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Proof of a Conjecture of Bollobás and Eldridge for Graphs of Maximum Degree Three

by

Béla Csaba[1,2]
Dept. of Computer Science
Rutgers University
New Brunswick, New Jersey 08903

Ali Shokoufandeh[3]
Dept. of Mathematics and Computer Science
Drexel University
Philadelphia, Pennsylvania 19104

Endre Szemerédi[4]
Dept. of Computer Science
Rutgers University
New Brunswick, New Jersey 08903

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[2]Research partially supported by a DIMACS fellowship.
[3]Research partially supported by a DIMACS fellowship.
[4]Permanent Member

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ABSTRACT

Let G_1 and G_2 be graphs on n vertices. If there are edge-disjoint copies of G_1 and G_2 in K_n, then we say there is a packing of G_1 and G_2. A conjecture of Bollobás and Eldridge [5] asserts that, if $(\Delta(G_1)+1)(\Delta(G_2)+1) \leq n+1$ then there is a packing of G_1 and G_2. We prove this conjecture when $\Delta(G_1) = 3$, for sufficiently large n.

1 Introduction

In this paper we will only consider graphs without loops and multiple edges. For a graph G, $V(G)$ and $E(G)$ will denote its vertex-set and edge-set, respectively. For any vertex v, $deg_G(v)$ is the degree of the vertex v, $deg_G(v, X)$ is the number of the neighbors of v in X, and $e(X, Y)$ is the number of the edges between X and Y. $N_G(v)$ is the set of neighbors of v and $N_G(v, X)$ is the set of neighbors of v in X. $\delta(G)$ and $\Delta(G)$ are the minimum degree and maximum degree of G (if it is clear from the context G might be omitted).

An important question in extremal graph theory is the sufficient condition on the minimum degree of a graph which guarantees the existence of a spanning subgraph. In [5] Bollobás and Eldridge conjectured that if $(\Delta(G_1) + 1)(\Delta(G_2) + 1) \leq n + 1$ then there will be a packing of G_1 and G_2. Their conjecture can be restated in the following complementary form:

Conjecture 1 (Bollobás-Eldridge) *If G is a graph on n vertices with*

$$\delta(G) \geq \frac{kn - 1}{k + 1}$$

then G contains any n vertex graph H with $\Delta(H) \leq k$.

Certain special cases of this conjecture have been proved by Corrádi and Hajnal [6], Hajnal and Szemerédi [7], Aigner and Brandt [3] and Alon and Fisher [4]. The goal of this paper is to show that the conjecture is true for all sufficiently large n when $k = 3$:

Theorem 2 *There exists an n_0 such that for all $n \geq n_0$ the following statement holds: Let H be a graph on n vertices and $\Delta(H) \leq 3$. If G is any n vertex graph such that*

$$\delta(G) \geq \frac{3n - 1}{4}$$

then G contains H as a spanning subgraph.

In proving Theorem 2 we distinct two cases depending on the structure of H. We call H *extremal* if except a small proportion of the vertices, H is the union of disjoint K_4's. The proof for this case is different from the proof of the *non-extremal* case.

2 Notations and Definitions

A bipartite graph G with color-classes A and B and edge-set E will be denoted by $G = (A, B, E)$. The *density* between disjoint sets X and Y is defined as:

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}.$$

In the proof of Theorem 2, Szemerédi's Regularity Lemma [9] plays a pivotal role. We will need the following definition to state the Regularity Lemma.

Definition 1 (Regularity condition) *Let $\varepsilon > 0$. A pair (A, B) of disjoint vertex-sets in G is ε-regular if for every $X \subset A$ and $Y \subset B$, satisfying*

$$|X| > \varepsilon|A|, \ |Y| > \varepsilon|B|$$

we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

This definition implies that regular pairs are highly uniform bipartite graphs; namely the density of any reasonably large subgraph is almost the same as the density of the regular pair.
We will use the following form of the Regularity Lemma:

Lemma 3 (Degree Form) *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set V into $\ell + 1$ clusters $V_0, V_1, \ldots, V_\ell$, and there is a subgraph G' of G with the following properties:*

- $\ell \leq M$,
- $|V_0| \leq \varepsilon|V|$,
- *all clusters V_i, $i \geq 1$, are of the same size $m \leq \varepsilon\lceil|V|\rceil$,*
- $deg_{G'}(v) > deg_G(v) - (d + \varepsilon)|V|$ *for all $v \in V$,*
- $G|_{V_i} = \emptyset$ *(V_i is an independent set in G') for all $i \geq 1$,*
- *all pairs (V_i, V_j), $1 \leq i < j \leq \ell$, are ε-regular, each with density either 0 or greater than d in G'.*

Often we call V_0 the *exceptional cluster*. A stronger one-sided property of regular pairs is super-regularity:

Definition 2 (Super-Regularity condition) *Given a graph G and two disjoint subsets of its vertices A and B, the pair (A, B) is (ε, d)-*super-regular*, if it is ε-regular and furthermore,*

$$deg(a) > d|B|, \text{ for all } a \in A,$$

and

$$deg(b) > d|A|, \text{ for all } b \in B.$$

We will apply Lemma 3 to the graph G of Theorem 2 with parameters $\varepsilon \ll d \ll 1$, creating a partition of $V(G)$ into clusters $V_0, V_1, \dots, V_\ell$. We define the *reduced graph* G_r as follows:
the vertices of G_r are the clusters $V_1, \dots, V_\ell$ and we have an edge between two clusters if they form an ε-regular pair in G', with density exceeding d. Since $\delta(G') \geq (3/4 - (d+\varepsilon))n$ it will follow that for $\eta = 10d$

$$\delta(G_r) \geq (\frac{3}{4} - \eta)\ell.$$

Using the theorem of Hajnal and Szemerédi [7], apart from $12\eta\ell$ clusters, G_r can be covered by vertex disjoint cliques of size four. Let us put the vertices of these $12\eta\ell$ clusters into the exceptional cluster V_0. For simplicity G_r denotes the remaining clusters and V_0 denotes the resulting exceptional vertices. Clearly, $|V_0| \leq 13\eta n$. In the sequel we will denote 13η by θ.

3 Properties of H

Without loss of generality, we might assume that the graph H is an "almost 3-regular graph" (with at most three vertices having degree less than 3). We denote the maximum independent set in H by I. Clearly $|I| \geq \frac{n}{4}$. The subgraph H' induced by $V(H) \backslash I$ is a graph with maximum degree 2. We define a partition $\{I_1, I_2, I_3\}$ of I based on the following conditions:

- $x \in I_1$ iff the neighbors of x in H' belong to three different components.
- $y \in I_2$ iff the neighbors of y in H' belong to exactly two different components.
- $z \in I_3$ iff all neighbors of z in H' belong to the same component.

The following characterizations of the components in H' is a direct consequence of I being the maximum independent set.

Lemma 4 *Every I_3-vertex is connected to either a triangle or a path of length 4 in H'. In the first case there can be one I_3-vertex connected to that triangle, in the second case there are two I_3-vertices adjacent to the length 4 path.*

Proof. Trivially follows from the maximality of I. **Q.E.D.**
For reasons which will be clear later we add those I_3-vertices which are connected to a length 4 path into I_2. These vertices will "behave" like I_2-vertices from our point of view.

By removing a subset Q of size at most $\sqrt{n}$ from H', we can guarantee that the remaining components in H' are all short (not longer than $\sqrt{n}$). Let I' be a maximum subset of I, with the property that any two points in I' are of distance at least 4. Since H has maximum degree 3, it can be seen that $|I'| \geq \frac{1}{18}|I|$. We let $I'_1 = I_1 \cap I'$, $I'_2 = I_2 \cap I'$, and $I'_3 = I_3 \cap I'$.

4 Non-Extremal Case

We assume that for some small positive constant c at least cn vertices do not belong to 4-cliques in H. After the necessary preparations, we will determine the exact value of c. Otherwise, if H is almost union of 4-cliques then it is an *extremal* graph. The following is the outline of the proof for the *non-extremal* case:

I. Assign the vertices of H to clusters in G_r such that if $x, y \in V(H)$ with $(x, y) \in E(H)$ are mapped to V_x and V_y in G_r, then $(V_x, V_y) \in E(G_r)$. There should be $\frac{n}{l} + o(n)$ vertices of H mapped to every cluster in G_r.

II. Let $L(V_i)$, $1 \leq i \leq \ell$, denote the set of vertices of H assigned to the cluster V_i. Clearly, $|L(V_i)| \geq |V_i|$ since no vertex in H has been assigned to V_0. (If V_0 is empty one can choose a random εm sized subset from each cluster to form V_0, which makes our discussion valid for an empty V_0 as well). By appropriately removing $|L(V_i)| - |V_i|$ vertices from every list $L(V_i)$, we form a set H_0. We will also add the vertices in Q to H_0. Consequently, H_0 will have the same size as V_0. For covering the vertices in V_0 with H-vertices we perform a preprocessing step, during which we may change some of the vertices in H_0, but the size of H_0 and the vertices in V_0 will remain the same. After preprocessing, H_0 will be embedded into V_0 and the subgraph $H \backslash H_0$ will be embedded into the vertices of $V(G) \backslash V_0$. For embedding $H \backslash H_0$ into $V(G) \backslash V_0$ we will use a procedure very similar to the one used in the Blow-up Lemma [8].

4.1 Mapping H' to Clusters of G_r

We start by assigning the vertices of $H' \backslash Q$ to clusters of G_r. For every component $C \in H' \backslash Q$, randomly and independently choose a triangle T from the K_4-cover of G_r. If C is 2-colorable, choose a random edge (V_i, V_j) of T, and distribute the vertices of C between the two sets $L(V_i)$ and $L(V_j)$ randomly. Otherwise, if C is an odd cycle, distribute its vertices among the clusters of T randomly. In both cases adjacent vertices in C should be mapped to adjacent clusters in T. This procedure will guarantee an almost even distribution of the vertices of $H' \backslash Q$ into the clusters of G_r:

Lemma 5 *Let $L(V_i)$, $1 \leq i \leq \ell$ denote the set of vertices assigned to V_i after distributing the vertices of $H' \backslash Q$ using the above procedure. Then with high probability $|L(V_i)| = \frac{n'}{\ell} \pm o(n)$, where $n' = |H' \backslash Q|$.*

Proof. Applying Chebyshev's inequality gives the proof of the lemma. **Q.E.D**

In the sequel, we will argue that an appropriate distribution of H' among the clusters of G_r will facilitate an even assignment of the vertices of I to the clusters of G_r. Let V be a cluster in G_r, we define the *associated list* $\mathcal{A}(V)$ as $\{y : y \in I, x \in L(V), (x, y) \in E(H)\}$, which are the set of independent vertices of H with a neighbor assigned to the cluster V. Given the associated lists $\mathcal{A}(V_i), \mathcal{A}(V_j), \mathcal{A}(V_k)$, for each $1 \leq p \leq 3$, the random variables $R'_p(V_i, V_j, V_k) = |I'_p \cap \mathcal{A}(V_i) \cap \mathcal{A}(V_j) \cap \mathcal{A}(V_k)|$ and $R_p(V_i, V_j, V_k) = |(I_p \backslash I'_p) \cap \mathcal{A}(V_i) \cap \mathcal{A}(V_j) \cap \mathcal{A}(V_k)|$ will denote the number of independent vertices in I'_p, and respectively $I_p \backslash I'_p$, which have neighbors assigned to the clusters V_i, V_j and V_k.

Lemma 6 *For any three clusters V_i, V_j, and V_k in G_r the following inequalities hold:*

$$\Pr\left[|R'_1(V_i, V_j, V_k) - \mathrm{E}[R'_1(V_i, V_j, V_k)]| > n^{\frac{4}{5}}\right] = o(1),$$

$$\Pr\left[|R_1(V_i, V_j, V_k) - \mathrm{E}[R_1(V_i, V_j, V_k)]| > n^{\frac{4}{5}}\right] = o(1).$$

Moreover, for all $1 \leq i \leq \ell$ if (V_j, V_k) is an edge in the K_4-cover of G_r then

$$\Pr\left[|R'_2(V_i, V_j, V_k) - \mathrm{E}[R'_2(V_i, V_j, V_k)]| > n^{\frac{4}{5}}\right] = o(1),$$

$$\Pr\left[|R_2(V_i, V_j, V_k) - \mathrm{E}[R_2(V_i, V_j, V_k)]| > n^{\frac{4}{5}}\right] = o(1).$$

Finally, if V_i, V_j, and V_k are the clusters of a triangle in the K_4-cover of G_r then

$$\Pr\left[|R'_3(V_i, V_j, V_k) - \mathrm{E}[R'_3(V_i, V_j, V_k)]| > n^{\frac{4}{5}}\right] = o(1),$$

$$\Pr\left[|R_3(V_i, V_j, V_k) - \mathrm{E}[R_3(V_i, V_j, V_k)]| > n^{\frac{4}{5}}\right] = o(1).$$

Proof. Similar to the proof of Lemma 5, again we omit the details. **Q.E.D.**

In short, the lemma states that for a proper distribution of the vertices in H', the mapped vertices to any three clusters in G_r will have almost the same number of neighbors in any of the classes I'_1, I'_2, I'_3, and their complements with respect to I.

4.2 Assigning I and Q

In this section we will present a consistent assignment of the vertices in $I \cup Q$ to the clusters of G_r, where consistency means that for any pair of vertices x, y in the graph H assigned to clusters V_i and V_j, respectively, if $(x, y) \in E(H)$ then $(V_i, V_j) \in E(G_r)$. As we will see, such assignments can be formulated as special matching problems. In order to finish the embedding of H into G, some of the H-vertices should be assigned to the exceptional cluster V_0, which will be done through a *preprocessing* procedure, discussed later. For a bipartite graph $A = (V, T, E(A))$ where $|T| = q|V|$ for some positive integer q, $M \subset E(A)$ is a *proportional matching* if every $v \in V$ is adjacent to exactly q vertices in T and every $u \in T$ is adjacent to exactly one V vertex in M. In order to show that A contains a proportional matching we will check the König - Hall conditions, that is, for every subset U of V its neighborhood in T should satisfy $|N_A(U, T)| \geq q|U|$. One can easily see this from the construction of an auxiliary graph: substitute every $v \in V$ with q instances $v_1, \ldots, v_q$, and if (v, u) $(u \in T)$ was an edge, then connect the v_i's to u for all $1 \leq i \leq q$. This auxiliary graph has a perfect matching if and only if A has a proportional matching.
Besides this kind of matching we are going to need another kind of matching in which the "load of the vertices" are distributed more evenly. We say A *allows a strong proportional matching with respect to* μ $(0 < \mu \ll 1)$ if there is a proportional matching in the following bipartite graph A'. Its color classes are V and T'. For every vertex $u \in T$ we add $\frac{\ell}{\mu}$ copies, $u_1, \ldots, u_{\frac{\ell}{\mu}}$, to T'. If $N_A(u) = \{v_1, \ldots, v_s\}$ then we will have the following edges: (u_i, v_i) for $1 \leq i \leq s$, and (u_j, v_i) where $1 \leq i \leq s$ and $s < j \leq \frac{\ell}{\mu}$. In other words, the first s copies of u have degree 1, while the others have the same degree, s. The existence of a strong proportional matching can be checked through the König-Hall conditions: one can see that if for $U \subset V$, $|U| < (1 - \mu)\ell$, then $|N_A(U)|(1 - \mu) \leq |N_{A'}(U)|$. Using this fact we can prove the existence of a strong proportional matching, and at the same time the existence of a proportional matching as well. As we will see later, both these matchings are needed to assign the vertices of I and Q to clusters of G_r.

Let us start by constructing our first bipartite graph A. For the reduced graph G_r we define $A = A(G_r)$ in the following way: $A = A(V, T, E(A))$, with $V = V(G_r)$, $T = \{\{v_i, v_j, v_k\} : v_i, v_j, v_k \in V(G_r), v_i, v_j, v_k \text{ all distinct}\}$, and $E(A) = \{(u, \{v_i, v_j, v_k\}) : (u, v_i), (u, v_j), (u, v_k) \in E(G_r)\}$. Let $0 < \theta \ll \mu \ll 1$.

Lemma 7 *There is a proportional matching in the bipartite graph A, and it allows a strong proportional matching with respect to μ.*

Proof. We will check the "strong" König - Hall conditions for all possible size subsets of V.

- For any $v \in V$ we have

$$|N_A(v,T)| \geq \binom{(\frac{3}{4}-\theta)\ell}{3}.$$

- Let $U \subset V$ with $|U| = (\frac{3}{4}-\theta)^3(1-\mu)\ell$. By the minimum degree condition in G_r for every $v \in V$ there exists a $u \in U$ such that $(u,v) \in E(G_r)$. This in turn implies that among all triplets in T containing v at least $(\frac{3}{4}-\theta)^2$ proportion belongs to $N_A(u,T)$. Summing over all v we get:

$$|N_A(U,T)| > (\frac{3}{4}-\theta)^2|T|.$$

- Next, assume U is a subset of V with size $|U| = (\frac{3}{4}-\theta)^2(1-\mu)\ell$. By the minimum degree condition in G_r, any two vertices $v, w \in V$ have a common neighbor $u \in U$. This implies that among all the triplets in T containing v and w at least $\frac{3}{4}-\theta$ proportion belongs to $N_A(u,T)$. Once again, summing over all the pairs $v, w \in V$ we will have

$$|N_A(U,T)| > (\frac{3}{4}-\theta)|T|.$$

- Let $U \subset V$ with $|U| = (\frac{3}{4}-\theta)(1-\mu)\ell$. First we calculate the number of edges from U to the set of all possible pairs: $|U|(\frac{3}{4}-\theta)^2$. From this one can conclude that more than $\frac{5}{16}\binom{\ell}{2}$ pairs have more than $(\frac{1}{4}+\theta)\ell$ neighbors in U. None of those pairs can appear in any triplet of $\overline{N_A(U,T)}$. Besides, no pair can participate in more than $(\frac{1}{4}+\theta)\ell$ triplets in $\overline{N_A(U,T)}$. Thus, $|\overline{N_A(U,T)}| < \frac{1}{5}|T|$.

- Observing that any triplet in T has a neighborhood of size at least $(\frac{1}{4}-3\theta)\ell$ in V, we are done with proving the existence of a proportional matching.

- For finishing the proof for the existence of a strong proportional matching let us suppose that U has size $(1-\mu)\ell$. It is possible that triplets with degree 1 have no neighbor in U. But those are already matched to vertices in V. This implies that we have a strong proportional matching, too. **Q.E.D.**

For constructing our second bipartite graph $B = B(V,T,E(B))$ we are going to need the disjoint K_4-cover of G_r. Let $M \subset E(G_r)$ denote the edge set of this K_4-cover. $V = V(G_r)$ and $T = \{\{v_i, v_j, v_k\} : v_i, v_j, v_k \in V(G_r), (v_i, v_j) \in M(G_r)\}$, $E(B) = \{(u, \{v_i, v_j, v_k\}) : (u, v_i), (u, v_j), (u, v_k) \in E(G_r)\}$. We call v_k the third vertex of the triplet. Again, $0 < \theta \ll \mu \ll 1$.

Lemma 8 *There is a proportional matching in the bipartite graph B, and it allows a strong proportional matching with respect to μ.*

Proof. We check the "strong" König-Hall conditions.

- Any $v \in V$ has $(\frac{1}{2} - 2\theta)\frac{3}{2}\ell$ edges of M as neighbors (i.e. connected to both endpoints of those edges). This implies, that
$$|N_B(v,T)| \geq (\frac{3}{4} - \theta)\ell(\frac{1}{2} - 2\theta)\frac{3}{2}\ell.$$
- Let $U \subset V$ with $|U| = (\frac{3}{4} - \theta)(\frac{1}{2} - 2\theta)(1-\mu)\ell$. Now for any $v \in V$ there exists a $u \in U$ such that $(u,v) \in E(G_r)$. This vertex u will have at least $\frac{1}{2} - 2\theta$ proportion of the edges in M as neighbors, and all of them with v will form a triplet in T which are connected to u. Summing over all $v \in V$ we get that
$$|N_B(U,T)| > (\frac{1}{2} - 2\theta)|T|.$$
- Let $U \subset V$ with $|U| = (\frac{1}{2} - 2\theta)(1-\mu)\ell$. We want to prove that $|N_B(U,T)| > (\frac{1}{2}+2\theta)|T|\frac{1}{1-\mu}$. Compute first the number of vertices having more than $(\frac{1}{4}+2\theta)\ell$ neighbors in U. A simple calculation shows, that this number is "almost" $\frac{\ell}{2}$ (tends to $\frac{\ell}{2}$ when θ, μ tends to 0). Actually, the limit is $\frac{\ell}{2}$ in an extremal case, otherwise it is even bigger. Also, we have strictly more than $\frac{\ell}{2}$ of them if some "not too small proportion" have less neighbors than, say, 99% of U. Denote these V-vertices having big degree to U by S. Observe, that an S-vertex with an arbitrary other vertex have a common neighbor in U. Our second observation is that out of any three vertices at least two have a common neighbor in U. It simply follows from the minimum degree condition of G_r. Now, if in a K_4 there is exactly one S-vertex, then the three edges adjacent to it are U-neighbors, and because of our second observation there is another edge having a U-neighbor. If there are two S-vertices in a K_4, then five edges will have some U-neighbor, and three or four S-vertices in a K_4 means that all the edges of it have some U-neighbor. Using these remarks a simple calculation shows, that if the S-vertices are not concentrated in "almost one half" of the 4-cliques of the cover, then strictly more than $\frac{2}{3}$ portion of the edges will have at least one U-neighbor. If an edge has a U-neighbor, then it cannot appear in more than $(\frac{1}{4} + \theta)\ell$ triplets of $\overline{N_B(U,T)}$. Hence, if the S-vertices are not concentrated in "almost one half" of the 4-cliques of the cover, then $|N_B(U,T)|$ will be strictly bigger than $\frac{|T|}{2}$, and this is what we want to prove. So, assume, that the S-vertices are "concentrated well enough" according to the K_4-cover.

But there is one more thing we now about S: its size cannot be strictly bigger than $\frac{\ell}{2}$. In this case we were done again, we would have more than $\frac{2}{3}$ portion of the edges having a U-neighbor. Summarizing our knowledge about S we know that S is concentrated in "almost" one half of the cliques of the K_4-cover, and almost all vertices in S have actually almost full degree to U. But an edge with endpoints having more common neighbor in U than $(\frac{1}{4}+\theta)\ell$ will constitute a U-neighboring triplet with any third vertex. Hence, even if the S-vertices are well concentrated, $|N_B(U,T)|$ will be big enough.

- Let $U \subset V$ with $|U| = (\frac{1}{2}+2\theta)\ell$. Then any edge in M will have a neighbor in U, thus we will have $(\frac{3}{4}-\theta)|T|$ triplets in $N_B(U)$.

- Let $U \subset V$ with $|U| = (\frac{3}{4}-\theta)(1-\mu)\ell$. First we calculate the number of edges from U to M: $|U|(\frac{1}{2}-2\theta)$. From this one can conclude that more than $\frac{1}{5}|M|$ edges have more than $(\frac{1}{4}+\theta)\ell$ neighbors in U. None of those edges can appear in any triplet of $\overline{N_B(U,T)}$. Besides, no edge can participate in more than $(\frac{1}{4}+\theta)\ell$ triplets in $\overline{N_B(U,T)}$. Thus, $|\overline{N_B(U,T)}| < (\frac{1}{5}+\theta)|T|$.

- Observing that any triplet in T has a neighborhood of size at least $(\frac{1}{4}-3\theta)\ell$ in V, we are done with proving the existence of a proportional matching.

- For finishing the proof for the existence of a strong proportional matching let us suppose that U has size $(1-\mu)\ell$. It is possible that triplets with degree 1 have no neighbor in U. But those are already matched to vertices in V. This implies the existence of a strong proportional matching. **Q.E.D.**

We arrived to discussing the assignment of the vertices in I to G_r. Let us start by assigning the vertices in $I_1 \backslash I_1'$. Assume $\mathcal{M}_1$ denotes the proportional matching of Lemma 7 with respect to the graph A. For a cluster V_t let $\{V_i, V_j, V_k\}$ be one of the triplets matched to it in $\mathcal{M}_1$. We will assign the vertices of $(I_1 \backslash I_1') \cap A(V_i) \cap A(V_j) \cap A(V_k)$ to the cluster V_t by adding them to the list $L(V_t)$. By Lemma 6, $|(I_1 \backslash I_1') \cap A(V_i) \cap A(V_j) \cap A(V_k)|$ is almost the same for all choices of $\{V_i, V_j, V_k\}$, which in turn implies that the lists $L(V_t)$ for all $V_t \in G_r$ will have almost the same length after the distribution of $I_1 \backslash I_1'$. Also, note that the construction of $A(G_r)$ and the matching structure of the proportional matching $\mathcal{M}_1$ provides that if $x \in I_1 \backslash I_1'$ is assigned to $L(V_t)$ then the $N_H(x)$-vertices are assigned to neighboring clusters of V_t. The distribution of $I_2 \backslash I_2'$ also can be carried out in the same manner using the matchings provided by Lemmas 7 and 8.

For the vertices in I_3, since all of their neighbors in H' are assigned to a single triangle in the K_4-cover of G_r, we can assign them to the fourth cluster of the same

4-clique. By Lemma 6 almost the same number of such vertices will be assigned to each cluster in G_r.

One may think that so far we have not considered all possible matching cases, e.g., sometimes the three clusters of the triplets may collapse to two clusters. But it is not hard to see that finding these matchings are rather obvious comparing to the much harder ones.

In the next section we will handle the assignment of I'_1 and I'_2 vertices. For this we will use both proportional and strong proportional matchings. As we will see, for assigning a proportion of these vertices we will need to use the strong proportional matchings, and the leftover will be distributed among the clusters of G_r by the proportional matchings.

4.3 Embedding of a non-extremal H

The embedding of H will follow a similar procedure to that of the Blow-up Lemma [8]. Observe that in the Blow-up Lemma the reduced graph is fixed, does not depend on the parameters, and all the edges of the reduced graph are super-regular pairs. Obviously, these conditions are too strong in our case. Besides, as we will see, there will be restrictions for the embedding of certain H-vertices. Hence, we have to make changes in the embedding procedure of the Blow-up Lemma.

Every vertex in a cluster of a super-regular pair has big degree to the other cluster. Our edges in G_r are regular pairs, some vertices may have just a small number of neighbors in the other cluster (this number can be even zero). To avoid problems which can be caused by this fact, we are going to discard some vertices of the clusters, and put them into V_0. If a considerable portion of the independent vertices are in I_3 (say, more than $\frac{1}{3}|I|$), then we make all the edges of the K_4-cover super-regular. Removing at most $3\varepsilon m$ vertices from each cluster (those vertices, which has small degrees inside the K_4's), we put at most $3\varepsilon n$ new vertices to V_0. Otherwise, if I_3 is not big enough, then either I_1 or I_2 will be large enough, say I_1. Let $\mathcal{M}_1$ be the proportional matching provided by Lemma 7 for the clusters in G_r. For a fix cluster $V_t \in V(G_r)$ let $\mathcal{T}$ denote the set of triplets matched to V_t in $\mathcal{M}_1$. We call a vertex $v \in V_t$ *bad*, if v has degree less than $d - \varepsilon$ to at least half of the triplets in $\mathcal{T}$, i.e., to at least one of the clusters of those triplets.

Lemma 9 *No cluster in G_r can contain more than $6\varepsilon m$ bad vertices.*

Proof. For a cluster $V_t \in V(G_r)$ which is matched to the triplets of $\mathcal{T}$, let $\{v_1, \ldots, v_s\}$ denote the set of bad vertices. If $s > 6\varepsilon m$ then there should be a triplet in $\mathcal{T}$, to which more than $3\varepsilon m$ vertices of V have bad degrees. Thus to one of the clusters of this triplet there are more than εm vertices with degree less than $d - \varepsilon$, which contradicts the ε-regularity condition. **Q.E.D.**

By removing the $6\varepsilon m$ bad vertices from the cluster V_t we can guarantee that all its vertices have big degrees to at least half of the matched triplets, and overall at most $6\varepsilon n$ bad vertices will be added to V_0.

Observe that the number of vertices participating in the clusters of G_r is about $(1-\theta)n$, which means that θn vertices from H have to be mapped to vertices of the exceptional cluster V_0. In order to guarantee that the embedding of these θn vertices of H can be carried out, we perform a preprocessing step. This preprocessing is done as follows: first we assign the necessary number of I'_1-vertices (or I'_2-vertices, if I'_1 is too small) using the strong proportional matching, and second, we perform a *switching* procedure. Since for every $1 \leq i \leq \ell$ $|L(V_i)| \geq |V_i|$, we form H_0 by removing a subset of I' vertices which are assigned by the "ordinary" proportional matching, so that for every $1 \leq i \leq \ell$, $|L(V_i)| = |V_i|$. Also, we put the cut-out vertices to H_0. Let $\phi : H_0 \rightarrow V_0$ be any bijective mapping. We need to ensure that the assignment of H_0 to V_0 is consistent with $E(H)$; that is, for any $x \in H_0$, with $(x, y_1), (x, y_2)$, and $(x, y_3) \in E(H)$, if $v = \phi(x)$, $y_1 \in L(V_i)$, $y_2 \in L(V_j)$, and $y_3 \in L(V_k)$ then $deg_G(v, V_i)$, $deg_G(v, V_j)$, and $deg_G(v, V_k)$ are all at least $\frac{m}{10}$. If this condition does not hold for a pair (x, v), a switching will be performed. In the switching operation we will locate a vertex x' in $L(V_s)$ such that (V_i, V_s), (V_j, V_s), (V_k, V_s) are all edges in $E(G_r)$. Furthermore, if $(x', y'_1), (x', y'_2)$, and $(x', y'_3) \in E(H)$ with $y'_1 \in L(V_{i'})$, $y'_2 \in L(V_{j'})$, and $y'_3 \in L(V_{k'})$ then $deg_G(v, V_{i'})$, $deg_G(v, V_{j'})$, and $deg_G(v, V_{k'})$ are all at least $\frac{m}{10}$. We will see that such x' can be found among the preprocessed vertices.

Lemma 10 *For every $x \in H_0$ there exists an x' as required above.*

Proof. Let us discuss the case when this switching is done using I'_2-vertices; the case of I'_1-vertices is similar. It is easy to see that for $v \in V_0$, to at least 72% of all the clusters v has degree $\frac{m}{10}$. Let V_s be as above, the number of clusters in its neighborhood for which v has large degree ($\geq \frac{m}{10}$) is at least $\frac{46}{100}\ell$, which in turn will span at least $\frac{14}{100}\frac{3\ell}{2}$ edges from the K_4 cover of G_r. We can conclude that at least 6% of the edge-vertex triplets will be spanned by these clusters. Any edge-vertex triplet τ will allocate ω vertices during the preprocessing. This means V_s receives at least $\frac{\omega}{\ell}\mu$ from every triplet which is connected to it. The triplets spanned by the clusters in the common neighborhood of V_s and v is at least 6% of the edge-vertex triplets. Letting $\Omega = \frac{3\ell^2\omega}{2}$, the number of assigned vertices during the preprocessing to V_s determined by this 6% of all the triplets is at least $\frac{6}{100}\frac{\mu}{\ell}\Omega$. Finally, if the number of vertices assigned by the strong proportional matching to the common neighborhood of V_i, V_j, and V_k is at least $|V_0|$, then we can find x'. More precisely we need

$$\left(\frac{1}{4} - 3\theta\right) \ell \frac{\mu}{\ell} \frac{6\Omega}{100} \geq |V_0|.$$

This is certainly true for $\Omega \geq \frac{100}{\mu}|V_0|$. **Q.E.D.**

An important remark is that we can perform this switching procedure in such a way that the neighbors of the switched x's are scattered in a constant proportion of the triplets, and so in a constant proportion of the clusters.

We are ready to give the criterion for H being non-extremal: $|I_1' \cup I_2'|$ should be at least as big as $\frac{100}{\mu}\theta n$. What we need is $\frac{100}{\mu}|V_0|$ vertices for the strong proportional matching, the rest of $|I_1' \cup I_2'|$ is assigned by the "ordinary" proportional matchings.

Now we are after consistently assigning the vertices of $H \backslash H_0$ to clusters of G_r, and we have already decided the actual embedding of the H_0-vertices in the preprocessing step. Note, that these embeddings will restrict us when embedding their neighbors.

The rest of the embedding procedure will follow the same line as the Blow-up Lemma. We will choose the parameters ε', ε'', ε''', δ''', δ'', and δ' so that

$$\varepsilon \ll \varepsilon' \ll \varepsilon'' \ll \varepsilon''' \ll \delta''' \ll \delta'' \ll \delta' \ll d.$$

For simplicity we will assume, that all the densities of the regular pairs are d, there are no bigger densities. This can be achieved by randomly discarding edges with the appropriate probabilities from the denser regular pairs. For every $1 \leq i \leq \ell$, the vertices of $L(V_i)$ will be embedded to the cluster V_i. Let $n' = |V(H \backslash H_0)|$, we order the vertices of $H \backslash H_0$ into a sequence $\mathcal{S} = (x_1, x_2, \ldots, x_{n'})$ which is almost the order in which $V(H \backslash H_0)$ will be embedded. For each $1 \leq i \leq \ell$, choose a subset B_i of $L(V_i)$ of size $\delta' m$ from I' (these are at distance at of least 4 from each other). Observe that by Lemma 6 the vertices $B = \cup_i B_i$ have almost the same number of neighbors in every list $L(V_j)$, $1 \leq j \leq \ell$. Let $M = |B|$, and $b_1, b_2, \ldots, b_M$ be the buffer vertices, then they will form the last part of $\mathcal{S}$. The sequence $\mathcal{S}$ starts with the vertices of $N_H(H_0)$, then comes $\{N_H(b_1),\ N_H(b_2), \ldots, N_H(b_M)\}$. We let $T_0 = \sum_{i=1}^{M} |N_H(b_i)|$. Then we add all the other vertices to the sequence, in such a way, that the buffer vertices form the tail of $\mathcal{S}$. For technical reasons we assume, that $\mathcal{S}$ is ordered evenly according to the $L(V_i)$ lists, i.e., the consecutive segments of length $\delta'' n'$ have the same number of vertices from every list. Later we may place some vertices forward, but then we rearrange $\mathcal{S}$ to maintain this property.

The embedding of the vertices of $H \backslash H_0$ has two separate phases. In the first phase we are going to embed the vertices of $N_H(H_0)$, and then will come the embedding of the next vertices of $\mathcal{S}$ after each other according to their position in the sequence (some reordering is possible), until we reach that just buffer vertices are left in $\mathcal{S}$. In the second phase, by a matching argument we embed the remaining buffer vertices.

4.4 Outline of the embedding algorithm

For an unembedded vertex $x \in L(V_i)$ let $H_{t,x}$ denote its monotonically shrinking *host set* in V_i at time t. Also, for technical reasons we keep track of another set, $C_{t,x}$. By Z_t we denote the set of occupied vertices (note that $Z_0 = V_0$), and we also maintain a set Bad_t of exceptional pairs in $H \backslash H_0$.

At time 0, we set $C_{0,x} = H_{0,x} = V_i$, where $x \in L(V_i)$, and x does not have any neighbor in H_0. For those vertices having neighbors in H_0 the setup is different. Let x in H_0 have neighbors $y_1 \in L(V_i)$, $y_2 \in L(V_j)$ and $y_3 \in L(V_k)$, and $v = \phi(x)$. By the end of the switching process, we have ensured that v has at least $\frac{m}{10}$ neighbors in V_i, V_j, and V_k. These neighborhoods give $C_{0,y_1} = H_{0,y_1}$, $C_{0,y_2} = H_{0,y_2}$ and $C_{0,y_3} = H_{0,y_3}$, respectively.

We let $T_1 = \delta'' n'$ and $T_2 = |N_H(H_0)|$. Given the initial host sets the embedding algorithm will go as follows:

Phase 1. For $t \geq 1$ repeat the following steps

Step 1. Embed the vertex x_t from the sequence $\mathcal{S}$. Using the *Selection Algorithm* choose an appropriate vertex v_t randomly, uniformly and independently from the previous choices from the set H_{t-1,x_t} as x_t's image.

Step 2. Update

$$Z_t = Z_{t-1} \cup \{v_t\}$$

and for all unembedded vertex x_i, with $t < i \leq n'$

$$C_{t,x_i} = \begin{cases} C_{t-1,x_i} \cap N_G(v_t) & \text{if } (x_i, x_t) \in E(H), \\ C_{t-1,x_i} & \text{otherwise,} \end{cases}$$

and

$$H_{t,x_i} = C_{t,x_i} \backslash Z_t$$

Step 3. Exceptional vertices in G

1. If $t \neq T_0 + T_2$ go to step 4.

2. If $t = T_0 + T_2$ then for every cluster V_i form a set E_i containing those uncovered vertices satisfying

$$|\{b : b \in B_i, v \in C_{t,b}\}| < \delta'' |B_i|.$$

We will cover them right after the neighbors of the buffer vertices, this way eliminating a possible objection to embed the buffer vertices in *Phase 2.* We slightly change the ordering of $\mathcal{S}$. From every list $L(V_i)$

we take $|E_i|$ vertices belonging to I', and place them forward. We will maintain the even ordering of $\mathcal{S}$, which is possible since the $|E_i|$'s are very small, as we will see later. The choice of these vertices is not arbitrary. First, these are from I', because this way they do not "disturb" each others neighborhood, and the neighborhoods of the buffers. Also, we want to cover the exceptional v vertex by such an I'-vertex, which has all its neighbors assigned to clusters to which v has degree big enough. Recall, that if I_3' is not small, then we make the edges of the K_4-cover super-regular pairs. In this case, we choose vertices from I_3' to cover the exceptional G-vertices. Note, that by Lemma 6 these I_3'-vertices are evenly distributed, thus, in every list $L(V_i)$ there is enough candidate to choose. Otherwise, if I_3' is not big enough, we are going to use vertices from I_1' (or I_2') for our purpose, and those, which are assigned by a proportional, but not a strong proportional matching. If H has this structure, we put to V_0 all the vertices of a cluster which have small degree to at least half of the triplets, which are determining, say, the I_1'-vertices of the list corresponding to that cluster. Thus, if v turns out to be an exceptional G-vertex, then v has degrees big enough to at least half of the "I_1'-sending" triplets, and we can choose vertices from I_1' to cover the exceptional G-vertices. Let us remark, that even in this case the neighbors of the H-vertices we use to cover are almost evenly distributed among all the lists. In no list can we find more than twice as many as the average.

Step 4. Exceptional vertices in $H\backslash H_0$

1. If T_1 does not divide t, then go to Step 5.

2. If T_1 divides t, we will find all exceptional unembedded vertices $y \in H\backslash H_0$ such that $|H_{t,y}| \leq (\delta')^2 m$. We again slightly change the order of the remaining vertices in $\mathcal{S}$ by bringing these exceptional vertices forward in $\mathcal{S}$ (including the exceptional buffer vertices) and will maintain the even distribution of vertices assigned to different clusters. This is possible because of the very small number of exceptional vertices we can find in this step.

Step 5. If the unembedded vertices are all buffer vertices, go to *Phase 2.*, otherwise set $t \leftarrow t+1$ and go back to *Step 1.*

Phase 2. Find a system of distinct representatives of the sets $H_{t,y}$ for all unembedded vertices.

4.5 Selection Algorithm

There can be two possible cases.

Case 1. $x_t \notin E_H$.
As the image of x_t, we will choose some $v_t \in H_{t-1,x_t}$ such that the following conditions are satisfied for every unembedded vertex y with $(x_t, y) \in E(H)$:

$$(d-\varepsilon)|H_{t-1,y}| \leq deg_G(v_t, H_{t-1,y}) \leq (d+\varepsilon)|H_{t-1,y}|, \quad (3)$$

$$(d-\varepsilon)|C_{t-1,y}| \leq deg_G(v_t, C_{t-1,y}) \leq (d+\varepsilon)|C_{t-1,y}|, \quad (4)$$

and

$$(d-\varepsilon)|C_{t-1,y} \cap C_{t-1,y'}| \leq deg_G(v_t, C_{t-1,y} \cap C_{t-1,y'}) \leq (d+\varepsilon)|C_{t-1,y} \cap C_{t-1,y'}|, \quad (5)$$

for at least $(1-\varepsilon')$ portion of the unembedded vertices y' so that y and y' are assigned to the same cluster V_i, and $\{y, y'\} \notin Bad_{t-1}$. The set Bad_t will be formed as the union of Bad_{t-1} and those pairs $\{y, y'\}$ which does not satisfy (5) for v_t. Clearly, at most $3\varepsilon' m$ new vertices will be added to Bad_t.

Case 2. $x_t \in E_H$.
As explained in the second part of *Step 3*, we can assign $x_t \in L(V_i)$ to an exceptional $v_t \in E_i$ so that for all unembedded $y \in N_H(x_t)$ the following is satisfied:

$$deg_G(v_t, C_{t-1,y}) = deg_G(v_t) \geq (d-\varepsilon)m \geq (d-\varepsilon)|C_{t-1,y}|, \quad (6)$$

and

$$deg_G(v_t, H_{t-1,y}) \geq deg_G(v_t) - 3\delta' m - |E_i| \geq (d-\varepsilon)m - 4\delta' m \geq \frac{d}{2}m. \quad (7)$$

In (7) we used the fact that for each i $|E_i| \leq \delta' m$. We will prove this in Lemma 12.

4.6 Correctness

We start by proving that *Phase 1.* of the algorithm succeeds. First consider the case when $1 \leq t \leq T_2$.

Let $x \in N_H(H_0)$, then $|H_{0,x}| \geq \frac{m}{10}$. In Lemma 16 we will prove that if *Phase 1.* succeeds up to time t and $|H_{t,x}| \geq \delta'' m$, then we can find a v_t to embed x. Since no two vertices in $N_H(H_0)$ are adjacent, the only way the host set of x decreases is that we cover some vertices of it by other $N_H(H_0)$-vertices. Note, that we choose the host vertex randomly, uniformly and independently, and the $3|V_0|$ restricted vertices are distributed among at least 6% of the triplets, as evenly as possible. Thus, the number of covered vertices in a host set at time t is proportional to the number of embedded vertices up to time t ($t \le T_2$). Now using the statement of Lemma 16 one can easily conclude that up to time T_2 all the unembedded $N_H(H_0)$-vertices have a host set of size at least $\frac{m}{20}$.

A vertex from the rest of H can have three neighbors embedded by time T_2. This means these sets may shrink three times, each time the sizes are multiplied by a number between $d-\varepsilon$ and $d+\varepsilon$, and we also lose some places because of the embedded $N_H(H_0)$-vertices. Again, the randomness in the *Selection Algorithm* helps us, just a little bit more than the expectation can be covered in those host sets when embedding an $N_H(H_0)$-vertex. Hence, for every unembedded vertex x at time t ($1 \le t \le T_2$) $|H_{t,x}| \ge d^4 m$.

For $t > T_2$ we need a more thorough analysis. At time t for the cluster V_i and a subset of the unembedded vertices $L_i \subseteq L(V_i)$ we define a bipartite graph $U_t = (V_i, L_i, E(U_t))$. Here if $x \in L_i$, $v \in V_i$, and $v \in C_{t,x}$ then $(x, v) \in E(U_t)$.
The following lemma is pivotal for the proof of the correctness of *Phase 1*.

Lemma 11 *For every $1 \le i \le \ell$ and $T_2 + 1 \le t \le T_0 + T_2$ and any set of unembedded vertices $L_i \subseteq L(V_i)$ at time t, with $|L_i| \ge (\delta''')^2 m$, if Phase 1 succeeded for all $t' \le t$ then apart from an exceptional set F of size at most $\varepsilon'' m$ the following will be true for every $v \in V_i$:*

$$deg_{U_t}(v) \ge (1 - \varepsilon'')d(V_i, L_i)|L_i|.$$

Proof. We are going to use the so called "defect form" of the Cauchy-Schwarz inequality, which states that if for some $p \le q$

$$\sum_{i=1}^{p} \alpha_i = \frac{p}{q} \sum_{i=1}^{q} \alpha_i + \beta$$

then

$$\sum_{i=1}^{q} \alpha_i^2 \ge \frac{1}{q} \left(\sum_{i=1}^{q} \alpha_i \right)^2 + \frac{\beta^2 q}{p(q-p)}.$$

Assume to the contrary that the lemma is not true, that is $|F| > \varepsilon'' m$. Choose $F_0 \subset F$ with $|F_0| = \varepsilon'' m$. Define $\nu(t, x)$ as the number of embedded neighbors of x by time t. Observe that if x has a neighbor in H_0 then $\nu(0, x) = 1$, otherwise it is 0.

$$|E(U_t)| = \sum_{x \in L_i} |C_{t,x}| \ge \sum_{x \in L_i} (d - \varepsilon)^{\nu(t,x)} m. \quad (8)$$

We also have

$$\begin{aligned}
&\sum_{x\in L_i}\sum_{x'\in L_i}|C_{t,x}\cap C_{t,x'}|\\
&\le \sum_{x\in L_i}\sum_{x'\in L_i}(d+\varepsilon)^{\nu(t,x)+\nu(t,x')}m+|L_i|m+6|L_i|m+3\varepsilon' m^3\\
&\le \sum_{x\in L_i}\sum_{x'\in L_i}(d+\varepsilon)^{\nu(t,x)+\nu(t,x')}m+4\varepsilon' m^3 \qquad (9)
\end{aligned}$$

For each pair $\{x,x'\}$ we can upperbound $|C_{t,x}\cap C_{t,x'}|$ by m. The diagonal terms ($x=x'$) result in error $|L_i|m$. For the nondiagonal terms for which $N_H(x)\cap N_H(x')\neq\emptyset$ we have the term $6|L_i|m$. If $\{x,x'\}\in Bad_t$, by **Case 1** of the *Selection Algorithm* either x or x' can appear in at most $3\varepsilon' m$ bad pairs. Hence there will be at most $3\varepsilon' m^2$ bad pairs associated with the cluster V_i. Using the Cauchy-Schwarz inequality with $p=\varepsilon'' m$, $q=m$ and the variables $\alpha_k=deg_{U_t}(v_k)$, $1\le k\le m$ with $v_k\in V_i$ and the first $\varepsilon'' m$ values set to degrees in F_0, we have:

$$\begin{aligned}
|\beta| &= \varepsilon''\sum_{v\in V_i}deg_{U_t}(v)-\sum_{v\in F_0}deg_{U_t}(v)\\
&\ge \varepsilon''\sum_{v\in V_i}deg_{U_t}(v)-\varepsilon''(1-\varepsilon'')d(V_i,L_i)|L_i|m\\
&= (\varepsilon'')^2\sum_{v\in V_i}deg_{U_t}(v). \ (10)
\end{aligned}$$

Then using (8), (10) and the Cauchy-Schwarz inequality we get

$$\begin{aligned}
&\sum_{x\in L_i}\sum_{x'\in L_i}|C_{t,x}\cap C_{t,x'}|\\
&=\sum_{v\in V_i}(deg_{U_t}(v))^2\\
&\ge \frac{1}{m}\left(\sum_{v\in V_i}deg_{U_t}(v)\right)^2+(\varepsilon'')^3 d(V_i,L_i)^2 m|L_i|^2\\
&\ge \frac{1}{m}\left(\sum_{x\in L_i}(d-\varepsilon)^{\nu(t,x)}m\right)^2+(\varepsilon'')^3(d-\varepsilon)^6 m|L_i|^2\\
&\ge \sum_{x\in L_i}\sum_{x'\in L_i}(d-\varepsilon)^{\nu(t,x)+\nu(t,x')}m+(\varepsilon'')^3(d-\varepsilon)^6 m|L_i|^2
\end{aligned}$$

which is a contradiction to (9), since $|L_i|\ge(\delta''')^2 m$,

$$(\varepsilon'')^3(d-\varepsilon)^6(\delta''')^2\gg 4\varepsilon'\gg 4\varepsilon$$

and

$$(d+\varepsilon)^{\nu(t,x)+\nu(t,x')}-(d-\varepsilon)^{\nu(t,x)+\nu(t,x')}\ll 4\varepsilon.$$

Q.E.D.

As a consequence we will have the following bound on the size of the exceptional sets E_i:

Lemma 12 *In Step 3, for each $1 \leq i \leq \ell$ we have $|E_i| \leq \varepsilon'' m$.*

Proof. Applying the previous lemma with $t = T_0$ and $L_i = B_i$, which means $|L_i| \geq (\delta''')^2 m$, we will have

$$(1-\varepsilon'')d(V_i, L_i)|L_i| \geq \frac{d^3}{2}|L_i| > \delta''|L_i|$$

and $E_i \subset F$. **Q.E.D.**

Next we will prove a result similar to Lemma 11 for $t > T_0 + T_2$.

Lemma 13 *For every $1 \leq i \leq \ell$ and $T_0 + T_2 < t \leq T$ and any set of unembedded vertices $L_i \subseteq L(V_i)$ at time t, with $|L_i| \geq (\delta''')^2 m$, if Phase 1 succeeded for all $t' \leq t$ then apart from an exceptional set of size at most $\varepsilon''' m$ the following will be true for every $v \in V_i$:*

$$deg_{U_t}(v) \geq (1-\varepsilon''')d(V_i, L_i)|L_i|.$$

Proof. The proof follows the same line of argument as Lemma 11 with parameter ε''', except those vertices in the neighborhood of E_H. The inequality in (8) will hold with the same parameters, since for all $x \in N_H(E_H)$ we will also have

$$|C_{t,x}| \geq (d-\varepsilon)^{\nu(t,x)} m.$$

In (9) there are more bad pairs, more precisely based on *Step 3* of the embedding algorithm there will be an additional error term of $6\varepsilon'' m^2 |L_i|$. Using the fact that

$$(\varepsilon''')^3 (d-\varepsilon)^6 (\delta''')^2 \gg \varepsilon''$$

we can see that the contradiction still holds. **Q.E.D.**

The following lemma is an easy consequence of Lemmas 11 and 13.

Lemma 14 *For every $1 \leq i \leq \ell$ and $T_2 < t \leq T$ and any set of unembedded vertices $L_i \subseteq L(V_i)$ at time t, with $|L_i| \geq (\delta'')m$ and a set $A \subset V_i$ with $|A| \geq (\delta''')m$, if Phase 1 succeeded for all $t' \leq t$ then apart from an exceptional set F of size at most $(\delta''')^2 m$, the following will be true for every $x \in L_i$:*

$$|A \cap C_{t,x}| \geq \frac{|A|}{2m}|C_{t,x}|.$$

Proof. Let us suppose, that the lemma is not true, there exists a set $F \subseteq L_i$ such that $|F| > (\delta''')^2 m$, and for every $x \in F$ the inequality of the statement does not hold. We again consider the bipartite graph $U_t = U_t(F, V_i)$.

$$\sum_{v \in A} deg_{U_t}(v) = \sum_{x \in F} |A \cap C_{t,x}| < \frac{|A|}{2m} d(F, V_i)|F|m.$$

Applying Lemmas 11 or 13 with F, we get

$$\sum_{v \in A} deg_{U_t}(v) \geq (1 - \varepsilon''')d(F, V_i)|F|(|A| - \varepsilon''' m),$$

which is a contradiction. **Q.E.D.**

In the following lemma we show that the host sets can never get too small.

Lemma 15 *For every $T_2 + 1 \leq t \leq T$ and for every H-vertex y which is unembedded at time t, if Phase 1 succeeded for all $t' \leq t$ then the following will be true:*

$$|H_{t,y}| > \delta'' m.$$

Proof. Let L_i be the set of all the unembedded vertices in V_i at time t, and let $A_t = V_i \backslash Z_t$. Applying Lemma 14 we can see, that for all $x \in L_i$ (except at most $(\delta''')^2 m$ vertices)

$$|H_{t,x}| = |A_t \cap C_{t,x}| \geq \frac{|A_t|}{2m}|C_{t,x}| \geq \frac{\delta'}{4}(d - \varepsilon)^3 m \gg (\delta')^2 m,$$

if $|A_t| \geq \frac{\delta'}{2} m$. Next we prove this statement. Let us suppose indirectly, that there is a $T_2 + 1 \leq T' < T$ such that

$$|A_{T'}| \geq \frac{\delta'}{2} m \text{ but } |A_{T'+1}| < \frac{\delta'}{2} m.$$

We know, that at any time t, where T_1 divides t, there are at most $(\delta''')^2 m$ exceptional unembedded vertices. Thus, up to time T' we can find at most

$$\frac{1}{\delta''}(\delta''')^2 m \gg \delta'' m$$

exceptional vertices. This implies, that at time T' there are many more than $(\delta' - \delta'')m$ unembedded buffer vertices, thus, on the contrary, $|A_{T'+1}| \gg (\delta' - \delta'')m$. Note, that we also proved, that $T \leq \ell m - \ell\delta' m + \ell\delta'' m$. Let us consider now an arbitrary $y \in L(V_i)$ unembedded at time t $(1 \leq t \leq T)$, and let $k\delta'' n' = kT_1 \leq t < (k+1)T_1$ for some $0 \leq k \leq T/T_1$. There are two cases to discuss:

Case 1. If y was not among the at most $(\delta''')^2$ exceptional vertices of *Step 4*, then

$$|H_{t,y}| \geq \left(\frac{d}{2}\right)^3 (\delta')^2 m - K,$$

where K is the number of vertices covered in V_i during the period between kT_1 and $(k+1)T_1$. Recall, that the sequence $\mathcal{S}$ is as balanced as possible, hence, $K \leq 2\delta'' m$, where $2\delta'' m$ comes from the reordering of $\mathcal{S}$ because of the exceptional vertices of G and H. Also, at time kT_1 we had, that $|H_{kT_1,y}| \geq (\delta')^2 m$. These facts imply that in this case the statement of the lemma holds.

Case 2. If y was among the at most $(\delta''')^2$ exceptional vertices of *Step 4*, then

$$|H_{t,y}| \geq \left(\frac{d}{2}\right)^3 (\delta')^2 m - K',$$

where K' is the number of vertices covered in V_i during the period between $(k-1)T_1$ and $(k+1)T_1$. Now K' can be as big as $(\delta'' + (\delta''')^2)m$, because at time $(k-1)T_1$ at most $(\delta''')^2 m$ exceptional vertices were placed forward. Again, by observing, that at time $(k-1)T_1$ we had, that $|H_{(k-1)T_1,y}| \geq (\delta')^2 m$, the proof of the lemma is finished. **Q.E.D.**

In what follows we show the succession of the *Selection Algorithm* in finding the v_t-vertices.

Lemma 16 *Assuming that Phase 1 succeeds for all t', with $t' < t \leq T$ then it succeeds for t.*

Proof. We only need to consider **Case 1** of the *Selection Algorithm.* The selected vertex $v_t \in H_{t-1,x_t}$ satisfies conditions (3), (4), and (5). By Lemma 15 we have

$$H_{t-1,x_t} \geq \delta'' m,$$

and by the ε-regularity we will have at most $2\varepsilon m$ vertices in H_{t-1,x_t} which do not satisfy (3), and the same holds for (4). For condition (5) we will define a bipartite graph $B = (W_1, W_2, E(B))$, where $W_1 = H_{t-1,x_t}$, and W_2 is the sets $C_{t-1,y} \cap C_{t-1,y'}$ for all pairs $\{y, y'\}$ where $(x_t, y) \in E(H)$, y and y' are both assigned to the same cluster, and $\{y, y'\} \notin Bad_{t-1}$. For $v \in W_1$ and $u \in W_2$, we have $(v, u) \in E(B)$ if (5) does not hold for v and the pairs corresponding to u. If we assume that there are more than $\varepsilon' m$ vertices $v \in W_1$ with $deg_B(v) > \varepsilon'|W_2|$, there should be a vertex $u \in W_2$ such that

$$deg_B(u) > \varepsilon' m \gg \varepsilon m.$$

But this is contradictory to the ε-regularity since the pair $\{y_u, y'_u\}$ corresponding to u does not belong to Bad_{t-1} and

$$|C_{t-1,y_u} \cap C_{t-1,y'_u}| \geq (\delta - \varepsilon)^6 m \gg \varepsilon m.$$

This in turn implies that H_{t-1,x_t} can contain at most $4\varepsilon m + \varepsilon' m \ll \delta'' m$ vertices which can not be used to map x_t, proving the succession of *Phase 1*. **Q.E.D.**

To prove that *Phase 2* of the algorithm succeeds, we will need to show that for all $1 \leq i \leq \ell$ there is a system of distinct representatives between the unembedded vertices of $L(V_i)$ and the remaining buffer vertices of V_i. Let $L_i \subset L(V_i)$ denote the set of unembedded vertices assigned to the cluster V_i, and $Y_i \subset V_i$ be the remaining vertices of the cluster V_i, with $M_i = |L_i| = |Y_i|$. Then by Lemma 15 for every $x \in L_i$ we will have $H_{T,x} > \delta''' M_i$. Furthermore, for all subsets $S \subset L_i$, if $|S| \geq \delta''' M_i$ then by Lemma 13

$$\left| \bigcup_{x \in S} H_{T,x} \right| \geq (1 - \delta''') M_i.$$

Finally, for any $v \in Y_i$, since v can not be exceptional in G, by *Step 3* there are at least $\delta''' M_i$ host sets $H_{T,x}$ containing v, which implies that for the subsets $S \subset L_i$ with $|S| \geq (1 - \delta''' M_i)$ we have

$$\left| \bigcup_{x \in S} H_{T,x} \right| = M_i,$$

which in turn implies the existence of the system of distinct representatives.

This finishes the proof in the non-extremal case. **Q.E.D.**

5 Extremal Case

Now we are assuming that apart from at most $\frac{100}{\mu}\theta n$ vertices of $\tilde{H} \subset H$, the rest of the $V(H)$ belong to disjoint K_4's. To handle this case we are going to apply a very useful general theorem, the Decomposition Theorem, see [1] or [2]. We will not use its full strength. On the other hand we will need to have a (very) little insight to the proof, but the details will be omitted.

Let us call a set $I \subset V$ of size p α-quasi independent, if I contains at most αp^2 edges.

Theorem 17 (Decomposition Theorem) *Let $0 < \varepsilon \ll d \ll \rho$, and k, h be any two positive integers. Then there exists an integer ℓ and a threshold n_0 such that if*

$G=(V,E)$ is an arbitrary n-vertex graph with $\delta(G)\ge \frac{k-1}{k}n$, and $n\ge n_0$, then either G contains a $(2k^3\rho)$-quasi independent set of size $\frac{n}{k}$, or we can partition V into ℓ clusters $V_1,\dots,V_\ell$ such that

$$\frac{n}{\ell}(1-\rho)\le |V_i|\le \frac{n}{\ell}(1+\rho).$$

Furthermore, this partition is decomposed into blocks $\mathcal{A}_1,\dots,\mathcal{A}_s,\mathcal{B}_1,\dots,\mathcal{B}_r$ such that the following properties hold:

- **Decomposition into k and $(k+1)$-blocks** *Each $\mathcal{A}$-block contains k clusters, and each $\mathcal{B}$-block contains k clusters. We will refer to an $\mathcal{A}_i$ as a k-block, and a $\mathcal{B}_j$ as a $(k+1)$-block.*

- **Super-regularity within blocks** *Any two clusters V_a and V_b in a k-block form an (ε,d)-super-regular pair. Similarly, any two clusters V_a and V_b in a $(k+1)$-block form an (ε,d)-super-regular pair.*

- **Extra $(k+1)$-blocks** *The number of the $(k+1)$-blocks is at least $\rho\ell$.*

- **Perfect balance in the k-blocks** *For every k-block $\mathcal{A}_i$ each cluster contains m_i vertices, and h divides m_i.*

- **Connectivity from k-blocks to $(k+1)$-blocks** *For every k-block $\mathcal{A}_i=\{V_1,\dots,V_k\}$ there is a $(k+1)$-block $\mathcal{B}_j=\{W_0,\dots,W_k\}$ (with its clusters ordered appropriately) such that (V_p,W_q) is (ε,d)-regular for all $p\ne q$.*

- **Connectivity among $(k+1)$-blocks** *Let us construct an auxiliary graph F on r vertices $\{b_1,\dots,b_r\}$, every vertex corresponding to a $(k+1)$-block. We connect b_i and b_j if there is a cluster $W\in\mathcal{B}_i=\{W_0,\dots,W_k\}$ such that there are at least $k-1$ (ε,d)-regular edges connecting W with the clusters of the $(k+1)$-block $\mathcal{B}_j$. Then F is a connected graph.*

Summarizing this theorem it states that either we can cover the reduced graph by a non-negligible number of $(k+1)$-cliques and the left by k-cliques, or G is extremal, there is a large quasi independent set in it. Besides, if G is not extremal we have some connectivity structure among the cliques.

Now we discuss what is important for us in the proof. It can be divided into two parts for the non-extremal case. First, we apply the Degree Form of the Regularity Lemma, and find the K_{k+1}-K_k cover of the clusters with enough K_{k+1}'s. During this process we may cut an original cluster into at most k equal pieces. Observe, that this way we do not "destroy" the regularity of the (ε,d)-regular edges – the only change is that instead of ε we will have $k\varepsilon$. In the sequel we will still use the same letter for that parameter. After finding such a cover we make the edges of the cliques super-regular pairs, and we put the discarded vertices into V_0. Then we distribute V_0 among the cliques in such a way that we do not ruin the super-regularity of the edges within a clique. This can be done because we have enough K_{k+1}'s, and using the minimum degree condition either a vertex has big degrees to all the k clusters of some k-cliques, or it should have big degrees to at least k clusters of some $(k+1)$-cliques. In the first case we can put the exceptional vertex to any cluster of that k-clique, while in the second case we put the vertex to the cluster in which it has the smallest number of neighbors (this is why we cannot expect the $(k+1)$-cliques to be perfectly balanced). In the case of the k-cliques we have to pay attention to the perfect balance of its clusters, but this can be done using the connectivity structure among the k and $(k+1)$-cliques.

In proving that an extremal H can be embedded into G we are going to distinguish two cases: the case of an extremal and a non-extremal G. The method for embedding H will be essentially different in these two cases.

5.1 G is non-extremal

Let us give the outline of the embedding in this case. First, we embed $\tilde{H}$ in a similar way what we did previously for a non-extremal H. We find the structure provided by the Decomposition Theorem, but we stop before distributing V_0. We do it so because we need the ε-regular edges. Using Lemmas 7 and 8 we almost evenly distribute $\tilde{H}$ among the clusters of G_r. Then we embed $\tilde{H}$ by a procedure similar to that of the Blow-up Lemma. Notice, that now we have a lot of room because $\tilde{H}$ is very small, thus we do not have to be careful, e.g., we do not need buffer vertices. Then we distribute V_0 among the 4 and 5-cliques of the cover. We can do so if the degrees are big enough. Note, that the exceptional vertices may lost some neighbors, so we have to be careful when embedding $\tilde{H}$. If $|\tilde{H}| > dn$, then we randomly choose $\frac{2}{\ell}|\tilde{H}|$ vertices from every cluster of G_r, and we embed $\tilde{H}$ into those sub-clusters. The leftover vertices of these sub-clusters we put back to the original clusters. If ε is small enough then we will have the necessary "regularity" for this embedding, although, instead of ε we will have a parameter at most $\frac{\varepsilon}{d}$. Now, because we choose a random subset for the embedding, we can assume that every exceptional vertex has degree big enough – we lose the neighbors proportionally to the size of $\tilde{H}$. If $|\tilde{H}| \le dn$ then we do not have any problem: just notice, that $d \ll \rho$, thus, we don't even need random subsets. Let us assume now, that we are after the embedding of $\tilde{H}$ and we distributed the V_0-vertices appropriately to achieve the structure described in the Decomposition Theorem. We claim that this subgraph G' can be covered by disjoint 4-cliques. It is an easy consequence of the Blow-up Lemma that we can cover the 4-blocks by 4-cliques: all the edges within a 4-block are super-regular pairs, and every 4-block is perfectly balanced. For finishing the K_4-cover we use the argument of [1] (there it was used to give an alternate proof for the Hajnal-Szemerédi theorem for a non-extremal G). We repeat this short and elegant proof here.

Lemma 18 *Let $K_{h_1,\ldots,h_5}$ be a complete 5-partite graph with color classes sized $h_1,\ldots,h_5$, respectively. If $h_1+\ldots+h_5=4m$ and $h_i\le m$ for all i, then $K_{h_1,\ldots,h_5}$ can be covered by m vertex disjoint 4-cliques.*

Proof. We proceed by induction on m. Let us denote the color classes of $K_{h_1,\ldots,h_5}$ by $V_1,\ldots,V_5$, respectively. For $m=0$ there is nothing to prove, so let $m>0$. We may assume that $h_1\le\ldots\le h_5$. First observe that the case $h_1=m$ is impossible, it implies that $h_1=\ldots=h_5$, hence $h_1+\ldots+h_5=5m$, i.e., $m=0$. Pick the first clique by choosing one-one vertex from the classes $V_2,\ldots,V_5$. Now we can apply the induction hypothesis, because $h_i\le m-1$ for all i. **Q.E.D.**

Applying Lemma 18 and the Blow-up Lemma we can find a 4-clique cover of a 5-block, if the sum of the cluster sizes is divisible by 4. For the "bad" 5-blocks we are going to use the connectivity structure among the 5-blocks, provided by the Decomposition Theorem. Recall, that the auxiliary graph F is connected. Let $\mathcal{B}_i$ be a bad block, then there is at least one more bad block $\mathcal{B}_j$ in the cover. This follows from the fact the the number of the vertices is divisible by 4. Denote the closest bad 5-block in F by $\mathcal{B}_j$. Consider any $(\mathcal{B}_i,\mathcal{B}_k)$ edge in F. It is easy to find a 4-clique using vertices from both blocks: if W is a cluster of a block having three regular edges to the clusters of the other block, then just take one-one vertex from W and the other three, which constitute a 4-clique. By extracting at most three 4-cliques with this method we can achieve that the sum of the cluster sizes of the block $\mathcal{B}_i$ is divisible by 4. On the other hand, we may destroy the divisibility of the other block. But repeating this clique extracting step starting from $\mathcal{B}_i$ going on the shortest path to $\mathcal{B}_j$ we can achieve that all the blocks in that path except possibly $\mathcal{B}_j$, the last block, are good. So, we could decrease the number of bad blocks. Note, that there cannot be exactly one bad 5-block, this contradicts with the divisibility by 4. Hence, in a finite number of steps, after extracting a finite number of K_4's we will have only good 5-blocks. This finishes the proof of the embedding into a non-extremal G.

5.2 G is extremal

We arrived to the point when we really need all the edges incident to the G-vertices. So far we could have managed to find the embedding even with all the degrees equal to $\frac{3}{4}n-o(n)$, but now we need the full strength of the minimum degree condition of G.

The rough outline of the embedding in this case is as follows. There is a quasi independent set $P \subset V(G)$ of size $\frac{n}{4}$. Either we can find the embedding of H into G in such a way, that almost all the vertices of $I(H)$ are in P, and the rest is embedded into $V(G)\backslash P$, or we can find another quasi independent set $Q \subset (V(G)\backslash P)$ of size $\frac{n}{4}$. Then either we can embed almost all the vertices of the maximum independent set in $H\backslash I(H)$ into Q, I into P, and the rest to $V(G)\backslash(P \cup Q)$, or we can find a quasi independent set $R \subset V(G)\backslash(P \cup Q)$ of size $\frac{n}{4}$. In this last case we can find the embedding easily. In the following we sometimes write n in the form $4t + r$, where $0 \le r \le 3$.

Case 1: one quasi independent set

By the Decomposition Theorem we know that there is a (128ρ)-quasi independent set P of size $\frac{n}{4}$ in $V(G)$, where $0 < \varepsilon \ll d \ll \rho$. More precisely, if $n = 4t + r$ and $r = 0$ then $|P| = t$, otherwise $P = t + 1$. Let us fix ρ to have that $128\rho = 10^{-100}$ (this number is small enough for our purposes, and convenient to calculate with it). It is useful to perform the following procedure: if there is a vertex $v \in V(G)\backslash P$ and a vertex $u \in P$ such that u has more neighbors in P than v, then we interchange them. There are two kind of exceptional vertices which may cause problems to us. First, the vertices of $Exc_1 \subset P$, which do not have enough neighbors in $V(G)\backslash P$, more precisely, $u \in Exc_1$, if $deg_G(u, V(G)\backslash P) \le (1 - 10^{-50})\frac{3n}{4}$. Second, $v \in Exc_2 \subset V(G)\backslash P$, if $deg_G(v, P) \le (1 - 10^{-50})\frac{n}{4}$. An easy calculation shows that neither $|Exc_1|$, nor $|Exc_2|$ can be bigger than $10^{-50}n$.

From the fact, that $\delta_G \ge \frac{3n-1}{4}$ it is easy to see, that the minimum degree in $V(G)\backslash P$ is always big enough to provide a triangle cover in it by the Hajnal-Szemerédi theorem. Depending on r we may have at most two vertices left out from these triangle cover. We take care of them later.

Our strategy is to embed the 4-cliques of $V(G)\backslash P$ in such a way, that for almost all the cliques one vertex is embedded into P and the three others into $V(G)\backslash P$, $I(\tilde{H})$ is embedded into P, and $\tilde{H}\backslash I(\tilde{H})$ into the triangles of $V(G)\backslash P$. Our procedure will guarantee that either we can do this, or we can find another quasi independent set of size $\frac{n}{4}$, this time in $V(G)\backslash P$.

We start by taking care of the exceptional vertices. Let us consider a $v \in Exc_2$, and first suppose, that v is the only exceptional vertex in the triangle $\tau = \{v, v', v''\}$ from the triangle cover of $V(G)\backslash P$. If we cannot extend it to a 4-clique with a P-vertex, then v has at most $2 \cdot 10^{-50}n$ neighbors in P. Hence, it has almost full degree to $V(G)\backslash P$. Note, that v' and v'' has at least $\frac{n}{4}$ common neighbors in $V(G)\backslash P$, almost all of them are the neighbors of v. If $u \in P$ is adjacent to v, then u also has at most $2 \cdot 10^{-50}n$ neighbors in P, otherwise we would have interchanged them in the beginning. Important to observe that if S is a small subset of $V(G)\backslash P$ then by the minimum degree condition $|P \cap N_G(S)| \ge |S|$, no matter what is the value of

r. Hence, there are enough neighbors for Exc_2 in P. (Here we need the full strength of the minimum degree condition.) ¿From these facts it follows, that we can find a "switching triangle" τ' for τ in such a way, that one vertex of τ' changes place with v, in the new τ' v has a common P-neighbor with the other vertices, and there is no exceptional vertex in the new τ.

Next we assume, that $v, w \in Exc_2$ are in the same triangle $\tau = \{v, w, v'\}$. If there is no common neighbor for τ in P, then v and w can be separated by placing them into disjoint triangles of the cover in such a way, that both triangles contain only one exceptional vertex. This is a simple consequence of the fact that vertices in Exc_2 have almost full degree to $V(G)\backslash P$.

The last case to consider is when τ is a triangle inside Exc_2, i.e., $\tau = \{v, w, u\}$, where $v, w, u \in Exc_2$. Assuming that there is no common neighbor in P for the vertices of τ one can find a "switching triangle" for τ to achieve that no triangle in the cover of $V(G)\backslash P$ contains more than two vertices from Exc_2. Again, this can be shown by a simple degree counting argument.

Thus, our strategy with the vertices in Exc_2 is first to find distinct triangles for them, and then if necessary to place them into such triangles, which have common neighbor in P.

Let us consider now the vertices of Exc_1. Assume, that $v \in Exc_1$ cannot be matched to a triangle, i.e., v may have degree 3 to some triangles, but we cannot match all Exc_1 to one-one triangle in $V(G)\backslash P$. What we want is to find 4-cliques containing the exceptional vertices of Exc_1. Note, that if for all $v \in Exc_1$ $deg_G(v, V(G)\backslash P) \geq 10^{-50}n + \frac{n}{2}$, then we can match the Exc_1-vertices to triangles. This implies, that when this matching is not possible, then Exc_2 is empty. Also, v has exactly two edges to most of the triangles in the cover: $\frac{n}{2} \leq deg_G(v, V(G)\backslash P) < 10^{-50}n + \frac{n}{2}$ implies, that v has exactly two neighbors in at least $\frac{n}{4} - 3 \cdot 10^{-50}n > \frac{n}{4} - 10^{-49}n$ triangles. We are going to consider only these triangles now, denote the set of them by T. In the ith triangle of T call the vertex a_i which is not connected to v, and b_i, c_i those, which are adjacent to v. If some a_i has degree 3 to another triangle in T, then we can interchange a_i with v. This way we still have the triangle cover in $V(G)\backslash P$, and we have one more 4-clique with one vertex in P. Let us now suppose, that this is not possible. We can conclude from it that no a_i has degree 3 to more than $3 \cdot 10^{-50}n$ triangles in the cover. Denote the set of a_i's by A ($|A| \geq \frac{n}{4} - 10^{-49}n$), and suppose that some a_j has $10^{-48}n$ neighbors in A. Let $a_k \in A$ be one such neighbor. We know that a_k is connected to b_j or c_j, but not to both of them, say, to b_j. If b_j is adjacent to b_k and c_k, then we find the following: $\{a_j, c_j, a_k\}$ constitutes a triangle, and $\{b_k, c_k, b_j\}$ is such a triangle, which is connected to v. If this is not possible, b_j has just one neighbor in the k-th triangle, then b_j "loses" one edge. Because the average is at least two edges into one triangle, either we can find the above configuration, or

b_j (or c_j) is connected to at least $10^{-48}n$ triangles in the cover with full degree. But in this case we can find two 4-cliques and one triangle in the following way: b_k and c_k has a lot of common neighbors in P, and v is connected to most of them, because it has small degree into $V(G)\backslash P$. These four vertices give one 4-clique. The other 4-clique is composed by b_j and the vertices of the triangle to which it is connected to. Finally, $\{a_j, c_j, a_k\}$ is again a triangle. In this procedure we used 3 vertices from P and 9 from $V(G)\backslash P$, hence, we could retain the balance. We can conclude, that either for all the vertices in Exc_1 we can do the above procedure, or we find a quasi independent set of size at least $\frac{n}{4} - 10^{-49}n$ with maximum degree at most $10^{-48}n$.

We now show, how to embed H into G, if we can "handle" the vertices of Exc_1. So far we could find 4-cliques for all the exceptional vertices. Let us continue the embedding of H with the "noise": $\tilde{H}$. What we do is the following: We decompose $\tilde{H}$ into a maximum independent set $I(\tilde{H})$ and the rest, $C(\tilde{H})$. Embed the vertices of $C(\tilde{H})$ into the triangles of $V(G)\backslash P$, using the least possible triangles and not touching the triangles which we used for the 4-cliques for the exceptional vertices. Then find P-vertices for embedding $I(\tilde{H})$, again avoiding those vertices which are used for cliques containing the exceptional vertices. That this is doable can be checked by using the König-Hall conditions. It is possible, that $\tilde{H}$ is not nicely balanced, i.e., $|I(\tilde{H})| > 3|C(\tilde{H})|$. To make the leftover balanced, we actually put the necessary number of vertices to P, taking into account those vertices we put aside in the beginning because of the value of r. Doing so we do not produce any new Exc_2-vertices, but possibly new Exc_1-vertices. The number of this new exceptional vertices is very-very small, less than $|V(\tilde{H})|$. Noting that we performed loose computations, and we needed the precise minimum degree condition only in the case of the Exc_2-vertices, we can find 4-cliques for these new exceptional vertices in P. All the other vertices in P have big degrees to $V(G)\backslash P$, and $V(G)\backslash P$ vertices have big degrees to P-vertices. It is easy to see, that the triangles of $V(G)\backslash P$ can be matched to the vertices of P. This can be shown by checking the König-Hall conditions. Thus the embedding can be finished.

Case 2: two quasi independent sets

Let's go back to the case when we could find a big quasi independent set in $V(G)\backslash P$. Call this quasi independent set Q', by putting at most $10^{-49}n$ new vertices to it from $V(G)\backslash(P \cup Q')$, we get a quasi independent set Q of size t or $t+1$: if $r = 0$, then $|P| = |Q| = t$, otherwise $|P| = t$ and $|Q| = t+1$. Note that we may have to put a vertex from P to Q, in such a case choose an Exc_1-vertex (we know, that $Exc_1 \neq \emptyset$). Note, that all vertices in Q have big degrees to P, because Exc_2 is empty. The Q-vertices can have low degrees to $V(G)\backslash(P \cup Q)$, but any vertex in G has enough neighbors in $V(G)\backslash(P \cup Q)$ to provide. a perfect matching in it (in fact one vertex can be left out, which we put aside to be dealt with at the end of the embedding). Like in the previous case, we interchange a Q-vertex u with a $V(G)\backslash(P\cup Q)$-vertex v if u has more neighbors in Q than v. The subset of the vertices of Q having less than $(1-10^{-45})\frac{n}{2}$ neighbors in $V(G)\backslash(P\cup Q)$ will be denoted by Exc_3. Similarly, we denote the subset of vertices in $V(G)\backslash(P\cup Q)$ having less than $(1-10^{-45})\frac{n}{4}$ by Exc_4.

By counting the number of the edges going in between P and $V(G)\backslash(P\cup Q)$ we can give a (rude) upper bound on $|Exc_4|$: it is smaller than $10^{-40}n$. As with Exc_2, one can see, that by a possible rearranging of the edges of the perfect matching, all the Exc_4-vertices has a common neighbor in Q with their matched pair. This follows from their degrees to P, because in every case $|P|=t$ while $|Q|=t$ or $t+1$ depending on r. Also, by counting the degrees as in the above, one can conclude that if there is not enough edges in $V(G)\backslash(P\cup Q)$ connected to an Exc_3-vertex, then every vertex in Exc_4 has a large neighborhood in Q. This is because in this case at least one $v\in Exc_3$ has less than $\frac{n}{4}+10^{-40}n$ neighbors. Out of the $\frac{n}{4}$ matching edges v has degree two to at most $10^{-40}n$, hence, it has degree one to at most $10^{-40}n$ matching edges, implying that to at least $\frac{n}{4}-10^{-39}n$ edges of the matching it has degree one. Consider now these edges, and denote the vertex of the ith edge not connected to v by a_i, and the other by b_i. Let $\{a_i,b_i\}$ and $\{a_k,b_k\}$ be two edges. If a_i is adjacent to a_k, then a $\{b_i,a_k\}$ edge would imply, that we can change v and a_k, v is in a matching edge with b_k, and a_k is connected to the matching edge $\{a_i,b_i\}$. If $\{b_i,b_k\}$ is an edge, then we can change v and b_i, having the matching edges $\{v,b_k\}$ and $\{a_i,a_k\}$. Hence, if even with this kind of procedure we cannot handle v, then no a_i can have more than $10^{-38}n$ other a_k neighbors. Hence, the set of a_i's is a quasi independent set of size at least $\frac{n}{4}-10^{-39}n$, and with maximum degree $10^{-38}n$.

Still, we haven't considered the set Exc_1. For a $v\in Exc_1$ we are looking for a neighbor $w\in Q$ and an edge $\{u_1,u_2\}$ from the matching of $V(G)\backslash(P\cup Q)$ such that these four vertices constitute a 4-clique. If v has almost full degree to Q, then it is possible, that it is not connected to any matching edge in $V(G)\backslash(P\cup Q)$ with two edges. Like in the case of Exc_3 we get, that to at least $\frac{n}{4}-10^{-39}n$ matching edges v has degree one. Out of those edges consider the ones, for which none of the endpoints are from Exc_4. This way we are considering at least $\frac{n}{4}-10^{-38}n$ matching edges. Using our notation above, we get that if some a_i is connected to another a_k, then if b_i is adjacent to a_k, then interchanging v with a_k, and using that a_k has big degree to Q, we can find a K_4. Otherwise, if b_i is adjacent to b_k, then we have the 4-clique we wanted given by the vertices v, b_i, b_k and a Q-vertex connected to them. Observing, that b_i has at least $\frac{n}{4}$ neighbors inside, and this way it loses one potential neighbor, we conclude, that this can happen at most $10^{-38}n$ times. Thus, the set of a_i's is a quasi independent set of size at least $\frac{n}{4}-10^{-39}n$ and maximum degree $10^{-38}n$.

If $v\in Exc_1$ has just some neighbors in Q (less, than $|Exc_1|+|Exc_3|$), then again, either there is a big quasi independent set in $V(G)\backslash(P\cup Q)$, or we can find one (or two) 4-cliques in the following way. Considering the matching edges in the neighborhood of v, if a_i is connected to a_k, then a $\{b_i, b_k\}$ edge means, that we can change v with say, a_k, where a_k has big degrees to Q, thus we can find a 4-clique containing a_k. If b_i is connected to a_k, then we have the 4-clique with v, b_i, a_i and a_k. Then either for b_k, or after rearranging the edges of the matching for another vertex we have to find a 4-clique containing it, having two vertices from Q and one from P. Note, that this is possible, because most of the vertices in Q are not connected to v, thus, they have an extra degree inside. Otherwise, if $\{a_i, a_k\}$ is an edge, we lose one potential neighbor of b_i. This implies, that it can happen only $|Exc_1| + |Exc_3|$ many times. Thus, in either case we have a quasi independent set of size at least $\frac{n}{4} - 10^{-39}n$ and maximum degree at most $10^{-38}n$.

We now show, how to embed H into G, if the vertices of Exc_1 and Exc_3 did not cause problem for us, i.e., we could find 4-cliques for all the exceptional vertices. We embed the "noise" $\tilde{H}$ in the following way. First, we take care of the vertices of $C(\tilde{H})$. We embed one vertex of a cycle (or path) to Q, and then the next two vertices to an edge of the perfect matching, not touching those vertices, which are in a 4-clique with some exceptional vertex. The degrees are big enough, so this can be easily done, and at most one vertex can be left out from a matching edge. Then we just choose their neighbors from P, again not touching any 4-clique with some exceptional vertex. As previously happened, after this we may arrive to an unbalanced situation. Also, we may have an extra vertex depending on the parity of $|V(G)\backslash(P \cup Q)|$. We put the necessary number of vertices to P and maybe one to Q. Notice that this way we do not produce any Exc_4-vertex, only Exc_1 or Exc_3 can be increased. We again can argue in the same way as above: we performed loose calculations, so we can find 4-cliques for the new exceptional vertices. Then we match the edges of the perfect matching in $V(G)\backslash(P \cup Q)$ to Q-vertices. That this is doable can be shown by checking the König-Hall conditions. Now we are having triangles, and we want to match them to P vertices. Here we can repeat the same argument we had in the previous case. Hence, the embedding can be finished.

Case 3: quasi four-partite graph

Only the last case is left now: to show the existence of the embedding, when G is a quasi four-partite graph. We arrive to this case when we cannot find 4-cliques for all the vertices of Exc_1 and Exc_3.

Let us denote the last two quasi independent sets by R and S. The fact that both are quasi independent is the consequence of R being quasi independent. Now all of our quasi independent sets are of size t or $t+1$.

Note that $V(\tilde{H})$ has an equipartition to 4 independent sets by the Hajnal-Szemerédi theorem. We distribute these vertices according to the sizes of P, Q, R and S. Thus, after embedding $\tilde{H}$ the remaining sets will be of the same size.

The embedding is now done as follows. First we find 4-cliques for the exceptional vertices. All the exceptional sets in the quasi independent sets will be small. Say, v is exceptional, if it has more than $10^{-20}n$ neighbors in its quasi independent set. Then the exceptional sets will be smaller than $10^{-20}n$. Let us consider e.g., what we do with a $v \in Exc_1$. If say, v has small degree to Q, than v has a lot of neighbor in P, R and S. Find one-one neighbor in these sets, which together with v constitute a 4-clique. Then, because v had small degree to Q, we can find two Q-vertices, which are connected, and one-one from R and S such that these four vertices constitute a 4-clique. The two vertices from Q should have good degrees to R and S, but this is not a significant restriction, since almost all of them has good degrees, and because with most of them v is not connected, they have an extra edge inside. For all the exceptional vertices one can find a 4-clique applying a similar procedure. Then one can embed $\tilde{H}$ easily: it is so small, that no problem can arise. Then the matching for the leftover is done in the following order. First, match the rest of S with the rest of R. Then match these edges with the vertices of Q. The last step is to match these triangles with the vertices of P. The existence of all these matchings can be verified easily, on the same way we did it previously. **Q.E.D.**

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