbf{E}_{N,t}[\,\cdot\,] = \mathbf{E}_{N,t,\kappa}[\,\cdot\,] = \mathbf{E}[\,\cdot\, e_N(t,\sigma)]/Z_N(t)$: this is the expectation for our system (1.3). A key lemma (analogue to Lemma 3.1 in [2]) is

Lemma 2.4. *For $(\kappa, t^{1/2}) \in D, \varepsilon > 0$ with $\sup_N \mathbf{E}_\kappa^{\otimes 2} \exp \frac{\beta^2+\varepsilon}{2} X_N(\sigma,\sigma')^2 < \infty$, we have as $N \to \infty$*

$$Z_N(t)^2 \left[\mathbf{E}_{N,t}^{\otimes 2} F(X_N(\sigma,\sigma')) - \mathbb{E}_\eta F(\eta)\right] \to 0 \quad \text{in } L^1(\Omega) \tag{2.9}$$

for any continuous function F such that $F(x) = o(\exp(\delta x^2/2))$ at $x = \pm\infty$ with some $0 < \delta < \varepsilon$. Here η denotes a mean 0 Gaussian variable with variance $\theta^2/(1-t\theta^2)$ and $\mathbb{E}_\eta$ the expectation in η.

Proof: The proof follows the line of the proof of Lemma 3.1 in [2], and it reduces to check (2.9) for $F(x) = G(x)\exp\{-tx^2/2\}$ with G bounded and such that $\mathbb{E}_\eta G(\eta) = 0$. This in turn is implied by the convergence to 0 of $w_N = \mathbf{E}^{\otimes 3}\{e^{\frac{t}{2}X_N(\sigma',\sigma'')^2} G(X_N(\sigma,\sigma'))G(X_N(\sigma,\sigma''))\}$. But the central limit (Lemma 2.2) and uniform integrability for $(\kappa, \sqrt{t}) \in D$ – by definition of D – imply that $\lim_N w_N = \mathbb{E}_\eta^{\otimes 3}\left[G(\eta'')G(\eta\prime)\exp(t\eta^2/2)\right] = 0$. $\square$

Define the logarithmic martingale of Z_N as the stochastic integral $M_N(t) = \int_0^t Z_N(s)^{-1} \mathrm{d}Z_N(s)$ This new process is itself a mean zero, L^2 martingale with quadratic variation

$$\langle\, M_N \,\rangle(t) = \frac{1}{2}\int_0^t \mathbf{E}_{N,s}^{\otimes 2}\left(X_N(\sigma,\sigma')^2\right) \mathrm{d}s \tag{2.10}$$

Lemma 2.4 with $F(x) = x^2$ implies that $V_N(t) = Z_N(t)^2 \frac{d}{dt}\left[\langle\, M_N \,\rangle(t) - \phi(t\theta^2)\right]$ tends to 0 in L^1 and is bounded in the L^1 norm uniformly for $N \geq 1$ and t in compact subsets of $[0, t_\kappa)$, where $t_\kappa := \sup\{t : (\kappa, \sqrt{t}) \in D\}$. Using the arguments in the proof of Proposition 3.2 in [2] it follows that

$$\sup_{s\in[0,t]} |\langle\, M_N \,\rangle(s) - \phi(s\theta^2)| \to 0 \qquad \text{in probability as } N \to \infty \tag{2.11}$$

for $t < t_\kappa$. Since ϕ is deterministic in this limit, it follows that the sequence M_N converges to a Gaussian process M_∞ on the time interval $[0, t_\kappa)$. We get finally a process convergence as stated below, which implies *ii*) in Theorem 2.1.

Theorem 2.5. *When $\kappa < \kappa_c$ the sequence of martingales M_N converges as $N \to \infty$ in distribution to the independent increments Gaussian process M_∞ on $[0, t_\kappa)$, with $M_\infty(0) = 0$, mean 0 and variance $\mathbb{E}[M_\infty(t)^2] = \phi(t\theta^2)$. Also the random process $(Z_N(t), t \in [0, t_\kappa))$ converges in law to*

$$\exp\{M_\infty(t) - \phi(t\theta^2)/2\} \ .$$

Proof of Theorem 2.1, iii): Using the assumption $\sum_i K(i) < \infty$, standard estimates yields

$$\frac{1}{|\Lambda_N|} \log \mathbf{E}\left(e^{-\beta H_N^{SK}}\right) = \frac{1}{|\Lambda_N|} \log \tilde{Z}_N(\kappa, \beta) - \frac{1}{|\Lambda_N|} \log \tilde{Z}_N(\kappa, 0) + o(1) \tag{2.12}$$

with $o(1) \to 0$ uniformly on Ω as $N \to \infty$. Note that $\lim_{N\to\infty}(1/|\Lambda_N|) \log \tilde{Z}_N(\kappa, 0) = p^I(\kappa)$ and that by Theorem 2.1, point *ii*), and a straightforward annealed computation

$$\mathcal{D}\text{–}\lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \tilde{Z}_N(\kappa, \beta) = \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{E}\left[\tilde{Z}_N(\kappa, \beta)\right] = \frac{\beta^2}{4} \tag{2.13}$$

when $(\kappa, \beta) \in D$. This ensures convergence in probability of the pressure. Almost sure convergence follows from the standard Gaussian concentration inequality (see e.g. [11]):

$$\mathbb{P}\left(\left|\frac{1}{|\Lambda_N|} \log \tilde{Z}_N(\kappa, \beta) - \mathbb{E}\left(\frac{1}{|\Lambda_N|} \log \tilde{Z}_N(\kappa, \beta)\right)\right| \geq \delta\right) \leq 2 \exp\left(-\frac{|\Lambda_N|\delta^2}{\beta^2}\right) \tag{2.14}$$

for $\delta \geq 0$. Notice now that $\{\mathbb{E}((1/|\Lambda_N|) \log \tilde{Z}_N(\cdot, \cdot))\}_{N\in\mathbb{N}}$ is a family of convex functions, uniformly bounded over compact sets. It is hence a compact family, in the topology of uniform convergence over compact sets and from this we conclude that the result obtained for $(\kappa, \beta) \in D$ can be extended by continuity to $\overline{D}$. $\square$

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