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On the Number of Vertices Belonging to all Maximum Stable Sets of a Graph

by

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ABSTRACT

Let us denote by $\alpha(G)$ the size of a maximum stable set, and by $\mu(G)$ the size of a maximum matching of a graph G, and let $\xi(G)$ be the number of vertices which belong to all maximum stable sets. We shall show that $\xi(G) \geq 1 + \alpha(G) - \mu(G)$ holds for any connected graph, whenever $\alpha(G) > \mu(G)$. This inequality improves on related results by Hammer, Hansen and Simeone (1982) and by Levit and Mandrescu (1998).

We also prove that on the one hand, $\xi(G) > 0$ can be recognized in polynomial time whenever $\mu(G) < |V(G)|/3$, and on the other hand that determining whether $\xi(G) > k$ is, in general, NP-complete for any fixed $k \geq 0$.

1 Introduction

Throughout this paper $G = (V, E)$ denotes a simple graph (i.e., finite, undirected, loopless and without multiple edges) with vertex set $V = V(G)$ and edge set $E = E(G)$. The open neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \ \ and \ (v, w) \in E\}$, and for a subset $A \subset V$ let us introduce the notations $N(A) = \cup\{N(v) : v \in A\}$ and $N[A] = A \cup N(A)$.

A subset $S \subseteq V$ is called a stable set of G, if no edge has both its endpoints in S, i.e. if $S \cap N(S) = \emptyset$. The size of a largest (maximum) stable set of G is denoted by $\alpha(G)$, and is called the *stability number* of G. Let $\Omega(G)$ denote the family of all maximum stable sets of the graph G, let $core(G) = \cap\{S : S \in \Omega(G)\}$ be the set of vertices belonging to all maximum stable sets, and let $\xi(G) = |core(G)|$. Similarly, let us define $corona(G) = \cup\{S : S \in \Omega(G)\}$ as the set of vertices belonging to some maximum stable sets of G, and let $\zeta(G) = |corona(G)|$.

A *matching* is a set of non-incident edges of G. Let us denote by $\mu(G)$ the size of a largest (maximum) matching in G. A maximum matching is called *perfect* if it covers all vertices.

In this paper we study the relationship between $\alpha(G)$, $\mu(G)$, and $\xi(G)$. Generalizing earlier results, we show in section 2 the following

Theorem 1 *If G is a connected graph with $\alpha(G) > \mu(G)$, then*

$$\xi(G) \geq 1 + \alpha(G) - \mu(G). \tag{1}$$

Hammer, Hansen and Simeone [5] proved that $\xi(G) \geq 1$, whenever $\alpha(G) > |V(G)|/2$. Levit and Mandrescu [7] further improved this by showing that for graphs G with $\alpha(G) > (|V(G)|+k-1)/2$ the stronger inequality $\xi(G) \geq k+1$ must also hold, moreover $\xi(G) \geq k+2$ is implied, whenever $|V(G)|+k-1$ is even. Gunther et al. proved in [3] that a tree T cannot have only one vertex in $core(T)$. Zito [10] has shown the same result independently, and proved also that $\zeta(T) < |V(G)|$ for a tree T, which is not an even path, i.e. for which $\alpha(T) > |V(G)|/2$. Levit and Mandrescu in [8] have shown that if $\alpha(T) > |V(G)|/2$ holds for a tree T, then $core(T)$ contains at least two leaves of T.

Next, in section 3, we analyze the complexity of computing the quantities $\xi(G)$ and $\zeta(G)$. The membership in the set $core(G) \cup (V(G) \setminus corona(G))$ was shown to be NP-complete by Hammer, Hansen and Simeone [5]. We prove that recognizing $\xi(G) > k$ or recognizing $\zeta(G) < |V(G)| - k$ are both NP-complete for any fixed integer $k \geq 0$.

2 Inequalities

Before proving our main theorem, we shall need to show a few simple lemmas.

Lemma 1 *If G is a connected graph, then*

$$\alpha(G) + \mu(G) \leq |V(G)| \leq \alpha(G) + 2\mu(G).$$

Proof. The first inequality follows immediately by observing that a stable set can contain at most one endpoint from every edge. The second inequality follows readily from the observation that if M is a maximum matching in G, then the vertices, not adjacent with the edges of M, must form a stable set. $\square$

Lemma 2 *For every graph G, there are no edges between the sets $core(G)$ and $corona(G)$, i.e.,*

$$N(core(G)) \subseteq V(G) \setminus corona(G).$$

Proof. For every vertex $v \in corona(G)$ there exists, by definition, a maximum stable set $I \in \Omega(G)$ containing v. Since I is a maximum stable set, it must contain $core(G)$, by the definiton of the core, and hence $I \supseteq core(G) \cup \{v\}$, implying $v \notin N(core(G))$, since I is stable. $\square$

Lemma 3 *For every graph G, and for any maximum stable set $S \in \Omega(G)$, there is a perfect matching from $(S \setminus core(G))$ into $(corona(G) \setminus S)$, implying in particular that*

$$|corona(G) \setminus S| \geq |S \setminus core(G)|.$$

Proof. We shall show the statement by Hall's marriage theorem (see e.g. [4]), and for this let us show that for any $A \subseteq S \setminus core(G)$ we must have

$$|A| \leq |N(A) \cap corona(G)| = |N(A) \cap (corona(G) \setminus S)|.$$

For if not, let us choose a minimal subset $A \subseteq S \setminus core(G)$ for which

$$|A| > |N(A) \cap corona(G)|. \tag{2}$$

Then for any maximum stable set $I \in \Omega(G)$ we have

$$N(A \cap I) \cap corona(G) \subseteq N(A) \cap corona(G) \setminus I,$$

and thus

$$|A \cap I| \leq |N(A \cap I) \cap corona(G)| \leq |N(A) \cap corona(G) \setminus I|,$$

since we chose A as a minimal subset violating $|A| \leq |N(A) \cap corona(G)|$. This implies $|A \setminus I| > |N(A) \cap I|$ by (2), and hence the set $A \cup I \setminus N(A)$ is a stable set of size greater than $\alpha(G)$, a contradiction, proving our claim. $\square$

Proof of Theorem 1. According to Lemma 3, for any maximum stable set $S \in \Omega(G)$ there exists a matching M in G of size $|M| \geq |S \setminus core(G)|$. This implies, since $\mu(G) \geq |M|$ and $\alpha(G) > \mu(G)$ is assumed that, $\alpha(G) > |S \setminus core(G)|$, and thus $core(G) \neq \emptyset$ follows.

Since the graph G is assumed to be connected, $N(core(G)) \neq \emptyset$, and thus there exists an edge $(u,v) \in E(G)$ with $u \in core(G)$ and $v \in N(core(G))$. According to Lemma 2, we have $N(core(G)) \cap corona(G) = \emptyset$, and hence (u,v) does not have a common endpoint with the edges of the matching M. Thus $M \cup \{(u,v)\}$ is also a matching, implying that

$$\mu(G) \geq |M| + 1 \geq |S \setminus core(G)| + 1 = \alpha(G) - \xi(G) + 1,$$

and thus the theorem follows. □

Corollary 1 *If G is a connected graph with $\alpha(G) > \mu(G)$, then*

$$\zeta(G) < |V(G)|.$$

Proof. By Theorem 1, $core(G) \neq \emptyset$ follows, and thus $N(core(G)) \neq \emptyset$ by the connectedness of G. By Lemma 2, $N(core(G)) \cap corona(G) = \emptyset$, and hence

$$|V(G)| - \zeta(G) \geq |N(core(G))| > 0$$

follows, proving the statement. □

Corollary 2 *If G is a connected graph with $\alpha(G) > |V(G)|/2$, then*

$$\xi(G) \geq 2 + |V(G)| - \alpha(G) - \mu(G) \text{ and } \zeta(G) < |V(G)|.$$

Proof. The assumption $\alpha(G) > |V(G)|/2$ implies $\alpha(G) > \mu(G)$, and thus Theorem 1 can be applied. Then the first inequality follows by Theorem 1 and from $\alpha(G) \geq 1 + |V(G)| - \alpha(G)$, which is implied by our assumption, while the second inequality follows from Corollary 1. □

Corollary 3 (cfv. [3, 10]) *If T is a tree, which is not an even path, then*

$$\xi(T) \geq 2 \text{ and } \zeta(T) < |V(T)|.$$

Proof. If the tree T is not an even path, then $\alpha(T) > |V(T)|/2$ is implied, and the statement follows by Corollary 2 and Lemma 1. □

Corollary 4 (cfv. [7]) *If G is a connected graph, and for some $k \geq 1$ the inequality*

$$2\alpha(G) \geq |V(G)| + k \tag{3}$$

holds, then

$$\xi(G) \geq k + 1.$$

Proof. Let us observe first that Theorem 1 and the first inequality in Lemma 1 imply that

$$\xi(G) \geq 1 + 2\alpha(G) - |V(G)| \tag{4}$$

whenever $\alpha(G) > \mu(G)$. In this case, $k \geq 1$ and (3) imply $\alpha(G) > |V(G)|/2 \geq \mu(G)$, hence (4) must hold, implying the statement by (3). □

3 Generating the core and the corona

In this section we shall prove that, extending a result of [5], both the membership in $core(G)$ and in $corona(G)$ are NP-complete problems.

For a graph G and a vertex $v \in V(G)$ let us denote by $G - v$ the subgraph induced by $V(G) \setminus \{v\}$.

Lemma 4 *If G is a connected graph, then*

$$v \in core(G) \text{ if and only if } \alpha(G - v) = \alpha(G) - 1.$$

Proof. The lemma follows immediately by observing that a stable set in the induced subgraph $G - v$ is also a stable set in G. □

Given a graph G and a nonnegative integer l, let us define a new graph B_G^l obtained from G by adding a stable set L of size l to G, $L \cap V(G) = \emptyset$, and connecting all vertices of L to $V(G)$, i.e. by defining $E(B_G^l) = E(G) \cup \{(v, x) | v \in V(G),\ x \in L\}$.

Lemma 5 *Using the notations and definitions above, we have*

$$\xi(B_G^l) = \begin{cases} \xi(G) & \text{if } l < \alpha(G), \\ 0 & \text{if } l = \alpha(G), \\ l & \text{if } l > \alpha(G). \end{cases}$$

Proof. It is easy to see that $\Omega(B_G^l) = \Omega(G)$ as long as $l < \alpha(G)$, implying $core(B_G^l) = core(G)$ in these cases.

If $l = \alpha(G)$, then $\Omega(B_G^l) = \Omega(G) \cup \{L\}$, and thus $core(B_G^l) = core(G) \cap L = \emptyset$.

Finally, if $l > \alpha(G)$, then $\Omega(B_G^l) = \{L\}$, and hence $core(B_G^l) = L$. □

Theorem 2 *The problems of computing $\xi(G)$ and $\alpha(G)$ are polynomially equivalent. Moreover, $\alpha(G)$ can be computed using only an oracle for the question $\xi(G) > 0$.*

Proof. From Lemma 4 it is obvious that by computing $\alpha(G - v)$ for all $v \in V(G)$, one can determine the set $core(G)$.

For the converse direction, let us test $\xi(B_G^l) > 0$ for $l = 1, 2, ..., |V(G)|$. Lemma 5 implies that we get $\xi(B_G^l) = 0$ iff $l = \alpha(G)$. □

Let us note that by using the values $\xi(B_G^l)$ computed, and Lemma 5, $\alpha(G)$ can be determined in $O(\log \alpha(G))$ steps, by applying a careful binary search.

Corollary 5 *For any fixed integer number $k \geq 0$ the question $\xi(G) > k$ is **NP**-complete.*

Next, we develop analogous results for $corona(G)$. For this end, let us characterize first the set $corona(G)$. For a graph G and a subset S of its vertices, let us denote by $G \setminus S$ the graph, induced by the vertex set $V(G) \setminus S$.

Lemma 6 *If G is a connected graph, then*

$$v \in corona(G) \text{ if and only if } \alpha(G \setminus N[v]) = \alpha(G) - 1.$$

Proof. Let us recall that $N[v] = N(v) \cup \{v\}$. It is easy to see that for every maximum stable set $S \in \Omega(G)$ and vertex $v \in V(G)$, either $v \in S$ or $S \cap N(v) \neq \emptyset$, but not both. Let us also note that if $S \cap N(v) = \{u\}$, then $(S \setminus \{u\}) \cup \{v\}$ is a maximum stable set, too.

Hence, $v \notin corona(G)$ if and only if $|S \cap N(v)| \geq 2$ for all $S \in \Omega(G)$, or in other words, if and only if $\alpha(G \setminus N[v]) < \alpha(G) - 1$. □

Given a graph G and an integer $h \geq 0$, let us define a new graph M_G^h on vertices

$$V(M_G^h) = V(G) \cup \bigcup_{v \in V(G)} S_v$$

where the sets S_v, $v \in V(G)$ are pairwise disjoint stable sets, have size $|S_v| = h$, and are disjoint from $V(G)$. The edge set of M_G^h consists of all the edges $(u, v) \in E(G)$, and all edges of the form (v, y) for all $v \in V(G)$, $y \in S_u$, $u \neq v$, and of the form (x, y) for $x \in S_v$, $y \in S_u$ for all $u, v \in V(G)$, $u \neq v$.

Lemma 7 *Using the notations and definitions above, we have*

$$\zeta(M_G^h) = \begin{cases} \zeta(G) & \text{if } h < \alpha(G) - 1, \\ |V(M_G^h)| & \text{if } h \geq \alpha(G) - 1. \end{cases}$$

Proof. Let us observe that the unique largest stable set in M_G^h, containing a vertex $x \in S_v$, is the set $S_v \cup \{v\}$, for all $v \in V(G)$. Hence, we have $\Omega(M_G^h) = \Omega(G)$ if $h < \alpha(G) - 1$, $\Omega(M_G^h) = \Omega(G) \cup \{S_v \cup \{v\} | v \in V(G)\}$ if $h = \alpha(G) - 1$, and $\Omega(M_G^h) = \{S_v \cup \{v\} | v \in V(G)\}$ if $h = \alpha(G)$ if $h \geq \alpha(G)$. Hence

$$corona(M_G^h) = \begin{cases} corona(G) & \text{if } h < \alpha(G) - 1, \\ V(M_G^h) & \text{otherwise} \end{cases}$$

follows, proving the claim of the lemma. □

Theorem 3 *The problems of computing $\zeta(G)$ and $\alpha(G)$ are polynomially equivalent. Moreover, $\alpha(G)$ can be computed using only an oracle for the question $\zeta(G) < |V(G)|$.*

Proof. One can test $v \in corona(G)$ by comparing $\alpha(G)$ and $\alpha(G \backslash N[v])$, according to Lemma 6, and hence $\zeta(G)$ can be computed by computing the stability number of $1+|V(G)|$ graphs.

For the reverse direction, let us test if $\zeta(M_G^h) < |V(M_G^h)|$ for $h = 1, 2, \ldots$. By Lemma 7, we shall have $\zeta(M_G^h) = |V(M_G^h)|$ the first time exactly when $h = \alpha(G) - 1$. Hence, we can determine $\alpha(G)$ by testing $\zeta(G') < |V(G')|$ for $\alpha(G)$ different graphs. □

Let us note that by using Lemma 7 and by applying binary search, $\alpha(G)$ can be determined via $O(\log \alpha(G))$ calls to the $\zeta(G) < |V(G)|$ oracle.

Corollary 6 *For any fixed integer number $k \geq 0$ the question $|V(G)| - \zeta(G) > k$ is* ***NP****-complete.*

Let us analyze finally situations when the above computational questions can be answered in polynomial time.

Corollary 7 *If G is a connected graph with $\mu(G) < \frac{|V(G)|}{3}$, then*

$$\xi(G) \geq 2.$$

Proof. It follows from Lemma 1 and Theorem 1. □

In other words Corollary 7 demonstrates that when $\mu(G) < \frac{|V(G)|}{3}$, then one can decide $\xi(G) > 0$ in polynomial time. Figure 1 shows the feasible area on the (α, μ) plane corresponding to general graphs. This area is bounded by three straight lines:

l_1: $\alpha(G) + \mu(G) = |V(G)|$,

l_2: $\alpha(G) + 2\mu(G) = |V(G)|$, and

l_3: $\mu(G) = |V(G)|/2$.

By Theorem 1, the triangle bounded by l_1, l_2 and $\alpha(G) = \mu(G)$ represents graphs with $\xi(G) \geq 2$. The triangle bounded by l_1, l_2, and $\mu(G) = |V(G)|/3$ represents a polynomially tractable subarea.

Exploiting Lemma 1 we can easily see the following:

Remark 1 *For hereditary families of graphs, where computing $\alpha(G)$ is polynomial, the questions $\xi(G) > 0$ and $|V(G)| - \zeta(G) > 0$ can be answered in polynomial time. Moreover, one can determine the sets $core(G)$ and $corona(G)$ in polynomial time. For instance, this is true for line graphs, circle graphs, circular arc graphs [1], and for all classes of perfect graphs [2].*

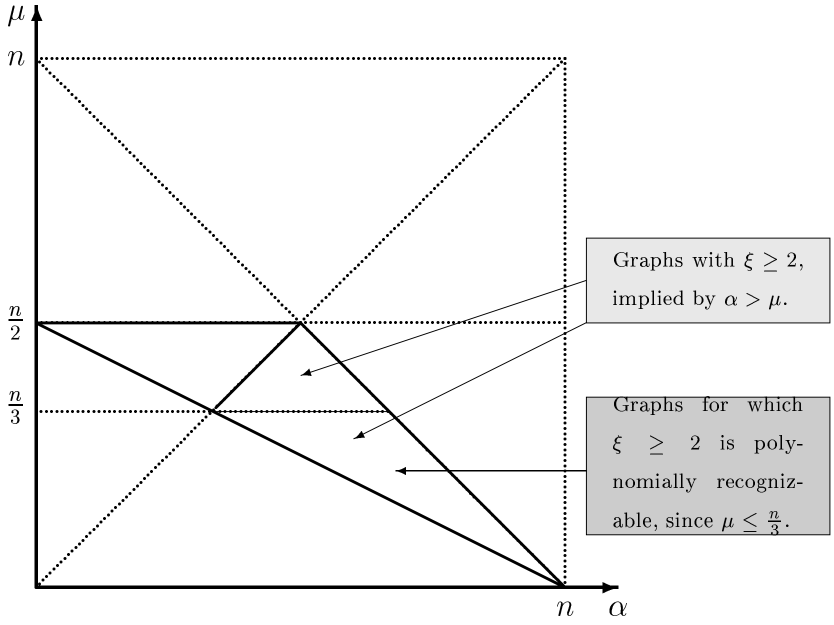

Figure 1: Feasible areas on the (α, μ) plane representing various types of graphs.

4 Conclusions

In this paper we have studied the relationships connecting the stability number and the matching number of a graph with the number of vertices belonging to all maximum stable sets and with the number of vertices belonging to no maximum stable sets of a graph. It would be interesting to see which of these results can be transferred to graphs satisfying $\alpha(G) \leq \mu(G)$. The following algorithmic problem seems attractive: how to find $core(G)$ and/or $corona(G)$ efficiently for some specific classes of graphs, which were not mentioned in Remark 1 and Corollary 7.

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