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On the Derivation of Hydrodynamics from the Boltzmann Equation

by

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Dedicated to Robert H. Kraichnan on the Occasion of his 70th birthday

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Abstract:

We review the main ideas on the derivation of hydrodynamical equations from microscopic models. The Boltzmann equation, which is a good approximation for the evolution of rare gases, provides an useful tool to test these ideas in mathematically controllable situations such as the Euler and incompressible Navier-Stokes limits, which we describe in some detail. We also discuss the heuristics and some few rigorous results available for stochastic particle systems.

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1. Introduction.

The kinetic description of a classical fluid is intermediate between a fully microscopic one and a purely macroscopic or hydrodynamical one. The microstate of the system, consisting of N point particles (atoms) in a box Λ is specified at a given instant of time by a point X in its $6N$-dimensional phase space Γ; $X = (\underline{r}_1, \underline{v}_1, \ldots, \underline{r}_N, \underline{v}_N)$ where $\underline{r}_i, \underline{v}_i$ are the coordinates and velocities of the i-th particle. As time changes, X evolves according to the Newtonian equations of motion which once the interaction between particles, the external force and some initial state X_0 are provided, gives X_t for all $t \in (-\infty, +\infty)$.

The fully macroscopic description on the other hand, appropriate for fluids, is given in terms of five hydrodynamical fields $\xi(\underline{r}, t) = (n(\underline{r}, t), \underline{u}(\underline{r}, t), e(\underline{r}, t))$ where $\underline{r}$ is a point in the three dimensional region of space Λ containing the fluid. The particle density n, momentum density $n\underline{u}$ and energy density ne correspond to the local quantities conserved by the interactions. (We have taken the mass of the particles to be unity.) The time evolution of $\xi(\mathbf{r}, t)$ is usually taken to be given by the compressible Navier-Stokes (NS) equations or some approximation to them, e.g. Euler (E), incompressible NS (INS), Boussinesq etc.

The kinetic description, intermediate between the microscopic and macroscopic ones, is specified by the density of particles with position $\underline{r}$ and velocity v, $f(\underline{r}, \underline{v}, t)$, in the six dimensional space, $\underline{r} \in \Lambda$, $\underline{v} \in \mathbb{R}^3$. The time evolution of f is given, for a gas, by the Boltzmann equation (BE) which has the form[1]

$$\frac{\partial f(\underline{r}, \underline{v}, t)}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{r}} + \underline{F}(\underline{r}, t) \cdot \frac{\partial f}{\partial \underline{v}} = Q(f, f) \qquad (1.1)$$

where $\underline{F}$ is an external force and $Q(f, f)$ is a bilinear collision operator whose exact form depends on the interaction[1].

While X_t clearly contains all the information necessary for the determination of ξ and f, the derivation of autonomous equations like NS and BE from the underlying Hamiltonian microscopic dynamics is a notoriously difficult problem only partially solved at the present time, see[2],[3],[4]. An intermediate problem with a long history is the derivation of hydrodynamical equations from the BE for gases, when the $\xi(\underline{r},t)$ are obtainable from f as integrals over $\underline{v}$,

$$\begin{aligned} n(\underline{r},t) &= \int f(\underline{r},\underline{v},t)d\underline{v}, \\ n\underline{u}(\underline{r},t) &= \int \underline{v} f(\underline{r},\underline{v},t)d\underline{v}, \\ ne(\underline{r},t) &= \int \frac{1}{2}(\underline{v}-\underline{u})^2 f(\underline{r},\underline{v},t)d\underline{v}. \end{aligned} \tag{1.2}$$

The collision term $Q(f,f)$ in the BE, like the interaction on the microscopic level, conserves all the ξ's. A direct evaluation of the evolution of the ξ's from the BE does not however lead to a closed autonomous system. Rather it involves higher order velocity moments of f which leads to the well known closure problem.

The derivation or closure problem was tackled via well known 'expansion' methods of Chapmann-Enskog and Hilbert. Both of these incorporate the assumption (or fact) that the spatial variation of f and thus of ξ is slow on the scale of the mean free path. In this note we shall describe recent work on making those derivations rigorous by introducing explicitly a spatial scaling parameter ε and controlling the remainders in the series expansion when ε is small. It will turn out, not surprisingly, that we get different hydrodynamical equations depending on the time scales and other circumstances we consider.

Thus, for times of order ε^{-1} the ξ's satisfy the Euler equations, while for times of order

ε^{-2} they satisfy the INS equations for certain classes of initial f's. A discussion of these results as well as comparison with results obtained from microscopic dynamics both for Hamiltonian systems and for particle models evolving according to stochastic dynamics will also be given.

2. The Boltzmann equation.

We shall not discuss here the problems involved in the derivation of the BE referring the reader to the books[2(,(1)] and references there. We will just write down the collision operator and state some properties which are important in the sequel. Since the collision operator acts only on velocities, position and time being just parameters, we omit such dependences.

We have, after symmetrization,

$$Q(f,g)(\underline{v}) = \frac{1}{2}\int_{\mathbb{R}^3} d\underline{v}_* \int_{S_2} d\underline{\omega} B(|\underline{v}-\underline{v}_*|,\underline{\omega})[f' g'_* + f'_* g' - f g_* - f_* g], \tag{2.1}$$

where S_2 is the $2d$-sphere in $\mathbb{R}^3$, $d\underline{\omega}$ is the surface measure on it,

$$f' = f(\underline{v}'), \quad f'_* = f(\underline{v}'_*), \quad f_* = f(\underline{v}_*), \quad f = f(\underline{v}).$$

The vectors $\underline{v}$, $\underline{v}_*$ are the outgoing velocities of a binary elastic collision between two equal mass particles with incoming velocities $\underline{v}'$ and $\underline{v}'_*$:

$$\begin{aligned} \underline{v}' &= \underline{v} - \underline{\omega}[\underline{\omega}\cdot(\underline{v}-\underline{v}_*)], \\ \underline{v}'_* &= \underline{v}_* + \underline{\omega}[\underline{\omega}\cdot(\underline{v}-\underline{v}_*)] \end{aligned} . \tag{2.2}$$

Finally, $B(|\underline{v}-\underline{v}_*|,\underline{\omega})$ is the differential cross section of the collision, depending on the interaction law. For hard spheres it is given by

$$B(|\underline{v}-\underline{v}_*|,\underline{\omega}) = \frac{1}{2}|(\underline{v}-\underline{v}_*)\cdot\underline{\omega}|, \tag{2.3}$$

while for power law interaction it can be easily computed using the rules of classical mechanics. For infinite range interaction the function B has a divergence corresponding to grazing collisions which we need to remove for technical reasons. We assume this to be done according to the Grad's angular cutoff procedure[5]. This assures that B is a bounded smooth function of $\underline{\omega}$, growing with $|\underline{v}-\underline{v}_*|$ at most linearly (hard spheres case).

Let f, g and h be functions of $\underline{v}$ such that the integrals below make sense. Then

a)

$$\int_{\mathbb{R}^3} d\underline{v} Q(f,g)(\underline{v})h(\underline{v}) = 0 \quad \text{iff} \quad h(\underline{v}) = \sum_{\alpha=0}^{4} c_\alpha \chi_\alpha(\underline{v}) \tag{2.4}$$

with c_α some real constants and

$$\chi_\alpha(\underline{v}) = \begin{cases} 1 & \alpha = 0, \\ v_i & \alpha = i = 1,\ldots,3, \\ \dfrac{v^2}{2} & \alpha = 4. \end{cases} \tag{2.5}$$

The functions χ_α, $\alpha = 0,\ldots,4$ are called *collision invariants*. The vanishing of the integrals in (2.4) expresses the conservation of mass ($\alpha = 0$), momentum ($\alpha = 1,2,3$) and energy ($\alpha = 4$) during a collision.

b) If $f \geq 0$, then

$$\int_{\mathbb{R}^3} d\underline{v} Q(f,f)(\underline{v}) \log f(\underline{v}) = N(f), \tag{2.6}$$

where the *entropy production* $N(f)$

$$N(f) = \frac{1}{8}\int_{\mathbb{R}^3} d\underline{v} \int_{\mathbb{R}^3} d\underline{v}_* \int_{S_2} d\underline{\omega} B(|\underline{v}-\underline{v}_{|},\underline{\omega})[f'f'_{-} f f_{]}[\log f f_{-} \log f' f'_{]} \tag{2.7}$$

is non positive and vanishes if and only if $\log f = \sum_{\alpha=0}^{4} c_\alpha \chi_\alpha$.

c) Let $f(\underline{r}, \underline{v}, t)$ be a non negative, normalized function. Then it was shown by Boltzmann that

$$Q(f, f)(\underline{r}, \underline{v}, t) = 0 \tag{2.8}$$

if and only if

$$f(\underline{r}, \underline{v}, t) = M(n(\underline{r}, t), \underline{u}(\underline{r}, t), T(\underline{r}, t); \underline{v}), \tag{2.9}$$

with

$$M(n, \underline{u}, T; \underline{v}) = \frac{n}{[2\pi T]^{3/2}} \mathrm{e}^{-\frac{(\underline{v}-\underline{u})^2}{2T}} \tag{2.10}$$

where we have set Boltzmann constant equal to one. Simple integrations show that n, $\underline{u}$ and $e = 3T$ fulfill the relations (1.2) for $f = M$ and this justifies the notation used for the parameters of M. The density $n(\underline{r}, t)$ and the temperature $T(\underline{r}, t)$ are non negative; $\underline{u}(\underline{r}, t)$ is the *stream velocity* vector. The functions $M(n, \underline{u}, T; \underline{v})$ satisfying (2.8) are called *local Maxwellians*. They play a crucial role in studying the hydrodynamical limit, because they are the *local equilibria* in the context of kinetic theory. We note that M is a stationary solution of the Boltzmann equation only if the parameters are independent of both $\underline{r}$ and t. This statement is correct when $F = 0$, under some assumption on the domain Λ containing the gas, e.g. Λ is a torus or a bounded open set of $\mathbb{R}^3$ with a smooth boundary were particles collide elastically. In this case M is the *global* Maxwellian or equilibrium solution of the (BE).

An immediate consequence of the property a) are the local conservation laws for the mass, momentum and energy. In fact, multiplying (1.1) by χ_α, $\alpha = 0, \ldots, 4$ and integrating on $\underline{v}$, by property a) we get

$$\begin{aligned}
&\partial_t n + \text{div}\, [n\underline{u}] = 0, \\
&\partial_t (n u_i) + \sum_{j=1}^{3} \partial_{r_j} \big(n u_i u_j + \Pi_{i,j} \big) = n F_i, \quad i = 1 \dots, 3 \\
&\partial_t \Big[n \big(\frac{u^2}{2} + e \big) \Big] + \text{div}\, \Big[n\underline{u} \big(\frac{u^2}{2} + \frac{5}{3} e \big) + \underline{q} \Big] = n\underline{F} \cdot \underline{u},
\end{aligned} \tag{2.11}$$

where

$$\Pi_{i,j} = \int_{\mathbb{R}^3} d\underline{v} f(\underline{r}, \underline{v}, t)(v_i - u_i(\underline{r}, t))(v_j - u_j(\underline{r}, t)) \tag{2.12}$$

is the *stress tensor* and

$$\underline{q} = \int_{\mathbb{R}^3} d\underline{v} f(\underline{r}, \underline{v}, t)(\underline{v} - \underline{u}(\underline{r}, t))^2 (\underline{v} - \underline{u}(\underline{r}, t)) \tag{2.13}$$

is the *heat current vector.*

Of course the above relations are *not* the hydrodynamical equations: the stress tensor and the heat current vector cannot be apriori computed only in terms of $n, \underline{u}, e$. The closure problem consists in finding such relations. The simplest case corresponds to assuming local equilibrium, i.e. that f is a local Maxwellian. In this case equations (2.11) are obviously closed and are just the Euler equations for a gas. The assumption however is not correct in general because as noted earlier local Maxwellians are not solutions of the Boltzmann equation. In fact an important point is missing in our discussion: the Boltzmann equation and the hydrodynamical equations hold on very different space scales, while (2.11) hold on the same scale of the Boltzmann equation. We shall see in next section that, after the introduction of different scales the assumption of local equilibrium is approximately correct on the hydrodynamical scale and provides the hydrodynamical equations.

We conclude this section by introducing the *linearized Boltzmann operator*, which will play a major role in the sequel and state some properties we will extensively use. We refer to[6] for the proofs.

We fix a Maxwellian $M(\underline{v}) = M(n, \underline{u}, T; \underline{v})$, with n, $\underline{u}$ and T possibly depending on $\underline{r}$ and t. Since the linearized Boltzmann operator acts on functions of the velocity, the space-time dependence is only parametric and we disregard it below. The linearized (around M) Boltzmann operator is defined formally by

$$\mathcal{L}g = 2Q(M, g). \tag{2.14}$$

It is convenient to introduce also

$$Lg = \frac{1}{\sqrt{M}}\mathcal{L}(\sqrt{M}g) = \frac{2}{\sqrt{M}}Q(M, \sqrt{M}g) \tag{2.15}$$

Simple calculations show that $\mathcal{L}$ and L can be decomposed as

$$\mathcal{L} = -\nu(\underline{v}) + \mathcal{K}, \quad L = -\nu(\underline{v}) + K, \tag{2.16}$$

where $\nu(\underline{v}) \geq \nu_0 > 0$ is a smooth function growing at large velocities as the cross section; K is an integral operator which is compact on the space of square integrable functions $\mathcal{H} = L_2(\mathbb{R}^3)$. Therefore the operator L is well defined in the domain $\mathcal{D} \subset \mathcal{H}$ of the functions $g(\underline{v})$ such that $\nu g \in \mathcal{H}$. Some properties of L are:

i) L is symmetric in $\mathcal{H}$:

$$(g, Lf) = (f, Lg), \quad f, g \in \mathcal{D}; \tag{2.17}$$

ii) L is non positive in $\mathcal{H}$:

$$(f, Lf) \leq 0 \quad f \in \mathcal{D}; \tag{2.18}$$

iii) Let

$$\psi_\alpha = \sqrt{M}\chi_\alpha, \quad \alpha = 0, \dots, 4, \tag{2.19}$$

$\mathcal{P}$ be the projector on the space $\mathcal{I}$ spanned by $\{\psi_\alpha, \alpha = 0, \dots, 4\}$ and $\mathcal{P}^\perp$ the projector on $\mathcal{I}^\perp$, the orthogonal complement of $\mathcal{I}$. Then

$$\text{Null } L = \mathcal{I}, \quad \text{Range } L \subset \mathcal{I}^\perp. \tag{2.20}$$

iv) There is $\mu > 0$ such that, if $f \in \mathcal{P}^\perp\mathcal{D}$, then

$$(f, Lf) \le -\mu(f, \nu f) \qquad (2.21).$$

3. Euler limit.

We introduce a scaling parameter $\varepsilon > 0$ which represents the ratio between the microscopic (kinetic) and macroscopic length: roughly ε is the size of the mean free path in macroscopic units. This scaling parameter arises in (1.1) when we consider initial data of hydrodynamical type. By this we mean an initial distribution $f_0^\varepsilon(\underline{r}, \underline{v})$ whose spatial dependence is so slow that one has to look at macroscopic distances to find appreciable variations of f_0^ε. This is obtained by choosing a smooth function $f_0(\underline{x}, \underline{v})$ and putting

$$f_0^\varepsilon(\underline{r}, \underline{v}) = f_0(\varepsilon\underline{r}, \underline{v}). \tag{3.1}$$

The variable $\underline{x} = \varepsilon\underline{r}$ represents the macroscopic position. We assume for simplicity periodic boundary conditions, that is $\underline{r}$ varies in the torus of size ε^{-1}, $\mathbb{T}_\varepsilon = [0, \varepsilon^{-1}]^3$ so that correspondingly $\underline{x}$ varies in the unit torus $\mathbb{T} = [0,1]^3$: i.e. ϵ is thus just the inverse length of the box expressed in microscopic units.

Since the initial datum is almost constant on the microscopic scale, the time evolution given by the Boltzmann equation (1.1) will not depart significantly from the uniform state before sufficiently long times. Just how long a time is required can be estimated from the fact that to depart from spatial uniformity a particle of the gas, moving with finite velocity $\underline{v}$, has to cover a macroscopic distance to detect the variations of the initial datum. This means that it needs a time of order ε^{-1}. Therefore, we shall look at the solution of (1.1) at a time $\varepsilon^{-1}t$: we set

$$f^\varepsilon(\underline{x}, \underline{v}, t) = f(\varepsilon^{-1}\underline{x}, \underline{v}, \varepsilon^{-1}t) \tag{3.2}$$

with $f(\underline{r}, \underline{v}, t)$ solving (1.1) with initial datum $f_0(\varepsilon\underline{r}, \underline{v})$. The space-time scaling used in (3.2) is called *hyperbolic* or *Euler* scaling.

We note that, in order to make the above considerations consistent, we also have to assume that the size of the external force is of order ε, because a finite size force would produce, on the time scale we are considering, very large accelerations of the particles. Therefore we assume in (1.1) a force of the type $\varepsilon\underline{F}(\varepsilon\underline{r}, \varepsilon t)$. Note that if the force is a potential one, with a potential U also slowly varying on microscopic scale, such a condition is certainly fulfilled. It is immediate to check that $f^\varepsilon(\underline{x}, \underline{v}, t)$ has to solve the *rescaled* BE: with the notation

$$D_t f^\varepsilon \stackrel{def}{=} \frac{\partial f^\varepsilon}{\partial t} + \underline{v} \cdot \frac{\partial f^\varepsilon}{\partial \underline{x}} + F \cdot \frac{\partial f^\varepsilon}{\partial \underline{v}}, \tag{3.3}$$

it is

$$D_t f^\varepsilon = \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon), \tag{3.4}$$

with initial datum $f_0(\underline{x}, \underline{v})$. We expect to recover the hydrodynamical equations in the limit $\varepsilon \to 0$. Formally it is clear that in this limit the right hand side of (3.4) would become singular. The only possibility to avoid this singularity is that

$$\lim_{\varepsilon \to 0} Q(f^\varepsilon, f^\varepsilon) = 0. \tag{3.5}$$

Condition (2.8),(2.9) then forces f^ε to converge to a local Maxwellian. As remarked before, this, together with the local conservation laws produces a set of closed equations for the hydrodynamical variables, which are the Euler equations for the ideal fluid.

A more accurate argument to get such a conclusion goes back to Hilbert[7], who proposed the famous *Hilbert expansion* as an example of his theory of linear transformations. The starting point is to look for the solution f^ε as a power series in ε:

$$f^\varepsilon = \sum_{k=0}^{\infty} \varepsilon^k f_k, \tag{3.6}$$

and try to find the functions f_k, $k \geq 0$ by imposing that (3.4) be satisfied at each order in ε. By plugging the series (3.6) in (3.4) and equating terms of the same order in ε we get the following conditions:

$$\begin{aligned}
\varepsilon^{-1}: &\qquad Q(f_0, f_0) = 0, \\
\varepsilon^{0}: &\qquad 2Q(f_0, f_1) = D_t f_0, \\
&\qquad \dots \\
\varepsilon^{k}: &\qquad 2Q(f_0, f_k) = S_k(f_0, f_1, \dots, f_{k-1}) \\
&\qquad \dots,
\end{aligned} \tag{3.7}$$

where

$$S_k(f_0, f_1, \dots, f_{k-1}) = D_t f_{k-1} - \sum_{\substack{(h,h'):\ h,h' \geq 1 \\ h+h'=k}} Q(f_h, f_{h'}). \tag{3.8}$$

The lowest order in (3.7) implies, using (2.8), (2.9), (2.10) that f_0 is a local Maxwellian $M = M(n, \underline{u}, T; \underline{v})$ with the functions $n, \underline{u}, T$ not yet determined. Definition (2.14) then allows to write the other equations (3.7) in the form

$$\mathcal{L} f_k = S_k(f_0, f_1, \dots, f_{k-1}), \quad k \geq 1. \tag{3.9}$$

We now discuss the solution of the equation

$$\mathcal{L} f = g. \tag{3.10}$$

By (2.4) a necessary condition for the solvability of (3.10) is that

$$\int_{\mathbb{R}^3} d\underline{v} \chi_\alpha(\underline{v}) g(\underline{v}) = 0, \quad \alpha = 0, \dots, 4. \tag{3.11}$$

With the notation $\tilde{f} = f M^{-1/2}$, $\tilde{g} = g M^{-1/2}$, (3.10) is equivalent to

$$L \tilde{f} = \tilde{g} \tag{3.12}$$

and the solvability condition becomes

$$(\tilde{g}, \psi_\alpha) = 0, \quad \alpha = 0, \dots, 4. \tag{3.13}$$

Hence, if $\tilde{g}$ is in $\mathcal{I}^\perp$, the space perpendicular to the invariant subspace $\mathcal{I}$, the solvability condition is fulfilled and, by the compactness of K and the application of the Fredholm alternative, we can conclude that there exists one and only one solution of (3.12) in $\mathcal{I}^\perp$. With an abuse of notation we denote it as $L^{-1}\tilde{g}$. In fact, since Null L is not trivial, L is not invertible. Only its restriction to $\mathcal{I}^\perp$ can be inverted. The general solution to (3.12) is therefore

$$\tilde{f} = L^{-1}\tilde{g} + \sum_{\alpha=0}^{4} c_\alpha \psi_\alpha. \tag{3.14}$$

Going back to (3.10), we can conclude that, provided that g satisfies (3.11) and $gM^{-1/2} \in L_2(\mathbb{R}^3)$, the general solution of (3.10) is given by

$$f = \mathcal{L}^{-1} g + \sum_{\alpha=0}^{4} c_\alpha \chi_\alpha M, \tag{3.15}$$

where we used the (slightly inaccurate) notation

$$\mathcal{L}^{-1} g = M^{1/2} L^{-1}(gM^{-1/2}). \tag{3.16}$$

We now return to the study of (3.9). Let us consider first $k = 1$. The solvability condition in this case is:

$$\int_{\mathbb{R}^3} d\underline{v} \chi_\alpha(\underline{v}) D_t M(n(\underline{x}, t), \underline{u}(\underline{x}, t), T(\underline{x}, t); \underline{v}) = 0, \quad \alpha = 0, \ldots, 4. \tag{3.17}$$

By simple Gaussian integrations we get, for $i, j = 1, \ldots, 3$,

$$\int_{\mathbb{R}^3} d\underline{v} M(n(\underline{x}, t), \underline{u}(\underline{x}, t), T(\underline{x}, t); \underline{v})(v_i - u_i(\underline{x}, t))(v_j - u_j(\underline{x}, t)) = n(\underline{x}, t) T(\underline{x}, t) \delta_{i,j}, \tag{3.18}$$

where $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$. Moreover

$$\int_{\mathbb{R}^3} d\underline{v} M(n(\underline{x}, t), \underline{u}(\underline{x}, t), T(\underline{x}, t); \underline{v})(v_i - u_i(\underline{x}, t))(\underline{v} - \underline{u}(\underline{x}, t))^2 = 0, \quad i = 1 \ldots, 3. \tag{3.19}$$

Therefore we can conclude that the solvability condition (3.17) is satisfied if and only if the parameters of the Maxwellian satisfy the equations

$$\begin{aligned}
&\partial_t n + \text{div } [n\underline{u}] = 0, \\
&n\partial_t \underline{u} + n(\underline{u} \cdot \nabla_{\underline{x}})\underline{u} = -\nabla_{\underline{x}} P + n\underline{F}, \\
&n\partial_t e + n(\underline{u} \cdot \nabla_{\underline{x}})e + P\text{div } \underline{u} = 0,
\end{aligned} \tag{3.20}$$

where

$$e = \frac{3}{2}T, \quad P = nT. \tag{3.21}$$

are the equations of state for the pressure and internal energy of the *ideal gas*. It should be noted that, since we are considering a situation in which the average distance between the particles is much larger than their interaction range these are indeed the appropriate equations of state.

The previous discussion shows that to the lowest order in ε the solution of the rescaled Boltzmann equation is a local Maxwellian whose parameters solve the Euler equations of the ideal fluid. The method allows to compute corrections of arbitrary order. In fact, once the solvability condition (3.17) is satisfied, f_1 is given by

$$f_1 = \mathcal{L}^{-1}[D_t M] + \sum_{\alpha=0}^{4} c_\alpha^{(1)} \chi_\alpha M, \tag{3.22}$$

with $c_\alpha^{(1)}(\underline{x}, t)$, $\alpha = 0, \dots 4$, arbitrary functions. The presence of arbitrary functions permits to continue the procedure and determine f_k for $k > 1$. Suppose that at some step k the function $f_1, \dots f_{k-1}$ are determined uniquely and f_k is determined up to arbitrary functions $c_\alpha^{(k)}(\underline{x}, t)$, $\alpha = 0, \dots, 4$:

$$f_k = \bar{f}_k + \sum_{\alpha=0}^{4} c_\alpha^{(k)} \chi_\alpha M. \tag{3.23}$$

In order to determine f_{k+1} we have to solve

$$\mathcal{L} f_{k+1} = S_{k+1}(M, f_1, \dots, f_k) \tag{3.24}$$

The solvability condition

$$\int_{\mathbb{R}^3} d\underline{v} \chi_\alpha(\underline{v}) S_{k+1}(\underline{v}), \quad \alpha = 0, \dots, 4 \tag{3.25}$$

can be satisfied taking advantage of the arbitrary functions $c_\alpha^{(k)}(\underline{x}, t)$: in fact (3.25) is a set of five linear partial differential equations PDE's in the unknown functions $c_\alpha^{(k)}(\underline{x}, t)$, which fully determine such functions once initial conditions are provided. Therefore f_{k+1} can be determined by (3.24), up to arbitrary functions $c_\alpha^{(k+1)}$ and the procedure can be iterated indefinitely.

In conclusion all the coefficients f_k are uniquely determined by above procedure. Unfortunately there is no indication that the power series (3.6) has a finite radius of convergence, so that the previous argument is only formal.

On the other hand, one can hope that finite truncations of the series are good approximations of the solution, if any. In order to see this, one fixes $N > 0$ and looks for a solution of (3.4) of the form

$$f^\varepsilon = \sum_{k=0}^{N} \varepsilon^k f_k + \varepsilon^m R, \tag{3.26}$$

with f_k, $k = 0, \dots, N$ determined by the previous procedure and R a suitable remainder function, depending on ε and N, to be chosen so that f^ε solves (3.4). Note that one would expect $m = N + 1$, but one can try to prove a weaker statement by choosing $m < N + 1$. Actually this is the case and the following theorem has been proved:

Theorem 3.1. *Assume that there exists a unique smooth solution in $H_s(\mathbb{T})$, with s sufficiently large. $(n(\underline{x}, t),\ \underline{u}(\underline{x}, t),\ T(\underline{x}, t))$ to the initial value problem associated to the Euler equations (3.21), at least in a finite time interval $[0, t_0]$. Denote by M_t the local Maxwellian with above parameters. Suppose that*

$$\|f_0 - M_0\| < \delta \tag{3.27}$$

for some $\delta > 0$ sufficiently small. Then there is an $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, there exists *a smooth solution $f^\varepsilon(t)$ to the rescaled Boltzmann equation, with*

$$\sup_{t\in(0,t_0]} \|f^\varepsilon(t) - M_t\| \le C\varepsilon \tag{3.28}$$

for some $C > 0$. The norm is defined as

$$\|g\| = \sup_{\underline{v}\in\mathbb{R}^3} (1+v^2)^k \|g(\,\cdot\,,\underline{v})\|_{H_s(\mathbb{T})}. \tag{3.29}$$

with $k, s > 3/2$, $\|\cdot\|_{H_s(\mathbb{T})}$ being the Sobolev norm of order s on $\mathbb{T}$.

Theorem 3.1 has been proved in the case of 1-dimensional space variations and $\delta = 0$ by Caflisch in[8], who used the truncated expansion with $N = 6$, $m = 3$. Note that such a choice of N and m is just the simplest possible choice and one could choose other pairs of values. Extension to higher space dimensions and $\delta > 0$ have been given in[9]. The supremum on time in (3.28) is restricted to $t > 0$. This is due to the fact that δ can be much bigger than ε. Even in this case, the solution at positive macroscopic times is close to the local Maxwellian up to $O(\varepsilon)$, because it is attracted to a neighborhood of the local Maxwellian on a time scale much shorter than the hydrodynamical one (*initial layer*).

Other results on the convergence of the solution of the rescaled Boltzmann equation to the Euler equations are due to[10],[11]. These results are based on the argument (3.5) and do not use the Hilbert expansion. They hold only for short times. In fact, it should be stressed that the locality in time of Theorem 3.1 is essentially of hydrodynamical type, in the sense that it is only due to the locality of the solutions of the Euler equation. If a global solution of the Euler equations is given then Theorem 3.1 holds with t_0 arbitrarily large (and the constant C possibly diverging for $t_0 \to +\infty$). On the contrary, the locality of the results in[10],[11] is of the kinetic type.

4. Navier-Stokes limit.

An noted in Section 3, the Hilbert expansion not only provides the limit as $\varepsilon \to 0$ of the solution of the rescaled BE, but also permits to compute the corrections to any order. We are mainly interested in the first order corrections, i.e. we look for f_1. While the explicit form of $\mathcal{L}^{-1}[D_t M]$ is only available for particular molecular interactions (Maxwell molecules) a general expression for f_1 can be obtained[1]: let $\underline{\widetilde{v}} = \underline{v} - \underline{u}$ and

$$A_{i,j}(\underline{v}) = \widetilde{v}_i \widetilde{v}_j - \frac{\widetilde{v}^2}{3}\delta_{i,j}, \quad B_i(\underline{v}) = (\frac{\widetilde{v}^2}{2} - \frac{5}{2}T)\widetilde{v}_i, \quad i,j = 1,\dots,3. \tag{4.1}$$

It is immediate to verify that, if we define $\widetilde{\mathcal{P}}^{\perp} g = M^{1/2}\mathcal{P}^{\perp}[M^{-1/2}g]$, where $\mathcal{P}^{\perp}$ is the projection into the space $I^{\perp}$, then

$$\widetilde{\mathcal{P}}^{\perp}[D_t M] = M\Big[\sum_{i,j=1}^{3} A_{i,j}\frac{\partial u_i}{\partial x_j} + \sum_{i=1}^{3} B_i \frac{\partial T}{\partial x_i}\Big]. \tag{4.2}$$

Moreover, there are two non negative functions, depending on $|\underline{\widetilde{v}}|$, Ψ_1 and Ψ_2 such that

$$\mathcal{L}^{-1}\big[A_{i,j}M\big] = -\Psi_1 A_{i,j}, \quad \mathcal{L}^{-1}\big[B_i M\big] = -\Psi_2 B_i](\underline{v}). \tag{4.3}$$

Therefore

$$\mathcal{L}^{-1}[D_t M] = -\Psi_1 \sum_{i,j=1}^{3} A_{i,j}\frac{\partial u_i}{\partial x_j} - \Psi_2 \sum_{i=1}^{3} B_i \frac{\partial T}{\partial x_i}. \tag{4.4}$$

This is the part of f_1 in $\mathcal{I}^{\perp}$. The part of f_1 in $\mathcal{I}$ represents the first order correction to the hydrodynamical quantities. A suitable resummation of the Hilbert expansion, known as the Chapmann-Enskog expansion[1], provides a closed set of equations, accurate to first order in f_1, for the hydrodynamical fields:

$$
\begin{aligned}
&\partial_t n + \text{div } [n\underline{u}] = 0, \\
&n\partial_t \underline{u} + n(\underline{u} \cdot \nabla_{\underline{x}})\underline{u} = -\nabla_{\underline{x}} P + n\underline{F} + \nabla_{\underline{x}} \cdot (\mu_\varepsilon \nabla_{\underline{x}} \underline{u}) + \nabla_{\underline{x}}(\zeta_\varepsilon \text{div } \underline{u}), \qquad (4.5) \\
&n\partial_t e + n(\underline{u} \cdot \nabla_{\underline{x}})e + P\text{div } \underline{u} = \nabla_{\underline{x}} \cdot (\kappa_\varepsilon \nabla_{\underline{x}} T) + \mu_\varepsilon (\nabla_{\underline{x}} \underline{u})^2 + \zeta_\varepsilon (\text{div } \underline{u})^2,
\end{aligned}
$$

with $\mu_\varepsilon = \varepsilon\mu$, $\zeta_\varepsilon = \varepsilon\zeta$, $\kappa_\varepsilon = \varepsilon\kappa$ and

$$
\mu = \int_{\mathbb{R}^3} d\underline{v}[A_{1,2}(\underline{v})]^2 \Psi_1(|\underline{v}|), \quad \zeta = \int_{\mathbb{R}^3} d\underline{v}[A_{1,1}(\underline{v})]^2 \Psi_1(|\underline{v}|) \tag{4.6}
$$

and

$$
\kappa = \int_{\mathbb{R}^3} d\underline{v}[B_1(\underline{v})]^2 \Psi_2(|\underline{v}|). \tag{4.7}
$$

The coefficients μ, ζ and κ are called *transport coefficients* and, in particular, μ is the *shear viscosity*, ζ the *volume viscosity* and κ the *heat conduction coefficient*. It is a well known result of the kinetic theory that the transport coefficients depend only on the temperature T and the molecular interaction, but not on the density, (see[(1)], p. 198). The above formulas are similar to the Green-Kubo formulas[(2)] for the transport coefficients in statistical mechanics. Equations (4.5), with $\varepsilon = 1$ are the well known Navier-Stokes equations for a viscous fluid. As derived from the Boltzmann equation, they contain the viscosity and heat conduction terms multiplied by the factor ε which, by definition, is essentially the mean free path in macroscopic units. There is no chance to prove that the arguments presented in the previous section are correct up to $\varepsilon = 1$ or at least some realistic value. So, we do not have a rigorous justification of (4.5) for relevant values of the transport coefficients: actually to show that the terms neglected in (4.5) are really small we are forced to consider ε a small number, eventually to be sent to 0. This is the main problem of the derivation of the Navier-Stokes equations as a scaling limit from a microscopic model: terms involving second order space derivatives are negligible in the scaling limit, when compared to terms involving first order space and time derivatives.

The typical way to enhance the second order derivatives is to consider from the beginning a longer time scale $\varepsilon^{-2}t$, instead of $\varepsilon^{-1}t$, that makes the time derivatives of the same order as the viscous terms. The heat equation is an example of an equation which is invariant under this *parabolic* or *diffusive* scaling. Unfortunately this is not sufficient to get rid of the ε's in (4.5). In fact, the nonlinear terms involving the first order space derivatives are now of order ϵ^{-1} while the other terms are of order 1.

In conclusion, both the hyperbolic and the parabolic scalings are not suitable to get the Navier-Stokes equations. It is not hard to realize that this is true for any *pure* space-time scaling.

On the other hand, if the velocity vector field $\underline{u}$ were of order ε, the first order terms would be as big as the time derivatives and the viscous terms in the second of (4.5). To make this consistent the force in (1.1) has to be assumed to be of order ε^3, because we want to keep $\underline{u}$ of order ε for times of order ε^{-2}, so the acceleration has to be of order ε^3. With such a scaling, equations (4.5) can be rewritten as

$$\begin{aligned}
&\partial_t n + \mathrm{div}\,[n\underline{u}] = 0,\\
&n\partial_t \underline{u} + n(\underline{u}\cdot\nabla_{\underline{x}})\underline{u} = -\varepsilon^{-2}\nabla_{\underline{x}}P + n\underline{F} + \Big[\nabla_{\underline{x}}\cdot(\mu\nabla_{\underline{x}}\underline{u}) + \nabla_{\underline{x}}(\zeta\mathrm{div}\,\underline{u})\Big],\\
&n\partial_t e + n(\underline{u}\cdot\nabla_{\underline{x}})e + P\mathrm{div}\,\underline{u} = \Big[\nabla_{\underline{x}}\cdot(\kappa\nabla_{\underline{x}}T) + \varepsilon^2\big(\mu(\nabla_{\underline{x}}\underline{u})^2 + \zeta(\mathrm{div}\,\underline{u})^2\big)\Big],
\end{aligned}\tag{4.8}$$

The study of the asymptotic behavior of (4.8) as $\varepsilon \to 0$ is well known in fluid dynamics as the *low Mach numbers limit.* Namely, in (4.8) ε plays the role of the Mach number. The equations (4.8) are often considered in the isentropic case, where the pressure is assumed to depend only on the density and hence the first two equations (4.8) can be solved independently of the third. In this case it has been proved in[12] that the limiting solution obeys the incompressible Navier-Stokes equations (INS)

$$\begin{aligned} &\text{div } \underline{u} = 0, \\ &\partial_t \underline{u} + (\underline{u} \cdot \nabla_{\underline{x}})\underline{u} = \underline{F} - \nabla_{\underline{x}} p + \eta \Delta \underline{u}, \end{aligned} \tag{4.9}$$

with η the *kinematic viscosity coefficient*. The pressure p appearing in (4.9) is no more given by a state equation, but is itself an unknown of the problem.

We notice that (4.9) is invariant under the scaling

$$\underline{x}' = \varepsilon^{-1}\underline{x}, \quad t' = \varepsilon^{-2}\tau, \quad \underline{u}' = \varepsilon \underline{u}, \quad p' = \varepsilon^2 p. \tag{4.10}$$

Such invariance is a necessary condition in order that an equation be the scaling limit of a more fundamental one.

With the above remarks in mind we can ask the question of the asymptotic behavior under the parabolic space time rescaling of the Boltzmann equation with initial data such that the average velocity $\underline{u}$ is of order ε (low Mach number assumption). In other words, we define

$$f^\varepsilon(\underline{x}, \underline{v}, t) = f(\varepsilon^{-1}\underline{x}, \underline{v}, \varepsilon^{-2}t), \tag{4.11}$$

and assume the force in (1.1) of the form $\varepsilon^3 \underline{F}$.

Equation (1.1), after rescaling becomes

$$\frac{\partial f^\varepsilon}{\partial t} + \frac{1}{\varepsilon}\underline{v} \cdot \frac{\partial f^\varepsilon}{\partial \underline{x}} + \varepsilon F \cdot \frac{\partial f^\varepsilon}{\partial \underline{v}} = \frac{1}{\varepsilon^2} Q(f^\varepsilon, f^\varepsilon). \tag{4.12}$$

We consider the following initial condition:

$$f_0^\varepsilon(\underline{x}, \underline{v}) = m(\underline{v}) + \varepsilon \Phi_0(\underline{x}, \underline{v}), \quad m(\underline{v}) = M(1, \underline{0}, 1; \underline{v}). \tag{4.13}$$

This means that at the lowest order the initial distribution is a global Maxwellian with vanishing average velocity. Therefore

$$\underline{u}_0(\underline{x}) = \varepsilon \int_{\mathbb{R}^3} d\underline{v} \Phi_0(\underline{x}, \underline{v})\underline{v}, \tag{4.14}$$

so the low Mach number assumption is fulfilled. We are also assuming constant initial density and temperature both set equal to one. We remark that the formal part of the argument below can be worked out even in the case of non constant T and $n^{(13)}$, where new interesting phenomena arise, e.g. mass flows induced by temperature gradients, which are not contained in the INS. We present only the case of constant temperature and density for sake of simplicity and because the theorem we state, has been proved only for constant initial density and temperature. The choice of the constants, $T = 1$, $n = 1$, is not restrictive, since can be easily removed by a suitable change of units.

Given the initial condition (4.13) we look for a solution of (4.12) with the same property: we set

$$f^\varepsilon(\underline{x}, \underline{v}, t) = m(\underline{v}) + \varepsilon \Phi^\varepsilon(\underline{x}, \underline{v}, t). \tag{4.15}$$

This assumption is consistent with (4.12) provided that Φ^ε solves the equation

$$\frac{\partial \Phi^\varepsilon}{\partial t} + \frac{1}{\varepsilon}\underline{v} \cdot \frac{\partial \Phi^\varepsilon}{\partial \underline{x}} + F \cdot \frac{\partial m}{\partial \underline{v}} + \varepsilon F \cdot \frac{\partial \Phi^\varepsilon}{\partial \underline{v}} = \frac{1}{\varepsilon^2}\mathcal{L}\Phi^\varepsilon + \frac{1}{\varepsilon}Q(\Phi^\varepsilon, \Phi^\varepsilon). \tag{4.16}$$

We try to solve this equation using a kind of Hilbert expansion along the same lines presented for the Euler limit. We look for a solution of (4.16) in the form

$$\Phi^\varepsilon = \sum_{k=0}^{\infty} \varepsilon^k \phi_k \tag{4.17}$$

and determine the unknown coefficients ϕ_k, $k = 0, \dots,$ by imposing equality of terms of the same order in ε. By plugging (4.17) into (4.16) we get the conditions:

$$
\begin{aligned}
&\varepsilon^{-2}: && \mathcal{L}\phi_0 = 0, \\
&\varepsilon^{-1}: && \mathcal{L}\phi_1 = \underline{v} \cdot \nabla_{\underline{x}}\phi_0 - Q(\phi_0, \phi_0) \\
&\varepsilon^{0}: && \mathcal{L}\phi_2 = \partial_t\phi_0 + \underline{v} \cdot \nabla_{\underline{x}}\phi_1 + \underline{F} \cdot \frac{\partial m}{\partial \underline{v}} - 2Q(\phi_0, \phi_1), \\
& && \dots \\
&\varepsilon^{k}: && \mathcal{L}\phi_{k+2} = Z_k(\phi_0, \phi_1, \dots, \phi_{k+1}) \\
& && \dots,
\end{aligned}
\tag{4.18}
$$

where

$$
Z_k(\phi_0, \phi_1, \dots, \phi_{k+1}) = \partial_t\phi_k + \underline{v} \cdot \nabla_{\underline{x}}\phi_{k+1} + \underline{F} \cdot \frac{\partial \phi_{k-1}}{\partial \underline{v}} - \sum_{\substack{(h,h'):\ h,h' \geq 0 \\ h+h'=k+1}} Q(\phi_h, \phi_{h'}). \tag{4.19}
$$

The first condition and (2.4) imply

$$
\phi_0(\underline{x}, \underline{v}, t) = m(\underline{v})\Big[\rho(\underline{x}, t) + \underline{v} \cdot \underline{u}(\underline{x}, t) + (\frac{v^2 - 3}{2})\vartheta(\underline{x}, t)\Big], \tag{4.20}
$$

where we have denoted by ρ and ϑ the first order corrections to the constant density and temperature and by $\underline{u}$ the first non vanishing contribution to the stream velocity vector, which is of order ε by construction.

The other equations (4.18) are again in the form (3.10). Therefore, in order to solve them, we have to check that the right hand sides satisfy the solvability condition of orthogonality to the null space of $\mathcal{L}$. To find ϕ_1 we need

$$
\int_{\mathbb{R}^3} d\underline{v}\chi_\alpha(\underline{v})\underline{v} \cdot \nabla_{\underline{x}}\phi_0(\underline{x}, \underline{v}, t) = 0, \quad \alpha = 0, \dots, 4. \tag{4.21}
$$

It is easy to check, by explicit Gaussian integrations that the above conditions are satisfied if and only if

$$\text{div}\ \underline{u} = 0, \quad \nabla_{\underline{x}}(\rho + \vartheta) = 0. \tag{4.22}$$

The first equality of (4.22) is the *incompressibility* condition. The second one, which ensure the constancy of the pressure up to the first order, is called the Boussinesq condition[(14)]. Once we choose $\underline{u}$, ρ and ϑ so that (4.22) is satisfied, the second equation of (4.18) can be solved and we get

$$\phi_1 = \mathcal{L}^{-1}[\underline{v} \cdot \nabla_{\underline{x}}\phi_0] - \mathcal{L}^{-1}[Q(\phi_0, \phi_0)] + \chi_1, \tag{4.23}$$

with $\chi_1 \in \text{Null}\ \mathcal{L}$, and hence of the form

$$\chi_1(\underline{x}.\underline{v}, t) = m(\underline{v})\Big[\rho_1(\underline{x}, t) + \underline{v} \cdot \underline{u}_1(\underline{x}, t) + (\frac{v^2 - 3}{2})\vartheta_1(\underline{x}, t)\Big]. \tag{4.24}$$

The quantity $\mathcal{L}^{-1}[\underline{v} \cdot \nabla_{\underline{x}}\phi_0]$ can be computed using the same arguments employed to get (4.4) and the result is

$$\mathcal{L}^{-1}[\underline{v} \cdot \nabla_{\underline{x}}\phi_0] = -\Psi_1 \sum_{i,j=1}^{3} A_{i,j}\frac{\partial u_i}{\partial x_j} - \Psi_2 \sum_{i=1}^{3} B_i \frac{\partial \vartheta}{\partial x_i}. \tag{4.25}$$

To evaluate the second term in (4.23) we note that, by simple calculations one can get

$$\begin{aligned} &- \mathcal{L}^{-1}[Q(\phi_0, \phi_0)] = \frac{1}{2}\widetilde{\mathcal{P}}^{\perp}\left(\frac{\phi_0^2}{m}\right) \\ &= \frac{1}{2}m \sum_{i,j=1}^{3} A_{i,j}u_i u_j + m\vartheta \sum_{i=1}^{3} B_i u_i + \frac{1}{2}\vartheta^2 \widetilde{\mathcal{P}}^{\perp}\Big[(\frac{v^2 - 3}{2})^2 m\Big], \end{aligned} \tag{4.26}$$

The evaluation of ϕ_1 is thus complete, up to χ_1. For this we need to evaluate ϕ_2. This requires the solvability condition for the third of (4.18). The expression just obtained for

ϕ_1 and the evaluation of some Gaussian integrals provides the explicit solvability condition:

$$
\begin{aligned}
&\partial_t \rho + \mathrm{div}\ \underline{u}_1 = 0,\\
&\partial_t \underline{u} + \underset{\sim}{u} \cdot \nabla_{\underset{\sim}{x}}\, \underline{u} = \underline{F} - \nabla_{\underline{x}}[\frac{u^2}{2} + \rho\vartheta + \rho_1 + \vartheta_1] + \eta\Delta\underline{u},\\
&\frac{3}{2}\partial_t\vartheta + \mathrm{div}\ \underline{u}_1 + \underline{u}\cdot\nabla_{\underline{x}}\vartheta = \bar{\kappa}\Delta\vartheta,
\end{aligned}
\tag{4.27}
$$

where η and $\bar{\kappa}$ are the the shear viscosity coefficient and the heat conduction coefficient given by (4.6) and (4.7), evaluated at temperature $T = 1$. We put

$$
p = \rho\vartheta + \rho_1 + \vartheta_1 + \frac{u^2}{2}
\tag{4.28}
$$

and replace div $\underline{u}_1$ in the third equation with $-\partial_t\rho = \partial_t\vartheta$. Note that the second eqn. of (4.22) only states that $\rho + \vartheta$ is constant in space. However, from this, it is easy to check that, with the periodic boundary conditions assumed here, also the time derivative of $\rho+\vartheta$ must vanish. Therefore we get the following system in the unknowns $\underline{u}$, ϑ and p:

$$
\begin{aligned}
&\mathrm{div}\ \underline{u} = 0,\\
&\partial_t \underline{u} + \underset{\sim}{u} \cdot \nabla_{\underset{\sim}{x}}\, \underline{u} = \underline{F} - \nabla_{\underline{x}} p + \eta\Delta\underline{u},\\
&\frac{5}{2}\partial_t\vartheta + \underline{u}\cdot\nabla_{\underline{x}}\vartheta = \bar{\kappa}\Delta\vartheta,
\end{aligned}
\tag{4.29}
$$

The first two equations are just the INS equations (4.9) for the unknowns $\underline{u}$ and p. As in the derivation of the Euler equations they arise as solvability conditions for the coefficients of the power series expansion (4.17). The third of (4.29) is the heat equation with a convective term due to the streaming velocity: its solution does not influence $\underline{u}$.

Suppose that, given appropriate initial conditions, a unique solution can be found for the system (4.29), at least in some time interval $(0, t_0]$. Then, by the Boussinesq condition, ρ is determined up to an additive constant and hence the first term in the expansion ϕ_0 is completely determined. The second term, ϕ_1, is determined up to χ_1, but with ρ_1, $\underline{u}_1$ and ϑ_1 not completely arbitrary because div $\underline{u}_1$ is prescribed by the first of (4.27) and the sum $\rho_1 + \vartheta_1$ is prescribed by (4.28). The procedure can be continued to determine the next term f_2, by taking advantage of the partial arbitrariness of χ_1 to satisfy the solvability condition for ϕ_2. In fact we are now in the same position as we were in the previous step. The only difference is that the equations (4.29) for ρ, $\underline{u}$ and ϑ are non linear, while those for ρ_1, $\underline{u}_1$ and ϑ_1 are linear. The argument goes on as for the Euler case and the coefficients ϕ_k can be determined for any k.

As in the Euler case, nothing is known about the convergence of the series. Also in this case it is possible to introduce a suitable truncation of the series and try to estimate the remainders. To do this, fix $N > 0$, and put

$$\Phi^\varepsilon = \sum_{k=0}^{N} \varepsilon^k \phi_k + \varepsilon^m R, \tag{4.30}$$

with ϕ_k, $k = 0, \dots, N$ computed according to the previous procedure and R, depending on ε, to be chosen so that Φ^ε solves (4.16). For sake of definiteness, assume $N = 6$ and $m = 3$. The existence of the function R is the main ingredient to prove the following

Theorem 4.1. *Assume that the initial value problem for the system (4.29) has a unique, sufficiently smooth solution in a time interval* $(0, t_0]$. *Then there is an* $\varepsilon_0 > 0$ *such that for* $\varepsilon < \varepsilon_0$, *there exists a smooth solution* $\Phi^\varepsilon(t)$ *to (4.16), having the property that*

$$\sup_{t \in (0,t_0]} \|\Phi^\varepsilon(t) - \phi_0(t)\| \le C\varepsilon \tag{4.31}$$

for some $C > 0$.

Theorem 4.1 has been proved in[13]. We remark that the existence of a solution to the hydrodynamical equations is among the hypotheses of the theorem, and hence the result inherits, as in the Euler case, the locality in time of the hydrodynamical problem. A different approach has been considered in[15], where the aim is to get existence of global weak solutions to the hydrodynamical equations as a consequence of the existence of weak solutions of the rescaled Boltzmann equation, proved in[16]. This interesting program is, to our knowledge, not yet complete.

Previous arguments are also applicable to derive the incompressible Euler equations. In fact it is enough to consider shorter time scales, such that the viscosity effects are still negligible. In other words, instead of looking at times $\varepsilon^{-2}t$, one has to look at times $\varepsilon^{-1-\sigma}t$, with $0 \le \sigma \le 1$, and scale the stream velocity as ε^{σ} and the force as $\varepsilon^{1+2\sigma}\underline{F}$. The cases $\sigma = 0$ and $\sigma = 1$ correspond to compressible Euler and incompressible Navier-Stokes equations, as already discussed. In general it is convenient to consider only values of σ such that $1 + \sigma^{-1}$ is an integer $N > 2$. With the notation $\tilde{\varepsilon} = \varepsilon^{\sigma}$, the rescaled Boltzmann equation becomes

$$\frac{\partial f^{\tilde{\varepsilon}}}{\partial t} + \frac{1}{\tilde{\varepsilon}}\underline{v} \cdot \frac{\partial f^{\tilde{\varepsilon}}}{\partial \underline{x}} + \tilde{\varepsilon} F \cdot \frac{\partial f^{\tilde{\varepsilon}}}{\partial \underline{v}} = \frac{1}{\tilde{\varepsilon}^{N}} Q(f^{\tilde{\varepsilon}}, f^{\tilde{\varepsilon}}). \tag{4.32}$$

We assume the low Mach number condition in the form

$$f^{\tilde{\varepsilon}} = m + \tilde{\varepsilon}\Phi^{\tilde{\varepsilon}}, \tag{4.33}$$

with $\Phi^{\tilde{\varepsilon}}$ satisfying

$$\frac{\partial \Phi^{\tilde{\varepsilon}}}{\partial t} + \frac{1}{\tilde{\varepsilon}}\underline{v} \cdot \frac{\partial \Phi^{\tilde{\varepsilon}}}{\partial \underline{x}} + F \cdot \frac{\partial m}{\partial \underline{v}} + \tilde{\varepsilon} F \cdot \frac{\partial \Phi^{\tilde{\varepsilon}}}{\partial \underline{v}} = \frac{1}{\tilde{\varepsilon}^{N}} \mathcal{L}\Phi^{\tilde{\varepsilon}} + \frac{1}{\tilde{\varepsilon}^{N-1}} Q(\Phi^{\tilde{\varepsilon}}, \Phi^{\tilde{\varepsilon}}). \tag{4.34}$$

Proceeding as before with the power series expansion, we can get the incompressible Euler equations

$$\begin{aligned} &\operatorname{div} \underline{u} = 0, \\ &\partial_t \underline{u} + (\underline{u} \cdot \nabla_{\underline{x}})\underline{u} = -\nabla_{\underline{x}} p + \underline{F}. \end{aligned} \tag{4.35}$$

The above ideas can be generalized in several directions. One of them concerns the scaling of the force $\underline{F}$. We have seen before that the low Mach number assumption and the parabolic scaling are consistent with a force of order ε^3. A remarkable exception to this rule is provided by gravity. An additional force of order ε^2, $\varepsilon^2 \underline{G}$, with $\underline{G}$ a constant vector, can be permitted in above discussion. The main difference is that in this case a buoyancy force $-\alpha\vartheta\underline{G}$, $\alpha > 0$, is also present in the second line of (4.28). Such a modification is particularly interesting when the top and bottom boundaries are not periodic, but at different temperatures (Benard problem). In this case the relevant adimensional parameter is the Rayleigh number, which depends on the viscosity coefficient, on $\underline{G}$ and the temperature difference. The scaling has to be arranged in order to leave the Rayleigh number invariant. For sufficiently large values of the Rayleigh number, well known convective instabilities arise. The same kind of instabilities have been found in numerical simulations for the Boltzmann equation, for sufficiently small values of ε[17]. Indeed, the numerical simulations show that for larger ε bigger critical values of the Rayleigh number arise and the usual instability portrait is obtained, also with some quantitative agreement, only for values of ε very small, where hydrodynamical solutions are a good approximation to the Boltzmann solutions.

The mathematical analysis of the Benard problem requires the extension of previous arguments to more general boundary condition, including the case of the thermal contact with a reservoir and to forces of order ε^2 of conservative type. Such extensions have been discussed in[18], where the analog of Theorem 4.1 has been established, together with results on the existence of the stationary solution matching the hydrodynamical conductive solution. It should be possible to extend these results to the convective solutions and study the stability properties as well as bifurcation phenomena at the kinetic level.

Stationary solutions corresponding to special geometries are the only examples where it has been possible to get some information on the problem of the derivation of the compressible Navier-Stokes equations from the Boltzmann equation. This is the case of a gas in a slab of macroscopic size between two infinite parallel plates in contact with thermal reservoirs at specified temperatures. There also acts on the gas a constant force parallel to the plates, of size $\varepsilon^2 \underline{F}$. The geometry of the problem is clearly one-dimensional. The macroscopic equations are the stationary one-dimensional compressible Navier-Stokes equations. In[19],[20] it is proved that, if $|\underline{F}|$ is sufficiently small, the local Maxwellian with parameters solving the hydrodynamical equations approximates the stationary solution in the sense of Theorem 3.1. The first paper deals with the case when the two plates are at rest and at the same temperature, while in the second such restrictions are dropped.

Open problems.

We conclude this section on the derivation of the hydrodynamical equations from the Boltzmann equation by mentioning a few of the many open problems in this field. To this end we fix a positive small ε, say $\epsilon = 10^{-5}$, and consider the time evolution of a hydrodynamical initial datum of the type (3.1) on the time scales $\tau = \varepsilon^{-\alpha} t$, for different values of α.

First of all, since the initial datum is almost constant on spatial regions of order ε^{-1}, the time evolution for $0 \leq \alpha < 1$ is essentially ruled by the homogeneous Boltzmann equation that drives the system locally to the equilibrium given by a local Maxwellian (2.10). This initial layer analysis has been carried out rigorously in[9] for $\alpha = 0$ but should be correct for any $0 < \alpha < 1$ too.

The time scale $\alpha = 1$, corresponding to the Euler scaling, has been discussed extensively in Section 3 and from the rigorous point of view we know that Theorem 3.1 holds. The main limitations of this result are the following: 1) it holds up to times t where the Euler equations have smooth solutions; 2) the space domain is assumed to be a torus.

It is known that, at least in one space dimension, the Euler equations develop singularities in finite times and the arguments proving Theorem 3.1 fail when this happens. The reason is that big space gradients develop in the process of creating singularities and the Navier-Stokes corrections discussed in Section 4 may become important.

The same problem arises when one considers more general boundary conditions than the periodic ones. Large gradients close to the boundary prevent the use of the Hilbert expansion uniformly in the domain. Therefore the expansion, even if still correct in the bulk, needs to be modified close to the boundary (or to the shock) by boundary layer (or shock layer) expansions to take care of big space variations. We refer to[21] and references quoted therein for a more extensive discussion of the role of such layers.

Unfortunately only few rigorous results are available on this at the present for the Boltzmann equation although there are some explicit solutions for some particle systems which give rise to the Burgers equation[22].

The only mathematical results we are aware of for fluids are for the case of stationary solutions in a slab discussed in[19],[20]. In these cases it is necessary and possible to use a mixed expansion: Hilbert expansion in the bulk and boundary layer expansion to fit the boundary conditions. It then comes out that the Navier-Stokes terms are essential: in fact, under the Euler equations no stationary solutions are possible because there is no mechanism to dissipate the energy provided to the system by the external force. Such a mechanism is present in the Navier-Stokes equations and the energy is dissipated at the boundaries by the viscosity, so that a stationary solution can be obtained. The size of the Knudsen boundary layer is of order ϵ on the macroscopic scale and does not affect the hydrodynamical equations. Other situations, such as the flow past an obstacle, can be studied along the same lines, but the technical difficulties are at the moment very great.

From the above considerations it is clear that going to longer time scales, $\alpha > 1$, is in general very difficult. Only the low Mach number situation has been studied with some success. Assume $1 < \alpha < 2$. If the initial datum has mean velocity of order $\varepsilon^{\alpha-1}$ while the density and temperature differ from constant values by corrections of order $\varepsilon^{\alpha-1}$, then, in the case of periodic boundary conditions the limiting behavior is, as discussed in Theorem 4.1, ruled by the incompressible Euler equations. Also in this case we have restrictions about the time of convergence, corresponding to development of singularities in the Euler solutions, if any. Boundary conditions may create large gradients which prevent the validity of the result close to the boundary. In both cases the Navier-Stokes corrections cannot be neglected, but the detailed analysis is not available.

The case $\alpha = 2$ is special, because the Navier-Stokes corrections are as big as the convective terms and the limiting equation is the incompressible Navier-Stokes equation. In this case no-slip boundary conditions can be included in the expansion, as shown in[18], and the boundary layer corrections are concentrated in a layer of order ε close to the boundary. Hence the solutions to the Boltzmann equation is not of hydrodynamical type in this region but matches the hydrodynamical solution in the bulk.

Finally, one should consider also the very long time scale $\alpha > 2$. Apriori higher order terms of the Chapman-Enskog expansion could become dominant on such time scales. This does not seem to be the case, because, as proved in[23], the presence of some of them makes the global equilibrium unstable. This corresponds to a violation of the positivity of the entropy production, a rather unpleasant feature which excludes the possibility that they attract the kinetic equation in some regime. On the other hand, the Navier-Stokes terms already contain the necessary mechanism to eventually drive isolated system to global equilibrium, so that, from the qualitative point of view, it does not seem necessary to conjecture more corrections to understand the approach to equilibrium of the system, although corrections may be required to get better agreement with experiments. Unfortunately, no precise statement about this has been proved at the moment.

5. Hydrodynamical limit for particle systems.

We describe now briefly how the ideas presented in the previous sections can be extended formally to particle systems with fully microscopic dynamics. The rigorous proofs however are not available for deterministic systems and the introduction of some artificial stochasticity is necessary, at the present time, to achieve them. The evolution X_t of a system of N particles of mass m in a box Λ under the action of a two body smooth potential is given by Newton's equations:

$$\begin{aligned} &\dot{\underline{r}}_j = \underline{v}_j, \\ &m\dot{\underline{v}}_j = -\Big[\sum_{k\neq j} \nabla_{\underline{r}_j} V(|\underline{r}_j - \underline{r}_k|)\Big], \qquad j = 1, \ldots, N. \end{aligned} \tag{5.1}$$

<u>Euler scaling</u>.

In order to get the Euler equations, we use the same scaling we adopted for the Boltzmann equation. We assume the particles to be in a box Λ of macroscopic size, so that its volume is of order ε^{-3}, e.g. in a three-dimensional torus of size ε^{-1}. To have a finite density, we also choose the number of particles $N = \varepsilon^{-3}$. This choice of N has to be compared with the choice $N = \varepsilon^{-2}$ adopted in Section 2 for the kinetic limit, where density has to be small. We set $\underline{x} = \varepsilon \underline{r}$, and look at $X_{\varepsilon^{-1}t}$, the solution of (5.1) at time $\varepsilon^{-1}t$. Let $\underline{x}_j^\varepsilon(t) = \varepsilon \underline{r}_j(\varepsilon^{-1}t)$, $\underline{v}_j^\varepsilon(t) = \underline{v}_j(\varepsilon^{-1}t)$ denote the position and velocity of the j-th particle on the macroscopic scale. Since we shall only use macroscopic variables, we drop the superscript ε from now on.

After this *hyperbolic rescaling* (5.1) become

$$\begin{aligned} &\dot{\underline{x}}_j = \underline{v}_j, \\ &m\dot{\underline{v}}_j = -\Big[\sum_{k\neq j} \nabla_{\underline{x}_j} V\Big(\frac{|\underline{x}_j - \underline{x}_k|}{\varepsilon}\Big)\Big], \qquad j = 1, \ldots, N. \end{aligned} \tag{5.2}$$

The hydrodynamical fields will be obtained as limits of the measure-valued fields

$\xi_\alpha^{(N)}(d\underline{x},t)$, $\alpha=0,\dots,4$, defined as:

$$
\begin{aligned}
&\xi_0^{(N)}(d\underline{x},t)=n^{(N)}(d\underline{x},t)\stackrel{def}{=}\frac{m}{N}\sum_{j=1}^{N}\delta_{\underline{x}_j(t)}(d\underline{x}),\\
&\underline{\xi}^{(N)}(d\underline{x},t)=\underline{p}^{(N)}(d\underline{x},t)\stackrel{def}{=}\frac{1}{N}\sum_{j=1}^{N}\underline{v}_j(t)\delta_{\underline{x}_j(t)}(d\underline{x}),\\
&\xi_4^{(N)}(d\underline{x})=e^{(N)}(d\underline{x},t)\stackrel{def}{=}\frac{1}{N}\sum_{j=1}^{N}e_j(t)\delta_{\underline{x}_j(t)}(d\underline{x}),
\end{aligned}
\tag{5.3}
$$

where $e_j(t)$ is the energy of the j-particle, given by

$$
e_j\stackrel{def}{=}\frac{1}{2}\Big[m\underline{v}_j^2+\sum_{k\neq j}V\Big(\frac{|\underline{x}_j-\underline{x}_k|}{\varepsilon}\Big)\Big]. \tag{5.4}
$$

One can evaluate the time derivatives of the fields (5.3) using (5.2). It is easy to obtain conservation laws of the form

$$
\frac{d}{dt}\xi_\alpha^{(N)}=\operatorname{div}\,\underline{J}_\alpha^N,\quad \alpha=0,\dots,4, \qquad (5.5),
$$

where the *currents* $\underline{J}_\alpha=(J_{\alpha,\ell},\ell=1,2,3)$ have the form

$$
\begin{aligned}
J_{0,\ell}^{(N)}(d\underline{x},t)&=p_\ell^{(N)}(d\underline{x},t),\\
J_{n,\ell}^{(N)}(d\underline{x},t)&=\frac{1}{N}\sum_{j=1}^{N}\Big[v_{j,\ell}(t)v_{j,n}(t)+\sum_{k\neq j}\Psi_{n,\ell}(\frac{\underline{x}_j(t)-\underline{x}_k(t)}{\varepsilon})\Big]\delta_{\underline{x}_j(t)}(d\underline{x}),\\
J_{4,\ell}^{(N)}(d\underline{x},t)&=\frac{1}{N}\sum_{j=1}^{N}\Big[e_j(t)v_{j,\ell}(t)\\
&\quad+\sum_{k\neq j}\sum_{n=1}^{3}\Psi_{n,\ell}(\frac{\underline{x}_j(t)-\underline{x}_k(t)}{\varepsilon})(v_{j,n}(t)+v_{k,n}(t))\Big]\delta_{\underline{x}_j(t)}(d\underline{x}),
\end{aligned}
\tag{5.6}
$$

where

$$
\Psi_{n,\ell}(\underline{x})=x_\ell\partial_{x_n}V(|x|). \tag{5.7}
$$

Similarly to the Boltzmann case, the currents cannot be expressed in terms of the hydrodynamical fields, so one would like to close such equations using the local equilibrium.

This, in particle systems is not just a local Maxwellian, but a local Gibbs state, defined as a probability distribution on the Γ-space $(\Lambda \times \mathbb{R}^3)^N$, with a density with respect to the Liouville measure given by

$$G^{(N)}(\underline{x}_1, \underline{v}_1, \dots, \underline{x}_N, \underline{v}_N; t) = Z_N^{-1} \prod_{j=1}^{N} \frac{n(\underline{x}_j, t))}{(2\pi T(\underline{x}_j, t))^{3/2}} \exp\Big[-\frac{(\underline{v}_j - \underline{u}(\underline{x}_j, t))^2}{2T(\underline{x}_j, t)} + \frac{1}{T(\underline{x}_j, t)} \sum_{k \neq j} V\Big(\frac{|\underline{x}_j - \underline{x}_k|}{\varepsilon}\Big)\Big], \tag{5.8}$$

where Z_N is a normalization factor and $n > 0$, $\underline{u}$, $T > 0$ are functions parameterizing the family of the local Gibbs states.

We formulate the *local equilibrium assumption* as follows: as $\varepsilon \to 0$ (and consequently $N \to \infty$) the probability distribution on Γ is "well approximated" by a local Gibbs state with smooth parameters. We leave vague the meaning of well approximated.

Given such assumption, the fields $\xi_\alpha^{(N)}$ and the currents $J_{n,\ell}^{(N)}$ converge, as $N \to \infty$, by the *law of large numbers*, weakly in probability to their averages, which can be computed using the local Gibbs state and hence can be expressed in terms of the fields n, $\underline{u}$ and T. Therefore, the local conservation laws (5.5) become in the limit the Euler equations (3.20), but the equations of state for the pressure and the energy

$$e = e(n, T), \quad P = P(n, T), \tag{5.9}$$

are now those corresponding to an interacting gas rather than to the ideal gas. We skip the details of such a derivation and refer the reader to the paper[24] and to the book[2] and references quoted therein.

As for the Boltzmann equation, the derivation of the Euler equations could be formally justified by an expansion in powers of ε similar to the Hilbert expansion. However, a rigorous justification is not available, because we do not know how to prove the analog of Theorem 3.1 for the N particle system. The only proof of the validity of the Euler equations for deterministic systems is, at our knowledge, contained in[25], where it is shown that the one dimensional hard rods model converges to a kind of degenerate hydrodynamics. A deep clarification of the situation has been given in[3]. In that paper it is shown that the derivation of the Euler equations can be really accomplished, at least in a region of the parameters where there is no phase transition, provided that one can prove a *local ergodic theorem*. For deterministic systems such a proof is not available with the present techniques, but in[3] it is shown that one can slightly modify the system to achieve such a result. In fact the authors add to Newton's equations a stochastic noise, small enough not to modify the macroscopic equations, but sufficient to provide the local ergodicity that they need to complete the proof. A technical limitation in the paper is that they cannot deal with high velocities so that, instead of the Hamiltonian of a classical particle system, they have to deal with Hamiltonians growing linearly at high velocities, an example being the relativistic Hamiltonian.

Navier-Stokes scaling.

For the derivation of the Navier-Stokes equations we may also proceed as for the Boltzmann case. By the considerations of Section 4, we confine ourselves to the incompressible case. We look at $X_{\varepsilon^{-2}t}$, the solution of (5.1) at time $\varepsilon^{-2}t$ and adopt the same notations as before, namely $\underline{x}^\varepsilon_j(t) = \varepsilon \underline{r}_j(\varepsilon^{-2}t)$, $\underline{v}^\varepsilon_j(t) = \underline{v}_j(\varepsilon^{-2}t)$ denote the position and velocity of the j-th particle on the macroscopic scale. Again, we drop the superscript ε from now on.

After this *parabolic rescaling* (5.1) become

$$\begin{aligned} &\dot{\underline{x}}_j = \varepsilon^{-1}\underline{v}_j, \\ &m\dot{\underline{v}}_j = -\varepsilon^{-1}\Big[\sum_{k\neq j} \nabla_{\underline{x}_j} V\Big(\frac{|\underline{x}_j - \underline{x}_k|}{\varepsilon}\Big)\Big], \qquad j = 1, \dots, N. \end{aligned} \tag{5.10}$$

The low Mach number assumption is embedded by choosing an initial distribution given by a local Gibbs state which is a perturbation of order ε of a global Gibbs state.

$$\begin{aligned} &G_0^{(N)}(\underline{x}_1, \underline{v}_1, \dots, \underline{x}_N, \underline{v}_N) \\ &= Z_N^{-1} \prod_{j=1}^N \frac{n_0(\underline{x}_j))}{(2\pi T_0(\underline{x}_j))^{3/2}} \exp\Big[-\frac{(\underline{v}_j - \underline{u}_0(\underline{x}_j))^2}{2T_0(\underline{x}_j)} + \frac{1}{T_0(\underline{x}_j)} \sum_{k\neq j} V\Big(\frac{|\underline{x}_j - \underline{x}_k|}{\varepsilon}\Big)\Big], \end{aligned} \tag{5.11}$$

with the parameters n_0, $\underline{u}_0$ and T_0 such that

$$n_0(\underline{x}) = 1 + \varepsilon\rho(\underline{x}), \quad \underline{u}(\underline{x}) = \varepsilon \underline{w}_0(\underline{x}), \quad T_0(\underline{x}) = 1 + \varepsilon\vartheta(\underline{x}), \tag{5.12}$$

where the reference temperature and density are set equal to unity without loss of generality. Even if one could prove that the distribution stays close to a local Gibbs state of the above form, this would not be sufficient to find the transport coefficients, which are crucial to get the Navier-Stokes equations, because their knowledge goes beyond the local equilibrium assumption: as we learned in dealing with the Boltzmann equation, in order to get them one has to know the corrections to local equilibrium. This can be achieved by an expansion technique inspired by the one presented in Section 4 and, as a result, one can obtain, at least heuristically, the INS equations and the heat equation as given by (4.29), with the factor 5/2 in the third of eqn. (4.29) replaced by c_p the *specific heath at constant pressure* of the fluid and the transport coefficients η and k given by the Green-Kubo formulas. We refer to[26] for details on this.

We remark that above discussion is totally formal and not even a rigorous stochastic version of it is available, as in the Euler case. The reason for that is related to the need of determining the transport coefficients in terms of the Green-Kubo formulas. Giving a rigorous sense to them is a highly non trivial task which has been accomplished in some cases with the help of the Varadhan's *non-gradient method*[27]. At the moment no effective application of this method to problems where the space variables are continuous is known.

For this reason we try to modify the deterministic model given by Newton's equations not by just adding some stochastic noise, but rather considering a particle system on a lattice. This means that particles are allowed to occupy only lattice sites and move from one site to another via random jumps. The most famous of such models is the *simple exclusion process* which we describe briefly because it is the basis of the construction of the model for the Navier-Stokes equations. We refer the reader to[28],[29]for a more extensive discussion and references.

Given $\varepsilon > 0$, consider the lattice Λ of points with integer coordinates contained in $[0, \varepsilon^{-1}]^d$ (we identify the edges in order to get periodic boundary conditions). On each point of Λ we allow at most one particle. $\eta(\underline{x}) \in \{0, 1\}$, is the occupation number at $\underline{x} \in \Lambda$. The dynamics is prescribed by giving the numbers $p_{\underline{e}} \geq 0$, $\sum_{\underline{e}} p_{\underline{e}} = 1$, where $\underline{e}$ are vectors in $\mathbb{Z}^d$. The dynamics is roughly described as follows: at all sites $\underline{x} \in \Lambda$ there are independent Poisson clocks which ring with intensity 1. When the first one rings, say at $\underline{x}$, a direction $\underline{e}$ is chosen with probability $p_{\underline{e}}$ and the particle at $\underline{x}$, if any, jumps in $\underline{x} + \underline{e}$, provided that the arrival site is empty. If $\underline{x}$ is empty or $\underline{x} + \underline{e}$ is already occupied, nothing happens. The evolution continues after this with the same rule. The only conservation law for this very simple system is the conservation of particle number.

Euler and Navier-Stokes limits can be investigated. On the Euler scale[30] the behavior of the simple exclusion process is determined by the inviscid Burgers equation, even when it develops shocks, the particle system following the solution which satisfies the entropy condition. The Navier-Stokes limit, in the case $p_{\underline{e}} = p_{-\underline{e}}$ (symmetric simple exclusion) is simply the diffusion equation with unit diffusion coefficient. When the symmetry condition is violated, no general answer is available, as in the case of the compressible Navier-Stokes fluid. However, if an analog of the low Mach number condition is satisfied, namely, if the initial density differs from a constant by terms of order ε, and if $d \geq 3$, the limiting behavior can be proved[31] to be determined by the viscous Burgers equation with a diffusion coefficient given by a Green-Kubo formula.

In order to get the usual hydrodynamical equations one needs more that just one conservation law. This is usually achieved by considering particles which, like real particles have a velocity. For stochastic particles on the lattice, the notion of velocity is replaced by one of drift. In the simple exclusion case, the drift is just the vector $\sum_{\underline{e}} \underline{e} p_{\underline{e}}$. Therefore, we fix the space dimension $d = 3$ and consider a set of three-dimensional vectors $\mathcal{V}$ of cardinality S, whose elements $\underline{v}$ have all the same modulus and are all the possible velocities of the model. The set $\mathcal{V}$ is assumed invariant under reflections and permutations of axes. For each $\underline{v} \in \mathcal{V}$ we consider a species of particles, labeled by $\underline{v}$, which jumps from one site to another according to a simple exclusion process with drift $\underline{v}$. For example we can choose $p_{\underline{e}}(\underline{v}) = \chi + (1/2)\underline{v} \cdot \underline{e}$, with $\chi > 0$ sufficiently big to ensure that $p_{\underline{e}}(\underline{v}) > 0$. The exclusion rule applies only to particles of the same species and the jump of the different species are independent of each other. Now we have S conserved quantities, the number of particles of each species, but they are too many. Therefore, we introduce a *collision process* to reduce the conserved quantities to the hydrodynamical conservation laws. The collision process is defined as follows: we say that a quadruple $q = (\underline{v}, \underline{w}, \underline{v}', \underline{w}')$ is a possible collision if the condition

$$\underline{v} + \underline{w} = \underline{v}' + \underline{w}', \tag{5.13}$$

i.e. the conservation of momentum is satisfied. Note that the conservation of energy is automatically satisfied because the velocities have the same modulus. In all the sites $\underline{x} \in \Lambda$ there are other Poisson clocks, independent of those used for the exclusion. When the first one rings, a possible collision, say $q = (\underline{v}, \underline{w}, \underline{v}', \underline{w}')$, is chosen at random, with equal probability and the collision q happens, provided that there is one particle with velocity $\underline{v}$ and one with velocity $\underline{w}$ at $\underline{x}$ and there are no particles with velocity $\underline{v}'$ and $\underline{w}'$ in $\underline{x}$. In this case after the collision the particles with velocities $\underline{v}$ and $\underline{w}$ disappear from the site $\underline{x}$, and particles with velocities $\underline{v}'$ and $\underline{w}'$ appear. After this nothing happens till the next ring of a Poisson clock. Such collisions are constructed in order to get the total particle number and the total momentum as the only conserved quantities. This is true only for suitable choices of the velocity set $\mathcal{V}$. In[4] it is proved that by choosing $\mathcal{V}$ as the set of vectors $(\pm 1, \pm 1, \pm\omega)$, up to permutation of axes, with ω irrational, the only conserved quantities are the total particle number and the total momentum. A formal calculation shows that the transport terms in the hydrodynamical equations for this system do not coincide with those of the Navier-Stokes equations due the presence of an extra Burgers type term. Its coefficient depends on ω and can be set equal to 0 by a suitable choice of the value of ω.

The results in[4] consist of two parts. The first part is about the existence of the transport coefficients. The currents $\underline{w}_\alpha$, $\alpha = 0, \ldots, 3$, the analog of the currents for the deterministic system, can be decomposed in a component proportional to the gradients of the conserved quantities, whose coefficients are the transport coefficients, and a component fast oscillating, which does not contribute to the hydrodynamical equations. This is the basic content of the non-gradient method of Varadhan, which, as shown in[4], can be successfully applied to a rather general class of models, including the model described above. The transport coefficients are obtained via some variational formulas which are equivalent to the Green-Kubo formulas, whose validity is therefore established. The viscosity matrix obtained in this way is not as isotropic as the one obtained from the Boltzmann equation and the one expected for Newtonian particle systems because the presence of the lattice creates some anisotropy, a phenomenon well known in the theory of cellular automata lattice gases[32]. We remark that the model in[4] closely resembles the models considered in the theory of cellular automata. The basic difference is that the jumps of the particles in those models are deterministic and they happen all simultaneously at integer values of the (microscopic) time rather that at stochastic Poisson times.

The second part of the result of[4] is a statement similar to the one of Theorems 3.1 and 4.1. Assume that a smooth solution to the initial value problem for the limiting equations does exist for some finite, not necessarily small time interval. Then, starting with an initial distribution for the model whose mean velocity is of order ε and total density constant up to terms of first order in ε, it is proved that in the stochastic evolution, after parabolic rescaling, the measure valued random fields

$$
\begin{aligned}
\nu_0^\varepsilon(\underline{z},t) &= \varepsilon^3 \sum_{\underline{x}\in\Lambda_L} \delta(\underline{z}-\varepsilon\underline{x}) I_0(\underline{x},t) \\
\nu_\alpha^\varepsilon(\underline{z},t) &= \varepsilon^2 \sum_{\underline{x}\in\Lambda_L} \delta(\underline{z}-\varepsilon\underline{x}) I_\alpha(\underline{x},t), \quad \alpha = 1,\dots,3,
\end{aligned}
\tag{5.14}
$$

converge, as $\varepsilon \to 0$, weakly in probability, to a constant for $\alpha = 0$ and to the solution of the INS equations for $\alpha = 1,\dots,3$. Here $I_\alpha(\underline{x},t)$ denote respectively the total number of particles, for $\alpha = 0$, and the total momentum, for $\alpha = 1,\dots,3$, at the macroscopic time t at the site $\underline{x} \in \Lambda$.

More recently it has been proved in[33] without assuming the existence of the solution to the Navier-Stokes equations, that subsequences of the random fields (5.14) converge, weakly in probability, globally in time, to weak solutions of the Navier-Stokes equations. This result, as remarked in Section 4, after Theorem 4.1, is not yet proved for the Boltzmann equation.

As a final remark, we note that the restriction of velocities to be all of the same modulus can be removed. In this way the conservation of energy is an independent law which permits to include thermal phenomena in the modeling and possibly treat convective problems. This has been done in[34], where a model with two different values of the modulus of the velocity is introduced. Moreover a suitable process is added to include the effects of external forces. Hydrodynamical equations similar to (4.29) with some anisotropy are obtained and results similar to those in[4] are proved.

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