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Full Color Theorems for L(2,1)-Colorings

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ABSTRACT

The span $\lambda(G)$ of a graph G is the smallest k for which G's vertices can be $L(2,1)$-colored, i.e., colored with integers in $\{0,1,\ldots,k\}$ so that adjacent vertices' colors differ by at least two, and colors of vertices at distance two differ. G is full-colorable if some such coloring uses all colors in $\{0,1,\ldots,\lambda(G)\}$ and no others. We prove that all trees except stars are full-colorable. The connected graph G with the smallest number of vertices exceeding $\lambda(G)$ that is not full-colorable is C_6. We describe an array of other connected graphs that are not full-colorable and go into detail on full-colorability of graphs of maximum degree four or less.

1 INTRODUCTION

Ordinary colorings of graphs that assign different integers to adjacent vertices have the property that the fewest colors needed can always lie in an interval all of whose integers are assigned to some vertex. The present paper considers special integer colorings called $L(2,1)$-colorings whose shortest interval containing all colors may necessarily have integers assigned to no vertex. The paper identifies a variety of graphs with this feature along with conditions which imply that every color in a shortest interval can be assigned to some vertex.

$L(2,1)$-colorings assign integers to vertices so that adjacent vertices' colors differ by at least 2 and the colors of any two nonadjacent vertices that are adjacent to a third vertex differ by at least 1. $L(2,1)$-colorings were first investigated extensively by Yeh [10] and Griggs and Yeh [4] as a generalization of T-colorings for the channel assignment problem that were introduced by Hale [5]: see also Roberts [7]. K_2 is the smallest graph with the feature that every interval which contains the integers of an $L(2,1)$-coloring also contains an integer assigned to no vertex.

The rest of this introduction specifies our notation and assumptions, notes prior results for $L(2,1)$-colorings and outlines our new results.

Let $\mathcal{G}^*$ be the set of finite simple graphs. An $L(2,1)$-*coloring* of $G = (V,E)$ in $\mathcal{G}^*$ is a vertex coloring $f : V \to \mathbb{Z}$ for which

$$\begin{aligned}
|f(u) - f(v)| &\geq 2 \quad \text{for all } \{u,v\} \in E, \\
|f(u) - f(v)| &\geq 1 \quad \text{whenever } u \neq v,\ \{u,v\} \notin E, \text{ and } \{u,t\},\ \{t,v\} \in E \\
&\qquad\ \ \text{for some } t \in V.
\end{aligned}$$

The *span* $\lambda = \lambda(G)$ of $G \in \mathcal{G}^*$ is the smallest $k \geq 2$ for which G has an $L(2,1)$-coloring $f : V \to \{0,1,\ldots,k\}$. A *span coloring* is an $L(2,1)$-coloring $f : V \to \{0,1,\ldots,\lambda\}$. We say that G is *full-colorable*, or *FC* for short, if some span coloring has $f(V) = \{0,1,\ldots,\lambda\}$, i.e., so that all colors in $\{0,1,\ldots,\lambda\}$ are used by f, and call such an f a *full coloring*. On the other hand, G is *NFC* if it is not full-colorable.

$L(2,1)$-colorings into $\{0,1,\ldots,\lambda\}$ have natural duals. If f is an $L(2,1)$-coloring, its *dual* g is defined by $g(x) = \lambda - f(x)$. Since span colorings must use color 0, g is a span coloring if and only if f is, and g is full if and only if f is.

Henceforth, f denotes a span coloring, and all colorings considered are span colorings. As usual, P_n, C_n, K_n and $K_{n,m}$ denote the n-vertex path, n-vertex cycle, complete n-vertex graph, and complete bipartite graph with part sizes n and m, respectively. Every K_n, $n \geq 2$, is NFC with $\lambda = 2(n-1)$ and $f(V) = \{0,2,4,\ldots,2(n-1)\}$. Moreover [4, 10],

$$\begin{aligned}
&\lambda(P_2) = 2,\ \lambda(P_3) = \lambda(P_4) = 3,\ \lambda(P_n) = 4 \quad \text{for} \quad n \geq 5\ ; \\
&\lambda(C_n) = 4 \quad \text{for all} \quad n \geq 3\ .
\end{aligned}$$

Although $K_2 = P_2$, K_3 and P_3 are NFC, the two-component graphs $K_2 + K_3$ and $P_2 + P_3$ are FC, as verified by $(f(K_2), f(K_3)) = (\{1,3\},\{0,2,4\})$ and $(f(P_2), f(P_3)) = (\{0,2\},\{0,1,3\})$.

Full colorability is impossible when $|V| \leq \lambda$ because $|\{0,1,\ldots,\lambda\}| = \lambda + 1$. We will note a variety of graphs that have $|V| \leq \lambda$, but will be more concerned with NFC graphs for which $|V| \geq \lambda + 1$. Aleksandar Pekeč [personal communication, 08 February, 2000] has observed that "almost all" graphs have $\lambda = |V| - 1$ and have all span colorings full. That is, if we choose a random graph $G_{n,.5}$ (a graph with n vertices and edges independently chosen with probability 1/2), then the probability that $\lambda = n - 1$ and the probability that all span colorings are full both approach 1 as n approaches ∞. In fact, this is true for a random graph $G_{n,p(n)}$ as long as $np(n)^2 - 2\log n$, $n^2[1 - p(n)]$, and $n[1 - p(n)] - \log n - \log\log n$ approach ∞ as $n \to \infty$.

We confine further attention to connected graphs and will base much of our analysis on $\Delta = \Delta(G)$, the *maximum degree* of a vertex in G. Accordingly, let

$$\begin{aligned} \mathcal{G} &= \{G \in \mathcal{G}^* : \ G \text{ is connected}\} \\ \mathcal{G}_\Delta &= \{G \in \mathcal{G} : \ G \text{ has maximum degree } \Delta\} \\ \mathcal{G}_\Delta(\lambda) &= \{G \in \mathcal{G}_\Delta : \ G \text{ has span } \lambda\} \text{ for } \lambda \geq \Delta + 1\ , \end{aligned}$$

where $\lambda \geq \Delta + 1$ is presumed because $\mathcal{G}_\Delta(\lambda)$ is empty if $\lambda \leq \Delta$. Also let $\mathcal{T}$, $\mathcal{T}_\Delta$ and $\mathcal{T}_\Delta(\lambda)$ be the subsets of trees in $\mathcal{G}$, $\mathcal{G}_\Delta$ and $\mathcal{G}_\Delta(\lambda)$, respectively. The following lemma and theorem from [4] involve the smallest possible span for Δ.

Lemma 1.1 *If $G \in \mathcal{G}_\Delta(\Delta+1)$, then $f(u) \in \{0, \Delta+1\}$ for every vertex u of degree Δ in G.*

Theorem 1.2 *If $T \in \mathcal{T}_\Delta$ then $\lambda(T) \in \{\Delta+1, \Delta+2\}$.*

Thus, every tree with maximum degree Δ has span $\Delta + 1$ or $\Delta + 2$. Chang and Kuo [1] present a polynomial time algorithm that determines $\lambda(T)$ for $T \in \mathcal{T}$. On the other hand, Griggs and Yeh [4] prove that determination of $\lambda(G)$ for $G \in \mathcal{G}$ is NP-complete. It is also known [1] that $\lambda(G) \leq \Delta^2 + \Delta$ for $G \in \mathcal{G}_\Delta$, but the exact value of $\max\{\lambda(G) : G \in \mathcal{G}_\Delta\}$ is unknown for most Δ. Tighter bounds on λ for special families of graphs in $\mathcal{G}_\Delta$ are established in [1, 2, 3, 4, 6, 8, 9, 10].

Henceforth let $n = |V|$. An outline of our results for $n \geq \lambda + 1$ follows. We will also identify graphs in $\mathcal{G}_\Delta$ with $n \leq \lambda$.

We show first in Section 2 that the only NFC graph in $\mathcal{G}_2$ with $n \geq \lambda + 1$ is C_6, where $f(C_6) = \{0, 2, 4\}$. This is followed in Section 3 by the observation that all trees in $\mathcal{T}_\Delta$ with $\Delta \geq 3$ and $n \geq \lambda + 1$ are FC.

We then focus on $\mathcal{C}_\Delta(\lambda)$, the set of connected non-tree graphs with maximum degree Δ and span λ, for $\Delta \geq 3$ and $\lambda \geq \Delta + 1$. A complete account of full coloring is given in Section 4 for $\mathcal{C}_3(4)$. It has exactly three NFC graphs with $n \geq 5$. Two involve C_6 with appended vertices, and the third is C_9 with three appended vertices equally spaced around C_9. We also identify every graph in $\mathcal{C}_3(4)$ that has both a full coloring and a non-full coloring.

Partial results are provided for $\mathcal{C}_4(5)$ and $\mathcal{C}_3(5)$ in Sections 5 and 6. The only NFC graph in $\mathcal{C}_4(5)$ with $6 \leq n \leq 8$ is an 8-vertex graph. $\mathcal{C}_4(5)$ contains at least six NFC graphs with $n = 9$, and at least 18 with $n = 10$. $\mathcal{C}_3(5)$ also has no NFC graph for $n = 6$ and at least one for $n = 8$, three for $n = 9$, and two for $n = 10$. Moreover, $\mathcal{C}_3(5)$ contains an infinite number of NFC graphs.

Our final two results, given in Section 7, involve NFC graphs for larger values of Δ and λ. The first says that $\mathcal{C}_\Delta(\lambda)$ has an NFC graph with $n \geq \lambda + 1$ whenever $\Delta \geq 3$ and $\Delta + 1 \leq \lambda \leq 2\Delta - 1$. The second says that every $\mathcal{C}_\Delta(\Delta + 2)$ for $n \geq 3$ has an NFC graph with $n \geq \Delta + 3$ all of whose colorings omit two colors in $\{0, 1, \ldots, \Delta + 2\}$.

Section 8 describes some open problems.

2 MAXIMUM DEGREE 1 or 2

Because $\mathcal{G}_1 = \{P_2\}$ with $\lambda(P_2) = 2$ and $f(P_2) = \{0, 2\}$, the only graph in $\mathcal{G}_1$ is NFC. The following theorem addresses $\Delta = 2$.

Theorem 2.1 *$\mathcal{G}_2 = \{P_3, P_4, \ldots\} \cup \{C_3, C_4, \ldots\}$ with $\lambda(P_3) = \lambda(P_4) = 3$ and $\lambda(G) = 4$ for all other $G \in \mathcal{G}_2$. All graphs in $\mathcal{G}_2$ are FC except for P_3, C_3, C_4 and C_6. Moreover, $f(C_6) = \{0, 2, 4\}$.*

Proof. It is easily seen that $\mathcal{G}_2 = \{P_3, P_4, \ldots\} \cup \{C_3, C_4, \ldots\}$. The spans of these graphs were noted earlier [4]. The members of $\mathcal{G}_2$ with $n \leq \lambda$ are P_3, C_3 and C_4, so they are NFC.

Suppose $G = P_4$ with $\lambda(P_4) = 3$. The $L(2,1)$-coloring 1302 of successive vertices shows that P_4 is FC. When $G = P_n$ for $n \geq 5$ with $\lambda(P_n) = 4$, the first n terms of $3024130241\ldots$ taken as successive vertex colors show that P_n is FC.

Suppose $G = C_6$. If a span coloring of C_6 uses color 1, then five successive vertex colors around C_6 must be 41302, and no color from $\{0, 1, 2, 3, 4\}$ for the sixth vertex is admissible for an $L(2,1)$-coloring. Hence $1 \notin f(C_6)$. Similarly (replace c by $4 - c$), $3 \notin f(C_6)$, and the successive colors around C_6 are 024024 or 042042.

Successive vertex colors 02413 for C_5 show that it is FC. The following successive vertex colors for the C_n with $n \geq 7$ show that they are FC:

$$\begin{aligned} n &\equiv 0 \pmod 3: \quad 024\ldots024130413\ ; \\ n &\equiv 1 \pmod 3: \quad 024\ldots0240314\ ; \\ n &\equiv 2 \pmod 3: \quad 024\ldots02413\ . \quad \blacksquare \end{aligned}$$

The only NFC graph in $\mathcal{G}_2$ with $n \geq \lambda + 1$ is C_6, where $f(C_6)$ contains neither 1 nor 3.

3 TREES WITH $\Delta \geq 3$

We assume henceforth that $\Delta \geq 3$. Our next theorem accounts for all trees that are not paths.

Theorem 3.1 *The only NFC trees in $\mathcal{T}$ with $\Delta \geq 3$ are the $K_{1,\Delta}$ for $\Delta = 3, 4, \ldots$.*

Proof: Let T be a tree with $\Delta \geq 3$. Given Δ, the tree T with the smallest number of vertices is $K_{1,\Delta}$. It has $f(K_{1,\Delta}) \in \{\{0, 2, 3, \ldots, \Delta + 1\},\ \{0, 1, \ldots, \Delta - 1, \Delta + 1\}\}$, so $K_{1,\Delta}$ is NFC.

We refer to the span coloring of star $K_{1,\Delta}$ that assigns 0 to its center and $2, 3, \ldots, \Delta + 1$ to its degree-1 vertices as its *base-coloring.*

Suppose T with maximum degree Δ and n vertices has $n \in \{\Delta + 2, \Delta + 3\}$. Then T consists of $K_{1,\Delta}$ and one or two other vertices, say x or $\{x, y\}$. We can assume without loss of generality that x is adjacent to the vertex in the base-coloring of $K_{1,\Delta}$ with color $\Delta + 1$. If $n = \Delta + 2$, $f(x) = 1$ shows that T is FC. Suppose $n = \Delta + 3$. Taking $f(x) = 1$ as before, we note that y is adjacent to exactly one of the vertices colored 1, $\Delta + 1$, or, without loss of generality, Δ. We then take $f(y)$ as Δ, 2, or 1, respectively, to obtain a full $L(2, 1)$-coloring with $\lambda(T) = \Delta + 1$.

Assume henceforth that $n \geq \Delta + 4$. By Theorem 1.2, $\lambda(T) \in \{\Delta + 1, \Delta + 2\}$. We deal with $\lambda(T) = \Delta + 2$ and then consider $\lambda(T) = \Delta + 1$.

Suppose $\lambda(T) = \Delta + 2$. We modify the first-fit greedy algorithm in [4] to show that T is FC. We will begin with a full coloring of $\Delta+3$ vertices $v_1, \ldots, v_{\Delta+3}$. There are three ways that $v_1, \ldots, v_{\Delta+3}$ might be chosen, but in each case the tree structure admits an ordering $v_1, \ldots, v_n$ of all vertices so that v_j for $1 < j \leq n$ is adjacent to exactly one vertex in $\{v_1, \ldots, v_{j-1}\}$. Once the first $\Delta + 3$ vertices have been colored, we proceed greedily by successively labeling $v_{\Delta+4}, \ldots, v_n$ with the smallest admissible color in $\{0, 1, \ldots, \Delta + 2\}$. Because each such v_j is adjacent to only one v_i for $i < j$ and that v_i is adjacent to at most $\Delta - 1$ others, at most $\Delta + 2$ of the $\Delta + 3$ colors are forbidden for v_j, so the greedy procedure completes an $L(2, 1)$-coloring.

Let $\{v_1, \ldots, v_{\Delta+1}\}$ be the vertex set of a $K_{1,\Delta}$ with center v_1. The choice of $v_{\Delta+2}$ and $v_{\Delta+3}$ depends on adjacencies to v_2 through $v_{\Delta+1}$. Three exhaustive cases are denoted as follows by (a), (b) and (c).

(a) Suppose a neighbor x of v_1 has degree 3 or more. Denote two non-v_1 neighbors of x by $v_{\Delta+2}$ and $v_{\Delta+3}$. Take $f(v_1) = 0$, $f(x) = \Delta + 2$, $f(v_{\Delta+2}) = 1$, $f(v_{\Delta+3}) = 2$, and use colors 3 through $\Delta + 1$ for the other neighbors of v_1. Then proceed greedily.

(b) Suppose (a) doesn't apply but there are degree-2 neighbors x and y of v_1. Let $v_{\Delta+2}$ and $v_{\Delta+3}$ be non-v_1 neighbors of x and y, respectively. Take $f(v_1) = 0$, $f(x) = \Delta + 1$, $f(v_{\Delta+2}) = 1$, $f(y) = \Delta + 2$, $f(v_{\Delta+3}) = 2$, and use colors 3 through Δ for the other neighbors of v_1. Then proceed greedily.

(c) Suppose neither (a) nor (b) applies. Then all but one of v_2 through $v_{\Delta+1}$ are terminal vertices, and the other, say v_2 for definiteness, has degree 2. Let $v_{\Delta+2}$ be the other vertex adjacent to v_2, and let $v_{\Delta+3}$ be a new vertex adjacent to $v_{\Delta+2}$. Take $f(v_1) = 0$, $f(v_2) = 2$, $f(v_{\Delta+2}) = \Delta + 2$, $f(v_{\Delta+3}) = 1$, and use colors 3 through $\Delta + 1$ for the terminal neighbors of v_1. Then proceed greedily.

Because (a)–(c) exhaust the possibilities and each produces an $L(2, 1)$-coloring that uses every color in $\{0, 1, \ldots, \Delta + 2\}$, the proof for $\Delta + 2$ is complete.

Suppose henceforth that $\lambda(T) = \Delta + 1$. By Lemma 1.1, every induced $K_{1,\Delta}$ in T for a span coloring has either the base-coloring or its dual in which the center has color $\Delta + 1$ and its neighbors have colors $0, 1, \ldots, \Delta - 1$.

Suppose T has only one degree-Δ vertex, i.e., only one induced $K_{1,\Delta}$ with center v_1 and neighbors $v_2, \ldots, v_{\Delta+1}$. Slight modifications in the preceding (a)–(c) procedure show that

T is FC. The greedy procedure for $v_{\Delta+4}$ through v_n works in this case with all colors in $\{0, 1, \ldots, \Delta+1\}$ because every vertex other than v_1 has degree $\Delta - 1$ or less. Details follow.

(a) Given the hypotheses of (a), take $f(v_1) = 0$, $f(x) = \Delta+1$, $f(v_{\Delta+2}) = 1$, $f(v_{\Delta+3}) = 2$, use colors 2 through Δ for the other neighbors of v_1, and proceed greedily. Here (a) implies that $\Delta \geq 4$.

(b) Given the hypotheses of (b), take $f(v_1) = 0$, $f(x) = \Delta$, $f(v_{\Delta+2}) = 1$, $f(y) = \Delta + 1$, $f(v_{\Delta+3}) = 2$, use colors 2 through $\Delta - 1$ for the other neighbors of v_1, and proceed greedily.

(c) Given the hypotheses of (c), take $f(v_1) = 0$, $f(v_2) = 2$, $f(v_{\Delta+2}) = \Delta+1$, $f(v_{\Delta+3}) = 1$, use colors 3 through $\Delta + 1$ for the terminal neighbors of v_1, and proceed greedily.

Assume henceforth that T has more than one degree-Δ vertex. Let M denote the set of degree-Δ vertices, call distinct x and y in M *M-neighbors* if the unique path between them has no interior vertex in M, let s denote a distance between M-neighbors, and let s^* be the minimum such s. If $s^* \in \{1, 2\}$ then T is FC because, with $x, y \in M$ at distance s^*, Lemma 1.1 implies that in any $L(2,1)$-coloring, one of x and y is colored 0 and the other is colored $\Delta + 1$.

Assume henceforth for $\lambda(T) = \Delta + 1$ that $s^* \geq 3$.

Suppose $\Delta = 3$. Fix $x \in M$ and assume with no loss of generality that x's $K_{1,3}$ is base-colored, so colors 0, 2, 3, 4 are used. The following path colorings are among the feasible colorings for paths from x to an M-neighbor of x at distance s from x:

$$\begin{aligned}
s = 3 &: \quad 0240, 0314, 0420 \\
s = 4 &: \quad 03140, 04130 \\
s = 5 &: \quad 024130, 031420, 041304 \\
s = 6 &: \quad 0241304, 0314204, 0413024 \\
s = 7 &: \quad 02413024, 03140240, 04130240\ .
\end{aligned}$$

The patterns for $s \in \{5, 6, 7\}$ extend three terms at a time to ensure a 1 in every path that begins 02, 03 or 04, has $s \geq 8$ terms, and ends in 0 or 4. The duals of the patterns apply if other members of M use the dual base-coloring. The *only* feasible path coloring from x for $s = 4$ not shown above is 04204, whose final 4 signifies the dual base-coloring for its $K_{1,3}$. Consequently, if any $y \in M$ has an M-neighbor at distance 4, color 1 must be used either through one of the $s = 4$ path colorings shown or because it appears in the dual base coloring with center 4. Hence, T is FC. If distance 4 never occurs between M-neighbors, we can color outward ¿from x through T using the above path colors to ensure that color 1 is in T and hence that T is FC.

Suppose henceforth for $\lambda(T) = \Delta + 1$ and $s^* \geq 3$ that $\Delta \geq 4$. With $b = \Delta + 1 \geq 5$, feasible paths from 0-colored $x \in M$ to an M-neighbor of x at distance s from x include the following:

$$\begin{aligned}
s = 3 &: \quad 0240, 031b, \ldots, 0\Delta 1b, 0b1\Delta \\
s = 4 &: \quad 0241b, 03140, 04130, \ldots, 0b130 \\
s = 5 &: \quad 024130, 031420, 04130b, \ldots, 0b130b
\end{aligned}$$

with three-term continuations for $s \geq 6$. Unlike $\Delta = 3$, there is always a feasible path from a member of M labeled 0 (or b) to an M-neighbor that begins $02, 03, \ldots,$ or $0b$. For $\Delta \geq 4$, we begin with base-colored $K_{1,\Delta}$ with center x that is distance s^* from some M-neighbor and color outward from x through T using the above path colors. In doing this we always go first from a newly-colored member of M along a shortest uncolored path to another member of M. Intermediate vertices along such a path can branch off to other as-yet-uncolored members of M, but since the branch vertex has degree $\Delta - 1$ or less there will always be an available color for the first vertex along that branch. Completion away from a newly-colored member of M or another vertex along a branch that encounters no more members of M can be done greedily. Because use of the preceding path colorings away from x can always ensure the appearance of 1, T is FC. ■

4 ANALYSIS OF $\mathcal{C}_3(4)$

Because Theorems 2.1 and 3.1 account for all trees with $\Delta \geq 2$, we focus henceforth on connected graphs that are not trees and let

$$\mathcal{C}_\Delta(\lambda) = \mathcal{G}_\Delta(\lambda) \setminus \mathcal{T}_\Delta(\lambda) \quad \text{for} \quad \Delta \geq 3 \quad \text{and} \quad \lambda \geq \Delta + 1 .$$

The only nonempty $\mathcal{C}_\Delta(\lambda)$ with $\Delta \geq 3$ that we have complete results for is $\mathcal{C}_3(4)$.

Theorem 4.1 *Every graph in $\mathcal{C}_3(4)$ is FC except G_1 through G_4 in Figure* 1.

Figure 1 about here

To prove Theorem 4.1, we note that the labels on the vertices of G_1-G_4, giving $L(2,1)$-colorings of span 4, are unique up to duality. This is obvious from $f(C_6) = \{0, 2, 4\}$ in Theorem 2.1 for G_2 and G_3. It follows by contradiction for G_4 if we use Lemma 1.1 and note that if on C_9 a label 1 is used and a vertex of degree 3 gets label 0, then we may assume that the first four vertices clockwise from the top of C_9 are labeled 0314. Thus, G_1-G_4 are in $\mathcal{C}_3(4)$ and are NFC.

We assume that G is connected with $\Delta = 3$ and $\lambda = 4$ and, because it is not a tree, has a cycle. The only G with these properties and $n < 5$ is G_1 of Figure 1 (C_4 with diagonal has $\lambda > 4$), so assume henceforth that $n \geq 5$.

To complete the proof of Theorem 4.1, we show that every $G \in \mathcal{C}_3(4) \setminus \{G_1, G_2, G_3, G_4\}$ is FC. Our proof proceeds through a series of observations to a key lemma, Lemma 4.2, that guides its completion. As usual, all colorings are span colorings.

Step 1. Because $\lambda = 4$ for all cycles [4], every cycle uses colors 0 and 4, so G uses these colors. By Lemma 1.1, every degree-3 vertex has color 0 or 4.

Step 2. Suppose C_m in G is colored without using color 2. Then C_m's successive colors can be assumed to be

$$0314\ 0314\ 0314\ \cdots ,$$

so $m = 4t$ for some $t \geq 1$. Connectedness and $\Delta = 3$ require at least one pendant edge or chord for C_m. A chord gives a contradiction because it would have to join vertices labeled

0 and 3, 0 and 4, or 1 and 4. But in the first case, 0 has a neighbor 3 on the cycle, which is impossible, and similarly for the other two cases. Hence C_m has a pendant vertex which must have color 2 because its edge-mate in C_m is either 0 (with adjacent colors 3 and 4) or 4 (with adjacent colors 0 and 1). Hence G is FC.

Step 3. Suppose C_m in G is colored without using color 1 or, equivalently (by duality), without using color 3. Then C_m's successive colors can be assumed to be

$$024\ 024\ 024\ \cdots\ ,$$

so $m = 3t$ for some $t \geq 1$.

Suppose $t = 1$, so $m = 3$. Then either C_3 has exactly two pendant edges to new vertices, which must have colors 1 and 3, or, since G isn't G_1, C_3 has a P_2 off of a C_3 vertex whose two new vertices must have colors 1 and 3. Thus $m = 3$ implies a full coloring.

Suppose $t \geq 2$ with $m = 3t$. A chord in C_{3t} gives a contradiction, so C_{3t} has at least one pendant edge to a new vertex. Suppose such a pendant vertex x has another neighbor y. If y is not adjacent to any vertex of C_{3t}, then the two new vertices x and y must have colors 1 and 3. If y is adjacent to a vertex of C_{3t}, let u be the vertex of C_{3t} adjacent to x and v the vertex of C_{3t} adjacent to y. A contradiction occurs if u and v receive the same color. Thus, u and v are colored using 0 and 4 and x and y are colored using 1 and 3. In both situations, we have a full coloring.

Step 4. Steps 2 and 3 imply that the only way to avoid a full coloring of G is for G to consist of exactly one C_m with $m = 3t$ and $t \geq 2$ that is colored $024024\cdots$, has no chord, has one or more pendant edges to distinct new vertices, and has no other vertices. Suppose G satisfies these restrictions. Number C_m's vertices successively as $1, 2, 3, \ldots, 3t$ and assume without loss of generality that the corresponding colors are $024024\cdots$ and that vertex 1 has degree 3. The pendant vertex adjacent to vertex 1 must have color 3. If another pendant edge goes off a color-4 vertex of C_{3t}, the pendant vertex there must have color 1 and we get a full coloring. We conclude that a full coloring can be avoided only if all other pendant edges (if any) besides the one off vertex 1 go off C_m vertices colored 0, i.e., vertices numbered 4, 7, 10, ... This brings us to the key lemma.

Lemma 4.2 *Suppose $G \in \mathcal{C}_3(4)$ with $n \geq 5$. Then G is FC unless it has the following structure with feasible coloring as indicated:*

(i). G has exactly one C_m, m must be in $\{3t : t \geq 2\}$, and the successive vertices of C_m are colored $024024\cdots$.

(ii). G has at least one other vertex besides the m of C_m, and each of its other vertices is a terminal vertex adjacent to a color-0 vertex of C_m (and has color 3).

Step 5. Assume that G has the structure described by (i) and (ii) of Lemma 4.2 and that, with the vertices of C_m numbered successively as $1, 2, \ldots, m$, the non-C_m vertices of G for (ii) are adjacent to C_m vertices in $\{1, 4, 7, 10, \ldots, m-2\}$.

Suppose $m = 6$. Then G is G_2 or G_3, and both are NFC.

Suppose $m = 9$. If there are three other vertices besides the nine in C_9, the most allowed by (ii), we get G_4, which is NFC. If G has one or two vertices not in C_9, we recolor C_9's vertices successively as 031420314, arranged so that the one other vertex is adjacent to vertex 1 (the first vertex in the labeling), or the two other vertices are adjacent to vertices 1 and 4. Then G is FC.

Suppose $m \geq 12$. Recolor C_m's vertices successively as

$$031\ 402\ 413\ 024\ 024 \cdots 024 .$$

Then G is FC because all pendant vertices are adjacent to vertices in positions $1, 4, 7, 10, \ldots, m-2$ that have colors in $\{0, 4\}$ and can therefore be colored to give an overall $L(2,1)$-coloring. This completes the proof of Theorem 4.1. ■

Some FC graphs in $\mathcal{C}_3(4)$ must be fully colored by every span coloring, whereas others also have non-full span colorings. The following result notes when the latter possibility arises.

Theorem 4.3 *Suppose $G \in \mathcal{C}_3(4) \setminus \{G_1, G_2, G_3, G_4\}$. Then G has both a full coloring and a non-full coloring if and only if it consists entirely of one C_{3t} for $t \geq 3$ plus one or more pendant edges to terminal vertices adjacent to vertices of C_{3t} that are spaced at distances that are multiples of 3 around the cycle.*

Proof. Assume that $G \in \mathcal{C}_3(4) \setminus \{G_2, G_3, G_4\}$ with $n \geq 5$, and let m denote any integer for which G has a C_m. It follows from Steps 2–4 that every coloring of G must be full unless G has the structure described in Lemma 4.2. In particular, unless $m = 3t$ for $t \geq 2$ and G has only the one cycle, a full coloring is unavoidable. For example, if $m = 8$ and C_8 itself is not full-colored, it can omit only color 2 (Step 2) because omission of color 1 or 3 requires $m = 3t$ (Step 3), and when color 2 is omitted from C_8 it must be assigned to a pendant vertex.

Hence, given the hypotheses of Theorem 4.3, a full coloring is unavoidable whenever G has a C_m with $m < 9$, or G has a C_m with $m > 9$ and m is not divisible by 3. Full colorings are also unavoidable when $m = 3t$ for $t \geq 3$ in the following cases:

(a) C_m has two pendant edges whose two C_m vertices are not distance $3j$ from each other for some $j \geq 1$;

(b) G has a vertex not in C_m that is adjacent to a C_m vertex and to another non-C_m vertex.

As can be seen from Step 5, many G's with a single C_m ($m = 3t, t \geq 3$) have both full and non-full colorings. In these cases, there can be only one omitted color (1 or 3).

The conclusion of Theorem 4.3 follows easily from Lemma 4.2 and the preceding observations. ■

The graph with the smallest number of vertices that satisfies the conclusion of Theorem 4.3 is C_9 plus a pendant edge. If C_9's vertices are colored successively as

$$024024024$$

and the pendant vertex is adjacent to the first of these and it gets color 3, the coloring is non-full. But if C_9's coloring is

$$024130413\ ,$$

and the pendant vertex is adjacent to the first of these and receives color 4, then the coloring is full.

5 ASPECTS OF $\mathcal{C}_4(5)$

The next cases for $\mathcal{C}$ are $\mathcal{C}_4(5)$ with $\lambda = \Delta + 1$, and $\mathcal{C}_3(5)$ with $\lambda = \Delta + 2$. We present partial results for these cases, then give examples of NFC graphs with $n \geq \lambda + 1$ and larger values of Δ and λ. The following theorem indicates that $\mathcal{C}_4(5)$ is already considerably more daunting than $\mathcal{C}_3(4)$.

Theorem 5.1 *$\mathcal{C}_4(5)$ has four graphs with $n = 5$ (and all are NFC). The only NFC graph in $\mathcal{C}_4(5)$ with $6 \leq n \leq 8$ is the 8-vertex graph G_5 in Figure 2. $\mathcal{C}_4(5)$ has at least six NFC graphs with $n = 9$ and at least 18 NFC graphs with $n = 10$.*

Proof. The four graphs with $n = 5$ in $\mathcal{C}_4(5)$ are the nonisomorphic ways that one or more edges can be added between terminal vertices of $K_{1,4}$ that admit $L(2,1)$-colorings of span 5. Figure 2 shows $L(2,1)$-colorings of the other graphs of Theorem 5.1. A dashed line in the figure indicates an optional edge that is $L(2,1)$-compatible but is not needed to show that G is NFC. With and without the optional edges, there are six graphs for $n = 9$ in the figure. The options for $n = 10$ give 3, 3, 7 and 6 nonisomorphic graphs for I, II, III and IV, respectively, but the two maximum-edge graphs in the upper $n = 10$ row are identical and the number of distinct graphs shown for $n = 10$ is 18. The upper right of the figure also shows an 8-point graph that is not NFC but plays a role for $n \in \{9, 10\}$.

We now verify that the graphs with $n \geq 6$ described here are in fact NFC.

Figure 2 about here

In what follows, we begin with the base-coloring of $K_{1,4}$ at the top of a graph and refer to its four terminal vertices as the *key vertices*. Note that a new vertex cannot be adjacent to both key vertices 2 and 4, or 2 and 5.

We verify that G_5 in Figure 2 is NFC, prove that it is the only NFC graph in $\mathcal{C}_4(5)$ with $6 \leq n \leq 8$, and then note that the graphs of Figure 2 for $n \in \{9, 10\}$ are NFC.

Suppose $G = G_5$. Vertices x and y must be adjacent to key vertices 2 and 3, and 4 and 5. Then $f(x)$ must be 5, as shown on the upper left of Figure 2, and $f(y) \in \{1, 2\}$. The bottom vertex must have color 0 or 1, but it is forced to have color 0 because of y, which then has $f(y) = 2$. This gives a unique (up to duality) coloring that is non-full. Note that no edges can be added to G_5 without forcing $\lambda \geq 6$, since adding any edge violates an $L(2,1)$-coloring requirement.

We now prove that G_5 is the only NFC graph in $\mathcal{C}_4(5)$ with $6 \le n \le 8$. Let G consist of $K_{1,4}$ plus sixth, seventh and eighth vertices x, y and z as needed. We use the base-coloring for $K_{1,4}$ and take x adjacent to one or more key vertices for connectedness. *Switch colors* means that colors $2, 3, 4, 5$ are replaced by $5, 4, 3, 2$, respectively. We consider $n = 6, 7, 8$ in turn.

n=6. If $f(x) = 1$, G is FC. If $f(x) \in \{2, 3\}$, change $f(x)$ to 1 for a full coloring. If $f(x) \in \{4, 5\}$, switch colors then change $f(x)$ to 1.

n=7. Suppose y as well as x is adjacent to a key vertex. Suppose $\min\{f(x), f(y)\} > 1$, else G is FC. If $\{f(x), f(y)\} \cap \{2, 3\} \ne \phi$, change the smaller color for x and y to 1 to get a full coloring. If $\{f(x), f(y)\} \subseteq \{4, 5\}$, switch colors, then repeat the preceding step.

Suppose y is adjacent only to x, and $f(x) > 1$. If $f(x) \ge 3$, take $f(y) = 1$. If $f(x) = 2$, set $f(y)$ to 0, switch colors, then change $f(y)$ to 1.

n=8. Let $F = \{f(x), f(y), f(z)\}$ and suppose $1 \notin F$.

Suppose $0 \notin F$. If $F \cap \{2, 3\} \ne \phi$, change those in $\{x, y, z\}$ with F's smallest color to color 1 to get a full coloring. If $F \subseteq \{4, 5\}$, switch colors and repeat the preceding step.

Suppose $0 \in F$, say $f(z) = 0$, so z has edges only to $\{x, y\}$. If z has only one edge to $t \in \{x, y\}$, take $f(z) = 1$ if $f(t) \ge 3$. If $f(t) = 2$, switch colors then set $f(z) = 1$ to get a full coloring. Assume henceforth that $\{x, z\}, \{y, z\} \in E$ so $f(x) \ge 2$ and $f(y) \ge 2$. If $\min\{f(x), f(y)\} \ge 3$, change $f(z)$ to 1; if $\max\{f(x), f(y)\} \le 4$, switch colors then change $f(z)$ to 1. It follows that a full coloring is impossible only if $\{f(x), f(y)\} = \{2, 5\}$.

With $f(z) = 0$, $\{f(x), f(y)\} = \{2, 5\}$ and $\{x, z\}, \{y, z\} \in E$, suppose $\{x, y\} \notin E$. If y has no edge to a key vertex, we can presume that $f(x) = 5$ and change $(f(z), f(y))$ to $(1, 3)$ to get a full coloring. To prevent this, assume that $f(y) = 2$ with y adjacent to one or both of the key vertices colored 4 and 5. If y is adjacent to only one key vertex, we can choose this key vertex to have color 5 and then change $(f(z), f(y))$ to $(1, 3)$ to get a full coloring. It follows that the only way to prevent a full coloring is to have x adjacent to the key vertices with colors 2 and 3, with $f(x) = 5$, and to have y adjacent to the key vertices with colors 4 and 5, with $f(y) = 2$. This gives G_5. We have already proved that G_5 is NFC and that any edge additions to G_5 force $\lambda > 5$.

With $f(z) = 0$, $\{f(x), f(y)\} = \{2, 5\}$ and $\{x, z\}, \{y, z\} \in E$, suppose $\{x, y\} \in E$. Assume without loss of generality that $f(x) = 5$. Then x is adjacent only to the key vertex with color 3, and y is adjacent to no key vertex or to the key vertex with color 4. If y is adjacent to no key vertex, interchange key vertex colors 2 and 3, then change $(f(z), f(y))$ to $(1, 3)$ to get a full coloring. If y is adjacent to the key vertex with color 4, we obtain the middle diagram on the top of Figure 2 and recolor as indicated there to get a full coloring. This completes the proof that G_5 is the only NFC graph in $\mathcal{C}_4(5)$ with $6 \le n \le 8$.

It remains to show that graphs for $n \in \{9, 10\}$ in Figure 2 are in fact NFC. This is immediate from our analysis of G_5 for the left diagram for $n = 9$ and diagram I for $n = 10$. The adjacencies of II for $n = 10$ force the non-full coloring there.

It is easily seen that the two colorings of the 8-point graph on the upper right of Figure 2 are the only $L(2,1)$-colorings of that graph using a base coloring for $K_{1,4}$. When a P_2 is added from the bottom vertex to an upper left key vertex, as in the rightmost diagram for $n = 9$, the full coloring of the upper right 8-point graph is not $L(2,1)$-feasible, so we must use the top middle coloring on the rightmost diagram for $n = 9$, and it is NFC. For a different reason, namely Lemma 1.1, diagram IV for $n = 10$ must also use the top middle coloring, so it too is NFC.

This leaves only the middle diagram for $n = 9$ and diagram III for $n = 10$. Careful analysis of the middle $n = 9$ diagram shows that if we delete its left $\{2,5\}$ edge then, using a base coloring for $K_{1,4}$, the bottom vertex must have color 0 or 1. One can try different values for the right-most vertex labeled 5 and show only 0 or 1 works for y. However, the presence of the left $\{2,5\}$ edge forces the bottom vertex to have color 0, and no vertex can have color 1. For diagram III for $n = 10$, where the left $\{2,5\}$ edge is now optional, suppose we have a base coloring for the top $K_{1,4}$. Lemma 1.1 forces the bottom vertex to be 0 or 5, but a careful analysis shows that 5 is not possible. If the bottom vertex is 0, then its neighbors do not include 1, so III never has color 1. ■

6 ASPECTS OF $\mathcal{C}_3(5)$

The following results for $\mathcal{C}_3(5)$ are like those for $\mathcal{C}_3(4)$ in that no graph in $\mathcal{C}_3(5)$ for $n = 6$ is NFC but at least one for $n = 8$ is NFC. It is true also that $\mathcal{C}_3(5)$ has no NFC graph for $n = 7$, but the proof is case intensive, so we omit $n = 7$ from the following theorem. We also show that $\mathcal{C}_3(5)$ has arbitrarily large NFC graphs, so it has an infinite number of such graphs in distinction to the small finite number of Theorem 4.1 for $\mathcal{C}_3(4)$.

Theorem 6.1 *$\mathcal{C}_3(5)$ has four graphs with $n \leq 5$ (and all are NFC). It has 17 graphs with $n = 6$, and all are FC. The number of NFC graphs in $\mathcal{C}_3(5)$ is at least one for $n = 8$, at least three for $n = 9$, and at least two for $n = 10$. Moreover, $\mathcal{C}_3(5)$ has at least one NFC graph for every $n \in \{10m - 1 : m = 2, 3, 4, \ldots\}$.*

Proof. The four graphs in $\mathcal{C}_3(5)$ with $n \leq 5$ are shown at the top of Figure 3, and the 17 with $n = 6$ appear below them. The NFC graphs for $n \in \{8, 9, 10\}$ are in the lower part of the figure along with the pattern used to prove NFC for $n = 10m - 1$.

Figure 3 about here

We begin the proof of these observations with comments that lead easily to the conclusion that the only members of $\mathcal{C}_3(5)$ with $n \leq 5$ are the four graphs in the top row of Figure 3. (Of course, all are NFC.) A triangle with two appended vertices and $\Delta = 3$ has $\lambda = 4$. The vertices of C_4 for $\lambda = 4$ are successively colored 0314, so the addition of a diagonal forces $\lambda = 5$. Two diagonals give K_4 with $\lambda = 6$. Addition of a vertex outside C_4 or outside C_4 plus a diagonal leads to the examples. The successive-vertex coloring of C_5 for $\lambda = 4$ is 03142. This allows one chord (0 to 4), but two disjoint chords require $\lambda = 5$. Two chords with a common vertex or more than two chords give $\Delta > 3$.

The graphs of Figure 3 for $n = 6$ are organized by cycle sizes. The only $\lambda = 4$ coloring of C_6 is 024024, so the addition of one or more chords forces $\lambda \geq 5$. Eight such additions have

$\Delta = 3$. Seven appear in the second row of Figure 3. The eighth has three diagonal chords and $\lambda = 6$. The rest of Figure 3 for $n = 6$ shows that $\mathcal{C}_3(5)$ has four graphs with a C_5 and no C_6, two with a C_4 plus diagonal and no C_5 or C_6, two with a C_4 and no diagonal and no C_5 or C_6, and two with a C_3 and no C_4, C_5 or C_6. This yields 17 graphs in $\mathcal{C}_3(5)$ with $n = 6$, and all are FC.

The rest of Theorem 6.1 is based on the 8-vertex graph G_6 in the lower left of Figure 3. It has exactly three span colorings with $\lambda = 5$: see Figure 4. These three result from a systematic examination of feasible colorings for the left diamond (C_4). Only the three of Figure 4 extend to the right diamond, and their extensions are unique. The original G_6 coloring yields the colorings for $n \in \{9, 10\}$ at the bottom of Figure 3. All are NFC.

Figure 4 about here

Let G_7 be the 9-vertex graph on the lower halves of Figures 3 and 4. It has only two $L(2,1)$-colorings of span 5, shown in Figure 4, because the rightmost G_6 coloring in the upper part of Figure 4 does not accommodate the new vertex. Denote by $G_7(m)$ the graph in $\mathcal{C}_3(5)$ with m copies of G_7 linked in series by $m-1$ new vertices, as illustrated for $m = 3$ in the bottom row of Figure 3. It has $9m+m-1 = 10m-1$ vertices. A new vertex adjacent to a degree-2 vertex of G_7 must have color 2 or 4 if we use the left coloring of G_7, and must have color 1 or 3 if we use the right coloring of G_7 on Figure 4. Because $\{2,4\} \cap \{1,3\} = \phi$, all copies of G_7 in $G_7(m)$ have identical colorings. If the left G_7 coloring is used, all $m-1$ linking vertices must have color 4, and if its right dual coloring is used, all $m-1$ linking vertices must have color 1. Hence $G_7(m)$ is NFC. ∎

7 MORE NFC GRAPHS

Our final two theorems follow the lead of G_6 on the lower left of Figure 3 to obtain NFC graphs for a variety of other (Δ, λ) pairs.

Theorem 7.1 *If* $\Delta \geq 3$ *and* $\Delta + 1 \leq \lambda \leq 2\Delta - 1$, *then* $\mathcal{C}_\Delta(\lambda)$ *contains an NFC graph with* $n \geq \lambda + 1$.

Proof. Let G be the graph of Figure 5 for $\Delta \geq 3$, $1 \leq k \leq \Delta - 1$ and $\lambda = \Delta + k$. G has 2λ vertices, which exceeds $\lambda + 1$. We arrange its vertices in four rows with k vertices in rows 1 and 4, and Δ vertices in rows 2 and 3. The top two rows form a $K_{k,\Delta}$, as do the bottom two rows, with edges between rows 2 and 3 as indicated. Each vertex in rows 1 and 4 has degree Δ, which is the maximum degree. The colorings shown and the fact that $\lambda(K_{k,\Delta}) = k + \Delta$ verify that its span is in fact λ.

Figure 5 about here

The colorings of Figure 5 are not full because they omit color k. It is easily seen that the only alternative span coloring of $K_{k,\Delta}$ is the dual of the one shown with colors $\Delta + 1$ through λ for row 1 (or 4) and 0 through $\Delta - 1$ for row 2 (or 3). This dual coloring for rows 1 and 2 is $L(2, 1)$-incompatible with the original coloring of rows 3 and 4, and if the dual coloring is used for both pairs of rows then color Δ is not used. It follows that G is NFC. ∎

The next theorem shows that there are graphs with $n \geq \lambda + 1$ whose span colorings must omit at least two colors in $\{0, 1, \ldots, \lambda\}$. Of course K_n colorings omit nearly half the colors in their span coloring sets, but they also have $\lambda = 2(n - 1) > n$. We leave open the question of whether there are graphs with $n \geq \lambda + 1$ whose span colorings must omit three or more colors in $\{0, 1, \ldots, \lambda\}$.

Theorem 7.2 *If $\Delta \geq 3$ and $\lambda = \Delta + 2$, then $\mathcal{C}_\Delta(\Delta + 2)$ contains an NFC graph with $n = 2(\Delta + 1)$ all of whose colorings use exactly $\lambda - 1$ colors.*

Proof. Let G be the graph of Figure 6 for $\Delta \geq 3$, $\lambda = \Delta + 2$ and $n = 2(\Delta + 1)$. The top two rows form a $K_{2,\Delta-1}$, as do the bottom two rows, with two edges between rows 2 and 3. The maximum degree is in fact Δ (rows 2 and 3), and $\lambda(G) = \Delta + 2$ even though $\lambda(K_{2,\Delta-1}) = \Delta + 1$. If we try to get $\lambda(G) = \Delta + 1$, then either

(a) row 1 has colors 0 through $\Delta - 2$, and row 2 has colors Δ and $\Delta + 1$, or

(b) row 1 has colors 3 through $\Delta + 1$, and row 2 has colors 0 and 1,

and similarly for rows 4 and 3. If (a) is used for rows 1 and 2, and for rows 4 and 3, we violate the distance-1 $L(2, 1)$ constraint; if (a) is used for rows 1 and 2, and (b) is used for rows 4 and 3, we violate the distance-2 $L(2, 1)$ constraint.

Figure 6 about here

The G coloring of Figure 6 omits colors $\Delta - 1$ and $\Delta + 1$. Its dual omits colors 1 and 3. Note that the coloring shown for rows 1 and 2 is $L(2, 1)$-incompatible with the dual coloring for rows 4 and 3.

It remains to prove that no other span coloring of G omits fewer than two colors. There are $\Delta + 3$ colors in the color set $S = \{0, 1, 2, \ldots, \Delta + 2\}$. Let A, B, C and D be the sets of colors used in rows $1, 2, 3$ and 4, respectively, of G, so $|A| = |D| = \Delta - 1$ and $|B| = |C| = 2$. Take

$$B = \{b_1 < b_2\} \quad \text{and} \quad C = \{c_1 < c_2\}$$

and let

$$\begin{aligned} A^* &= S \cap \{b_1 - 1, b_1, b_1 + 1, b_2 - 1, b_2, b_2 + 1, c_1, c_2\} \\ D^* &= S \cap \{b_1, b_2, c_1 - 1, c_1, c_1 + 1, c_2 - 1, c_2, c_2 + 1\} \ , \end{aligned}$$

so that the colors in A^* cannot be in A, and the colors in D^* cannot be in D. Because $(\Delta + 3) - |A| = (\Delta + 3) - |D| = 4$, we require $|A^*| \leq 4$ and $|D^*| \leq 4$.

Suppose $1 \leq b_1 \leq \Delta - 1$. If $b_2 > b_1 + 1$ then $|A^*| \geq 5$, so we require $b_2 = b_1 + 1$. Then

$$A^* = \{b_1 - 1, b_1, b_1 + 1, b_1 + 2, c_1, c_2\} \ ,$$

and therefore $|A^*| \leq 4$ implies $c_1, c_2 \in \{b_1 - 1, b_1, b_1 + 1, b_1 + 2\}$. The two edges between B and C then require $c_1 = b_1 - 1$ and $c_2 = b_1 + 2$. But this implies $|D^*| \geq 5$, a contradiction.

We conclude that $b_1 \in \{0, \Delta, \Delta + 1\}$. Similarly, $c_1 \in \{0, \Delta, \Delta + 1\}$ and, by duality, $b_2 \in \{1, 2, \Delta + 2\}$ and $c_2 \in \{1, 2, \Delta + 2\}$.

Suppose $b_2 = 1$, so $(b_1, b_2) = (0, 1)$. Then the edges between B and C require $c_1 \geq 2$, and it follows that $|D^*| \geq 5$. Hence $b_2 \neq 1$. Similarly, $c_2 \neq 1$ and, by duality, $b_1 \neq \Delta + 1$ and $c_1 \neq \Delta + 1$. Thus,

$$b_1, c_1 \in \{0, \Delta\} \quad \text{and} \quad b_2, c_2 \in \{2, \Delta + 2\} \ .$$

Suppose $(b_1, b_2) = (0, 2)$. Then $|A^*| \leq 4$ requires $c_1, c_2 \subseteq \{0, 1, 2, 3\}$. Neither c_1 nor c_2 can be 1 because of the edges between B and C, so $\{c_1, c_2\} \subseteq \{0, 2, 3\}$. If $c_2 = 3$, then $c_1 = 0$ or $c_1 = 2$ and $|D^*| \geq 5$. We therefore require $(c_1, c_2) = (b_1, b_2) = (0, 2)$, and colors 1 and 3 are absent from the coloring of G. A similar result obtains if $(b_1, b_2) = (\Delta, \Delta + 2)$, with colors $\Delta - 1$ and $\Delta + 1$ absent, as in Figure 6.

Suppose $(b_1, b_2) = (0, \Delta + 2)$. Then $|A^*| \leq 4$ requires $c_1, c_2 \subseteq \{0, 1, \Delta + 1, \Delta + 2\}$, and $|D^*| \leq 4$ then implies that $(c_1, c_2) = (b_1, b_2) = (0, \Delta + 2)$. Then $A = D = \{2, 3, \ldots, \Delta\}$ and colors 1 and $\Delta + 1$ are absent ¿from the coloring of G. This covers all remaining cases with $b_1 < b_2$ and $c_1 < c_2$, so the proof of Theorem 7.2 is complete. ■

8 OPEN PROBLEMS

Although the open problems of our study seem limitless, we suggest the following three questions as guides to further research.

1. What is the largest λ for which $\mathcal{C}_3(\lambda)$ has an NFC graph with $n \geq \lambda + 1$?
2. Is the set of NFC graphs in $\mathcal{C}_4(5)$ finite or infinite?
3. Is there a G with $n \geq \lambda + 1$ in some $\mathcal{C}_\Delta(\lambda)$ such that every span coloring of G omits at least three colors in $\{0, 1, \ldots, \lambda\}$?

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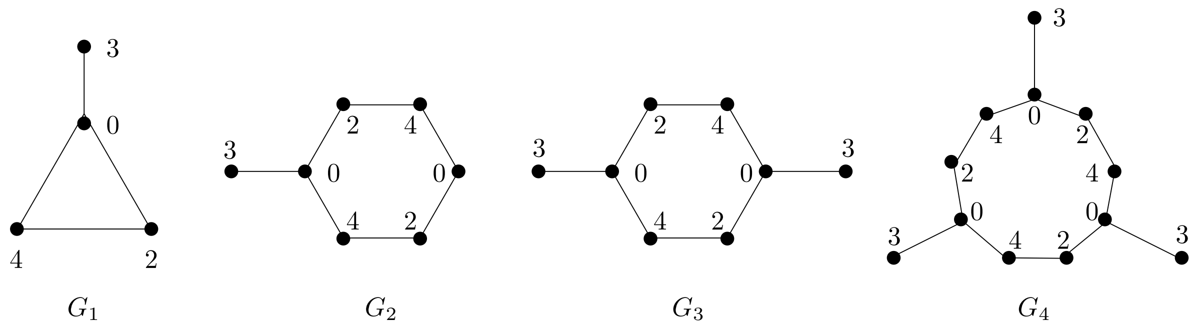


Figure 1. The NFC graphs in $\mathcal{C}_3(4)$. Numbers shown are $L(2,1)$-colorings of span 4.

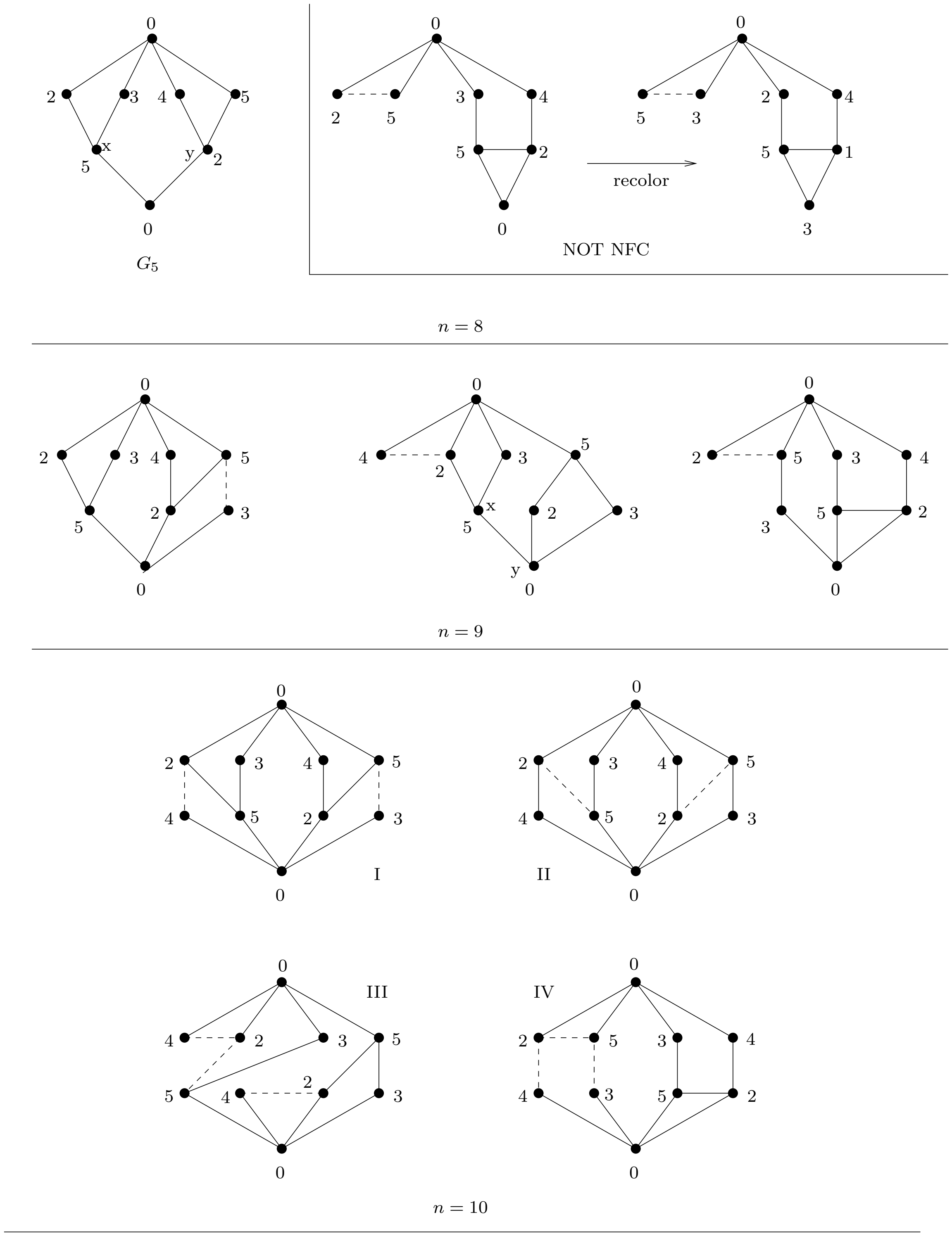


Figure 2. NFC graphs in $\mathcal{C}_4(5)$ with $L(2,1)$-colorings of span 5 shown, and one non-NFC graph with a recoloring. Dashed edges can be included in the graph or omitted.

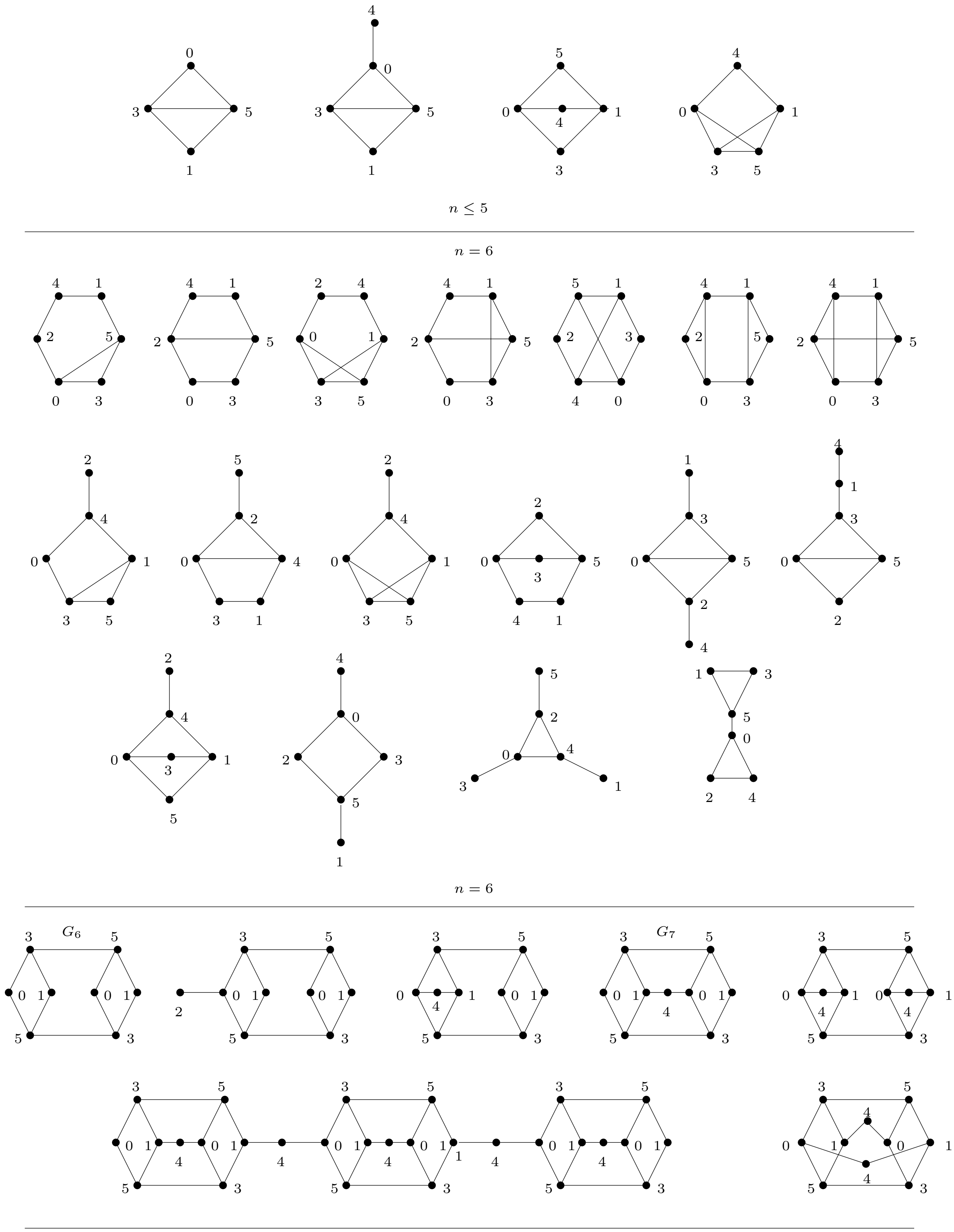
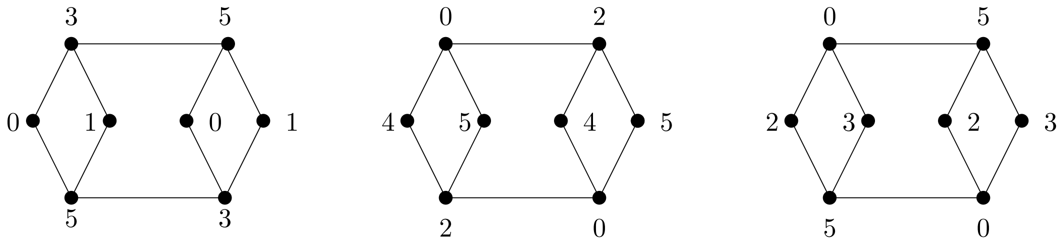
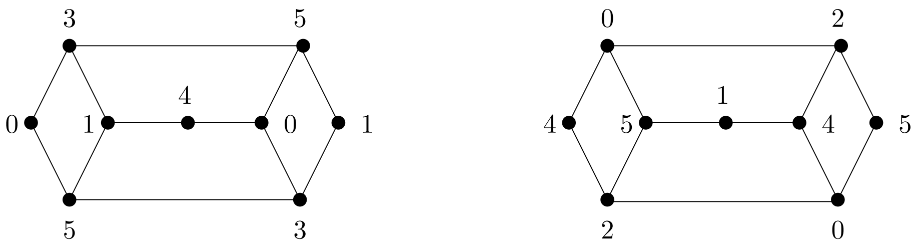


Figure 3. Graphs in $\mathcal{C}_3(5)$ with full colorings for $n = 6$ and essentially unique span colorings for $n = 8$, 9, 10 and $10m - 1$.



The three $L(2,1)$-colorings of G_6 of Figure 3 with $\lambda = 5$



The two $L(2,1)$-colorings of G_7 of Figure 3 with $\lambda = 5$

Figure 4

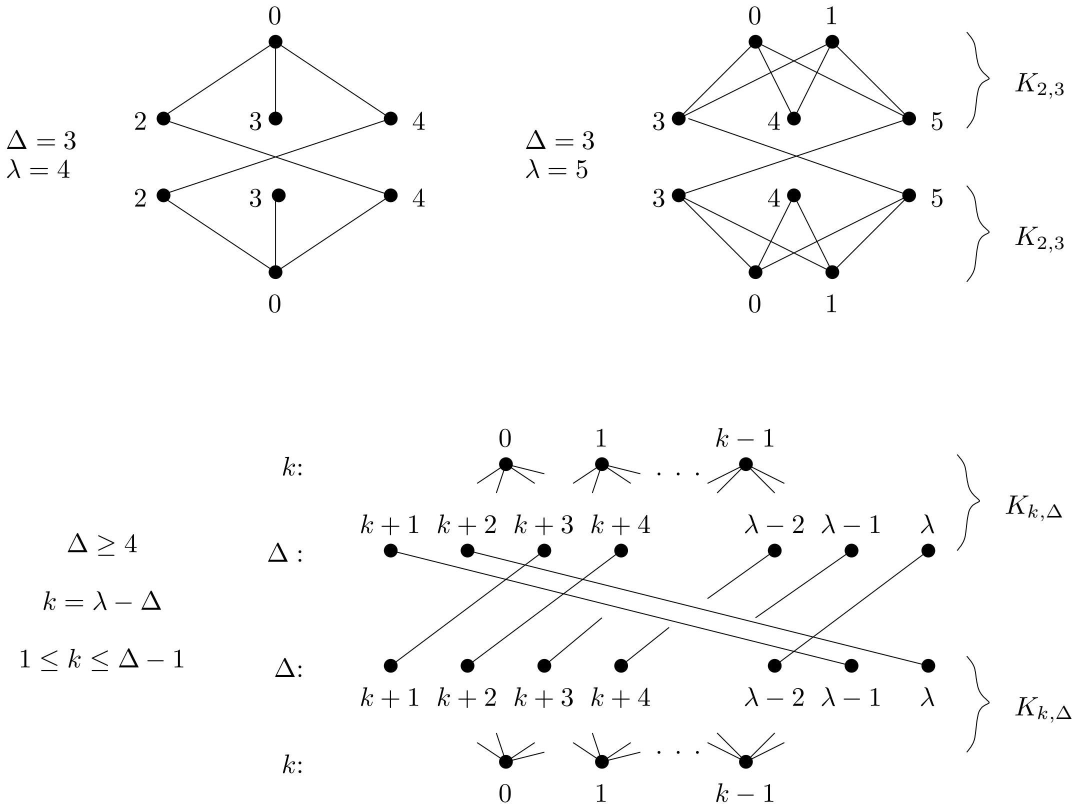


Figure 5. $L(2,1)$-colorings of graphs for proof of Theorem 7.1.

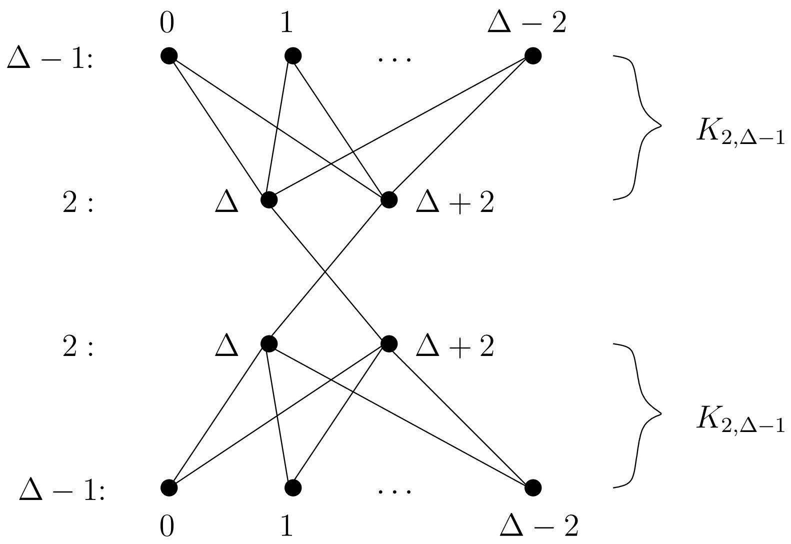


Figure 6. An $L(2,1)$-coloring for proof of Theorem 7.2.