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Recursive generation of partitionable graphs

by

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ABSTRACT

Results of Lovász (1972) and Padberg (1974) imply that partitionable graphs contain all the potential counterexamples to Berge's famous Strong Perfect Graph Conjecture. A recursive method of generating partitionable graphs was suggested by Chvátal, Graham, Perold and Whitesides (1979). Results of Sebő (1996) entail that Berge's conjecture holds for all the partitionable graphs obtained by this method. Here we suggest a more general recursion. Computer experiments show that it generates *all* the partitionable graphs with $\omega = 3, \alpha \leq 9$ (we conjecture that the same will hold for bigger α, too) and 'almost all' for $(\omega, \alpha) = (4, 4)$ and $(4, 5)$. Here α and ω are respectively the clique and stability numbers of a partitionable graph, i.e. numbers of vertices in its maximum clique and stable set. All the partitionable graphs generated by our method contain a *critical* ω*-clique*, that is an ω-clique which intersects only $2\omega - 2$ other ω-cliques. This property might imply that in our class there are no counterexamples to Berge's conjecture (c.f. Sebő (1996)), however this question is still open.

1 Introduction

Given a graph G, we denote by $n = n(G)$ the number of vertices in G, by $\omega = \omega(G)$ the clique number, that is the maximal number of pairwise connected vertices, by $\alpha = \alpha(G)$ the stability number, that is the maximal number of pairwise non-connected vertices, and by $\chi = \chi(G)$ the chromatic number, that is the minimal number of colors which allow a proper coloring.

In (1960) Claude Berge introduced the notion of perfect graph. A graph G is called *perfect* if $\chi(G') = \omega(G')$ for every induced subgraph G' in G. Naturally, a graph G is called *minimally imperfect* if it is a vertex-minimal non-perfect graph, i.e. if G itself is not perfect but every proper induced subgraph G' of G is perfect. It is not difficult to see that chordless odd cycles of length five or more (*odd holes*) as well as their complements (*odd antiholes*) are minimally imperfect. Berge conjectured that there are no other minimally imperfect graphs. This conjecture is called Strong Perfect Graph Conjecture and it is still open. A weaker conjecture, that the complement G^c of a perfect graph G is perfect was also suggested by Berge (1960) and was proved by Lovász (1972). (It is known as the Perfect Graph Theorem.)

We would like to recall here two important results from the paper by Lovász (1972). The first one is stating that a graph G is perfect if and only if $n(G') \leq \alpha(G')\omega(G')$ for every induced subgraph G' in G. Since the equalities $\alpha(G) = \omega(G^c)$ and $\omega(G) = \alpha(G^c)$ obviously hold for every graph G, the above inequality implies readily the Perfect Graph Theorem.

The second one states that every minimally imperfect graph G is *partitionable*, i.e. $n(G) = \alpha(G)\omega(G) + 1$, and for every vertex v the induced subgraph $G(V \setminus \{v\})$ can be partitioned into $\alpha(G)$ cliques of size $\omega(G)$, as well as into $\omega(G)$ stable sets of size $\alpha(G)$. If G is partitionable then clearly $\chi(G) = \omega(G) + 1$, $\chi(G(V \setminus \{v\})) = \omega(G) = \omega(G(V \setminus \{v\}))$, and thus the complementary graph G^c is partitionable, too.

Padberg (1974) derived from Lovász' result that for any minimally imperfect graph G the number of $\omega(G)$-cliques is $n(G)$ and every vertex belongs to exactly $\omega(G)$ of the ω-cliques. Their characteristic vectors are linearly independent, i.e. they form a basis in $\mathbb{R}^n$. Padberg also observed the following convenient way to list all $n(G)$ maximum cliques (of size $\omega(G)$) in G. Let us fix an arbitrary ω-clique C and for every vertex $v \in C$ consider a partition of $G(V \setminus \{v\})$ into α maximum cliques. Such a partition is unique. There are ω different vertices $v \in C$ and there are α maximum cliques in each partition. All these cliques appear to be different. Together with the clique C itself we get exactly $\alpha\omega + 1 = n$ maximum cliques of G. Of course, the analogous construction works for stable sets, too.

Bland, Huang and Trotter (1979) proved that all these properties hold not only for minimally imperfect but for arbitrary partitionable graphs as well.

Due to Padberg's construction, it is obvious that in every partitionable graph G every ω-clique C intersects at least $2\omega - 2$ other ω-cliques of G. Indeed, let us chose any two disjoint ω-cliques C and C' in G and consider the clique partitions corresponding to the vertices of C'. Every ω-clique of G (except C') appears in these partitions exactly once, hence exactly one of these partitions contains C. Thus, every other partition splits C in at least two parts. Thus C intersects at least $2\omega - 2$ other ω-cliques of G.

Let us call an ω-clique *critical* if it intersects *exactly* $2\omega - 2$ other ω-cliques. It follows from the above observations that the $2\omega - 2$ cliques intersecting a critical clique C can be combined into $\omega - 1$ pairs such that each of these pairs induces a partition of the vertices of C into two nonempty parts.

An edge $e \in E(G)$ of a partitionable graph G is called *critical* if $\alpha(G - e) = \alpha(G) + 1$, or in other words, if there exist two maximum stable sets S and S' which have $\alpha(G) - 1$ vertices in common and the two vertices in their symmetric difference are connected by the edge e.

Critical cliques and critical edges were studied by Sebő (1996). He proved that every critical ω-clique C of an (α, ω)-partitionable graph contains exactly $\omega - 1$ critical edges which form a spanning tree $T = T(C)$ on the vertices $V(T) = C$. Furthermore (see Lemma 3.1 of Sebő (1996)), the following claims are equivalent:

(i) C is a critical clique;

(ii) Critical edges in C form a spanning tree of C;

(iii) The induced subgraph $G(V \setminus C)$ is uniquely colorable.

(A graph is *uniquely colorable* if it has a unique partition into $\chi(G)$ stable sets.)

We can observe a further connection between a critical clique C and tree T formed by the critical edges in C. Obviously, the removal of any edge $e \in E(T)$ splits T into two connected components, hence splitting the vertices of C into two parts. The $2\omega - 2$ sets obtained in this way, corresponding to the $\omega - 1$ edges of T, are exactly the $2\omega - 2$ intersections of clique C with the other ω-cliques of G.

These observations suggest the following reduction. Given a partitionable (α, ω)-graph G which contains a critical clique C, let us consider the tree T formed by the critical edges in C. Let us now consider any pair of disjoint ω-cliques C' and C'', corresponding to an edge e of T, i.e. for which the intersections $C \cap C'$ and $C \cap C''$ are nonempty and form a partition of C. Let us now change the graph by changing the list of its maximum cliques in the following way. Remove the cliques C', C'' and instead of these two add only one new ω-clique $(C' \setminus C) \cup (C'' \setminus C)$. Let us repeat the same for all the $\omega - 1$ pairs of ω-cliques, corresponding to the edges of T. Finally, let us remove the clique C itself from the list. We shall show that this procedure *always* results in a new *partitionable* $(\alpha - 1, \omega)$-graph G'.

Let us remark that in the procedure above we specified the changes of the family of ω-cliques of the graph G only, rather than the changes with the graph itself. In particular, we paid no attention to updating the edge set, or updating the maximum stable sets of the graph. In Section 2 we shall show that such an approach is correct and the "partitionability" of the family of the ω-cliques in fact implies the "partitionability" of α-stable sets.

It is a natural idea to inverse the above reduction. For this we need first to generalize slightly the properties (i)-(ii). In Section 3 we shall prove that if $\mathcal{S}$ is a family of $2\omega - 2$ subsets of a finite set C of size ω satisfying that $S \in \mathcal{S}$ iff $\overline{S} = C \setminus S \in \mathcal{S}$, and for every point $v \in C$ there is a subfamily $\mathcal{P}_v \subset \mathcal{S}$ which forms a partition of $C \setminus \{v\}$, then there exists a unique spanning tree T on the vertex set C, such that the $2\omega - 2$ sets of $\mathcal{S}$ are exactly the vertex sets of the connected components, which one can obtain by the successive removal of $\omega - 1$ edges of T.

Using this characterization, in Section 4 we shall describe a constructive method to obtain a new partitionable $(\alpha+1, \omega)$-graph G' from a given partitionable (α, ω)-graph G. Unlike the reduction, the recursion is not *always* applicable. In Section 4 we obtain conditions necessary and sufficient for such a procedure to work. In Section 5 we specify these conditions for the case of webs and demonstrate that it is always possible "to substitute a spider in a web", that is given an (α, ω)-graph G which is a web and a tree T which is a spider, the recursion is always applicable.

But how many partitionable graphs have critical cliques? We conjecture that in case $\omega = 3$ they all have. Computations confirm this conjecture for $\alpha < 10$. We prove that this conjecture is equivalent to the following one: every partitionable $(\alpha, 3)$-graph contains an induced gem: $(a, b), (b, c), (c, d), (d, e), (a, c), (c, e), (b, d)$. However, it is not even known if every $(\alpha, 3)$-graph contains an induced diamond: $(a, b), (b, c), (c, d), (a, c), (b, d)$.

In case $\omega = 4$ there are partitionable graphs without critical cliques. There exist 5 partitionable $(3, 4)$-graphs and all 5 have critical cliques, there exist 132 partitionable $(4, 4)$-graphs and 126 have critical cliques, there exist 8340 partitionable $(5, 4)$-graphs and only 6909 have critical cliques.

Let us remark that our recursion generalizes an analogous one suggested by Chvátal, Graham, Perold and Whitesides (1979). We get their recursion as a special case when tree T is a simple path and $\omega - 1$ maximum cliques in G, which define the recursion, form a chain on $2\omega - 2$ vertices, i.e. satisfy that $C_k = \{v_k, v_{k+1}, ..., v_{k+\omega-1}\}$, for $k = 1, ..., \omega - 1$. In particular, every two successive ω-cliques in this chain have $\omega - 1$ vertices in common. For example, let $\omega = 3$. In this case there exists only one tree with 2 edges: this is the simple path P_3, but still we can chose two 3-cliques C_1, C_2 in three different ways, such that cardinality of the intersection $|C_1 \cap C_2|$ is 2,1 or 0. Chvátal, Graham, Perold and Whitesides (1979) demonstrated that in the first case, $|C_1 \cap C_2| = 2$, only 4 out of 5 partitionable (4,3)-graphs can be recursively generated. Our computation shows that the fifth one can be generated if we allow $|C_1 \cap C_2| = 1$, and all three ways, $|C_1 \cap C_2|$ equals 2,1 and 0, are necessary to generate all $(7, 3)$-graphs.

Every partitionable graph generated by our recursion has a critical clique. Sebő (1996) proved that no partitionable graph can be a counterexample to Berge's conjecture if this graph and its complement *both* contain critical cliques. This result is an argument that in our class there is no counterexample either, however this question is still open.

2 Axiomatics of partitionability

In their definition of partitionable graphs Bland, Huang and Trotter (1979) demand partitionability for both families of maximum cliques $\mathcal{C}$ and maximum stable sets $\mathcal{S}$. But in fact, it is sufficient to demand partitionability for only one of these two families and which then will imply the partitionability of the other one. This idea is not new, and some results in this direction can be found in literature. For completeness, we devout a special section to this problem, as well as to some other axiomatics which also imply the numerous properties of the PGs. In fact, this section plays a very important role in our paper, because the transformations, which we will introduce, are based on transformations of the family of ω-cliques only. The justification of this approach is based on the following subsection.

2.1 A one-axiom definition

Let us consider a finite set V of n elements, and a family $\mathcal{C}$ of its subsets.

Definition 1 *The family $\mathcal{C}$ will be called* partitionable *if $|\mathcal{C}| \leq |V| = n$ and for every $v \in V$ the set $V \setminus \{v\}$ is a union of some pairwise disjoint sets from $\mathcal{C}$, i.e. if there exists a subfamily $\mathcal{P}_v \subset \mathcal{C}$ for every $v \in V$ such that*

$$V \setminus \{v\} = \bigcup_{C \in \mathcal{P}_v} C \text{ and } C \cap C' = \emptyset \text{ for } C, C' \in \mathcal{P}_v, \text{ whenever} C \neq C'. \tag{A}$$

Let $\mathbb{B} = \{0, 1\}$, and let us consider the characteristic vectors $\mathbf{x}^C \in \mathbb{B}^V$ of the sets $C \in \mathcal{C}$, the vector of all ones $\mathbf{e} \in \mathbb{B}^V$, and the unit vectors $\mathbf{e}_v \in \mathbb{B}^V$ for $v \in V$. With this notation we can rewrite (A) as

$$\forall v \in V \exists \mathcal{P}_v \subset \mathcal{C} \text{ such that } bx^{V \setminus \{v\}} = \mathbf{e} - \mathbf{e}_v = \sum_{C \in \mathcal{P}_v} \mathbf{x}^C. \tag{A*}$$

Obviously, the vectors $\mathbf{e} - \mathbf{e}_v$, $v \in V$, form a basis in $\mathbb{R}^V$. If the family $\mathcal{C}$ is partitionable then by (A) every such vector is a linear combination (with (0,1)-coefficients) of some of the vectors $\mathbf{x}^C$, $C \in \mathcal{C}$, implying that these vectors form a generator of $\mathbb{R}^V$. Since $|\mathcal{C}| \leq |V|$ is also assumed, it follows that

$$|\mathcal{C}| = |V|. \tag{1}$$

$$\text{The vectors } \mathbf{x}^C, c \in \mathcal{C}, \text{ form a basis of } \mathbb{R}^V. \tag{2}$$

$$\text{The partition } \mathcal{P}_v \subset \mathcal{C} \text{ is unique for every } v \in V. \tag{3}$$

Let us now fix a set $C \in \mathcal{C}$ and let us sum up the equations of (A*) for $v \in C$. We obtain

$$\sum_{v \in C} (\mathbf{e} - \mathbf{e}_v) = |C|\mathbf{e} - \mathbf{x}^C = \sum_{v \in C} \sum_{C' \in \mathcal{P}_v} \mathbf{x}^{C'}$$

from which we can express $\mathbf{e}$ as

$$\mathbf{e} = \frac{1}{|C|} \left(\mathbf{x}^C + \sum_{v \in C} \sum_{C' \in \mathcal{P}_v} \mathbf{x}^{C'} \right). \tag{4}$$

Since the vectors on the right hand side of (4) are from a basis of $\mathbb{R}^V$ according to (2), the expression in (4) must be the unique representation of $\mathbf{e}$ in the basis $\{\mathbf{x}^C | C \in \mathcal{C}\}$. Since $C \notin \mathcal{P}_v$ for any $v \in C$ by definition, we obtain that the coefficient of $\mathbf{x}^C$ in the unique representation of $\mathbf{e}$ must be equal to $\frac{1}{|C|}$, for all $C \in \mathcal{C}$. On the other hand, looking at (4) for a fixed set $C \in \mathcal{C}$, we can observe that for any other set $C' \in \mathcal{C}$, the coefficient of the vector $\mathbf{x}^{C'}$ on the right hand side is an integer multiple of $\frac{1}{|C|}$, i.e. it can be equal to $\frac{1}{|C'|}$only if all sets appear exactly once on the right hand side of (4), and if all sets $C \in \mathcal{C}$ have the same size. Let us denote this common size of the sets in $\mathcal{C}$ by ω. It follows then that all the partitions $\mathcal{P}_v$, $v \in V$, are of the same size, which we shall denote by α.

Thus, we can draw the following chain of conclusions:

$$|C| = \omega \text{ for all } C \in \mathcal{C}, \text{ and } |\mathcal{P}_v| = \alpha \text{ for all } v \in V. \tag{5}$$

$$\text{The families } \mathcal{P}_v \text{ for } v \in C \in \mathcal{C} \text{ are pairwise disjoint.} \tag{6}$$

$$n = \alpha\omega + 1. \tag{7}$$

$$\text{Every point } v \in V \text{ belongs to exactly } \omega \text{ of the sets } C \in \mathcal{C}. \tag{8}$$

$$\begin{gathered} \text{For every } C \in \mathcal{C} \\ \text{the subfamilies } \mathcal{P}_v, v \in C \text{ together with } C \\ \text{form a partition of } \mathcal{C}. \end{gathered} \tag{9}$$

We can also rewrite (9) as

$$\forall C, C' \in \mathcal{C}, C \neq C',\ \exists! v \in C \setminus C' \text{ such that } C' \in \mathcal{P}_v. \tag{9'}$$

¿From this, by a simple counting argument we can conclude that

$$\text{Every set } C \in \mathcal{C} \text{ belongs to exactly } \alpha \text{ of the partitions } \mathcal{P}_v, v \in V. \tag{10}$$

To verify (10), let us introduce the notation

$$S_C = \{v \in V | C \in \mathcal{P}_v\} \tag{11}$$

for $C \in \mathcal{C}$. Clearly, $C \cap S_C = \emptyset$, by the definition. On the other hand, the set C must belong to exactly one of the partitions $\mathcal{P}_v$, $v \in C'$ for any other set $C' \in \mathcal{C}$, $C' \neq C$ by (9), implying thus

$$C \cap S_C = \emptyset \text{ and } |C' \cap S_C| = 1 \text{ for all } C, C' \in \mathcal{C}, C \neq C'. \tag{12}$$

Since a partition $\mathcal{P}_v$ for any $v \in C$ contains α pairwise disjoint sets $C' \neq C$, $|S_C| \geq \alpha$ is implied by (12). By counting the pairs $C \in \mathcal{P}_v$ first by $v \in V$, and second by $C \in \mathcal{C}$, we obtain

$$\sum_{v \in V} |\mathcal{P}_v| = \sum_{C \in \mathcal{C}} |S_C|.$$

from this, using (5) and the lower bound on $|S_C|$, we get

$$n\alpha = \sum_{v \in V} |\mathcal{P}_v| = \sum_{C \in \mathcal{C}} |S_C| \geq n\alpha,$$

which implies the equality

$$|S_C| = \alpha \text{ for all } C \in \mathcal{C}, \tag{13}$$

proving hence (10).

Remark 1 *Formula (11) is especially important for our approach. Given a partitionable family $\mathcal{C}$, we* introduce *a family $\mathcal{S}$ by formula (11), and then prove that this new family is partitionable, too. While Bland, Huang and Trotter (1979) introduce families $\mathcal{C}$ and $\mathcal{S}$ together and then define partitionability in terms of both.*

We are now ready to show that a partitionable family $\mathcal{C}$ is exactly the family of ω-cliques in a corresponding (α,ω)-partitionable graph. We can verify this, based on the results of Bland, Huang and Trotter (1979), and on the properties above, by showing that the family $\mathcal{S} = \{S_C | C \in \mathcal{C}\}$ forms a partitionable family of α-sets. For this we claim that the subfamily

$$\mathcal{Q}_v = \{S_C | C \in \mathcal{C}, C \ni v\}$$

is a partition of $V \setminus \{v\}$, for every $v \in V$.

Let us note first that if $v \in S_C \cap S_{C'}$, then by (11) both sets C and C' belong to the partition $\mathcal{P}_v$, and hence either $C = C'$, or $C \cap C' = \emptyset$. Thus, we get

$$S_C \cap S_{C'} = \emptyset \text{ whenever } C \cap C' \neq \emptyset \text{ and } C \neq C'. \tag{14}$$

This implies immediately that the sets $S_C \in \mathcal{Q}_v$ are pairwise disjoint. Since $v \notin S_C$ for $S_C \in \mathcal{Q}_v$ by definition, and since $|\mathcal{Q}_v| = \omega$ by (8), the subfamily $\mathcal{Q}_v$ forms a partition of a subset of $V \setminus \{v\}$ of size $\omega\alpha = n-1$, i.e. it forms a partition of $V \setminus \{v\}$.

We can now define a partitionable graph $G = G(\mathcal{C},\mathcal{S})$ on the vertex set $V(G) = V$, in which the sets $C \in \mathcal{C}$ are the ω-cliques, and the sets $S \in \mathcal{S}$ are the α-stable sets. In other words, for $u, v \in V$, $u \neq v$, let us say that $(u,v) \in E(G)$ if there is a set $C \in \mathcal{C}$ such that $\{u,v\} \subset C$, and let us define $(u,v) \notin E(G)$ if there is a set $S \in \mathcal{S}$ containing both u and v. We do not get any contradiction in this way, since $|C \cap S| \leq 1$ for all $C \in \mathcal{C}$ and $S \in \mathcal{S}$ according to (12). However, the graph $G(\mathcal{C},\mathcal{S})$ is not well defined yet, because there can be pairs of vertices which do not belong neither to ω-cliques nor to α-stable sets. Such pairs of vertices are called *indifferent edges*. An arbitrary subset of indifferent edges can be included in $G(\mathcal{C},\mathcal{S})$. Thus in fact, $G(\mathcal{C},\mathcal{S})$ is not one graph but a family of (partitionable) graphs. Each of these graphs has exactly n cliques $C \in \mathcal{C}$ of cardinality ω and exactly n stable sets $S \in \mathcal{S}$ of cardinality α. If $\omega \neq n-1$ then there cannot exist cliques of cardinality $\omega+1$, and similarly, if $\alpha \neq n-1$ then there are no stable sets of cardinality $\alpha+1$.

Remark 2 *In principle, partitionable families could have parameters $(\alpha,\omega) = (1, n-1)$ or $(\alpha,\omega) = (n-1, 1)$. However, when dealing with partitionable graphs the standard assumption is that $\alpha > 1$ and $\omega > 1$.*

2.2 Geometrical axioms

The following nice geometrical approach to partitionability was suggested by Temkin (private communications). Given a set $V = \{v_1, ..., v_n\}$ and two families of its subsets $\mathcal{C} = \{C_1, ..., C_n\}$ and $\mathcal{S} = \{S_1, ..., S_n\}$ such that $C_1 \cap S_1 = \emptyset, ..., C_n \cap S_n = \emptyset$, let us introduce a *projective biplane* whose n points are $v_1, ..., v_n$ and n lines are $L_1 = C_1 \cup S_1, ..., L_n = C_n \cup S_n$. The difference between the standard finite projective plane and biplane is as follows. The incidence function $F(L_i, v_j)$ for a standard plane takes two values: $F(L_i, v_j) = 1$ if $v_j \in L_i$ and $F(L_i, v_j) = 0$ if $v_j \notin L_i$, while for a biplane it takes three values: $F(L_i, v_j) = 1$ if $v_j \in C_i$, $F(L_i, v_j) = -1$ if $v_j \in S_i$, and $F(L_i, v_j) = 0$ if $v_j \notin L_i$.

Also the intersection of lines is understood in a rather unusual way. Given two lines $L_i = C_i \cup S_i$ and $L_j = C_j \cup S_j$, their intersection is $L_i \cap L_j = (C_j \cap S_i) \cup (C_i \cap S_j)$, that is only those points which belong to both lines and whose incidence functions with respect to these two lines have opposite signs are included, while the points from $(C_i \cap C_j) \cup (S_i \cap S_j)$ do not count. After these two radical innovations a finite projective biplane is defined by the following two more or less standard axioms.

$$\begin{array}{c}\text{Every two different lines } L_i = C_i \cup S_i \text{ and } L_j = C_j \cup S_j \\ \text{intersect in exactly two different points } v_k \text{ and } v_m \\ \text{such that } v_k \in C_i \cap S_j \text{ and } v_m \in C_j \cap S_i;\end{array} \tag{G1}$$

$$\begin{array}{c}\text{Every two different points } v_k \text{ and } v_m \\ \text{are connected by exactly two different lines} \\ L_i = C_i \cup S_i \text{ and } L_j = C_j \cup S_j \text{ such that } v_k \in C_i \cap S_j \text{ and } v_m \in C_j \cap S_i.\end{array} \tag{G2}$$

Let us prove that axioms ((G1), (G2)) and (A1) are equivalent. First, given a set $V = \{v_1, ..., v_n\}$ and a partitionable (i.e. satisfying (A1)) family $\mathcal{C} = \{C_1, ..., C_n\}$, let us generate the family $\mathcal{S} = \{S_1, ..., S_n\}$, according to (10), consider the corresponding biplane and prove that ((G1), (G2)) hold. Formula (G1) results directly from (11). To prove (G2) let us fix any two different points $v_k, v_m \in V$ and consider all the ω sets $C_j, j \in J(v_k)$ which contain v_m, see (9). According to (10), the corresponding ω sets $S_j, j \in J(v_k)$ are pairwise disjoint and each one contains α points, according to (12). Hence, together they contain $n-1$ points and must form a partition $\mathcal{P}(v_m)$, that is exactly one of these sets, let us say S_{j_0}, contains v_k. Thus, there exists a unique $j_0 \in [n]$ such that $v_m \in C_{j_0}$ and $v_k \in S_{j_0}$. In the same way we prove that there exists a unique i_0 such that $v_m \in S_{i_0}$ and $v_k \in C_{i_0}$. Thus, (G2) holds.

Now let us derive (A1) from ((G1), (G2)). That is given a biplane, let us prove that family $\mathcal{C} = \{C_1, ..., C_n\}$ must be partitionable. For this let us fix an arbitrary point $v \in V$ and consider all the lines $L_j = C_j \cup S_j$ $j \in J(v)$ such that $v \in S_j$. Then (14) means exactly that $C_j,\ j \in J(v)$ form a partition of $\mathcal{P}(v)$.

2.3 Matrix axioms

The following matrix approach to partitionability was suggested by Chvátal, Graham, Perold and Whitesides (1979). Let us consider equation

$$XY = J - I \tag{M}$$

in $n \times n$ (0,1)-matrices where I is the identity matrix, J is the matrix whose all n^2 entries are 1's, and X, Y are unknown.

Again, given a set $V = \{v_1, ..., v_n\}$ and two arbitrary families of its subsets $\mathcal{C} = \{C_1, ..., C_n\}$ and $\mathcal{S} = \{S_1, ..., S_n\}$, let us introduce X as (0,1) $n \times n$ incidence matrix of V (columns) and $\mathcal{C}$ (rows), and Y as (0,1) $n \times n$ incidence matrix of V (rows) and $\mathcal{S}$ (columns). And vice versa, to any two (0,1) $n \times n$ matrices X and Y we can assign a set V and two families $\mathcal{C}$ and $\mathcal{S}$ of its subsets such that the same incidence relations takes place. Thus we get two mutually inverse one-to-one mappings. Let us prove that axioms (M) for X, Y and (A) for $V, \mathcal{C}$ are equivalent. Firstly, (M) is an obvious consequence of (12) because for (0,1) vectors the intersection and the scalar product mean just the same. Secondly, (M) implies partitionability of the corresponding set-family $\mathcal{C}$. Indeed, from one hand, the rows of matrix $J - I$ are by the definition vectors $e - e_i; i = 1, ..., n$. ¿From the other hand, rows of the matrix product XY are linear combinations of the rows of X, and all the coefficients takes only values 0 and 1. Thus these linear combinations are just sums. But a sum of characteristic vectors is $e - e_i$ if and only if the corresponding sets from $\mathcal{C}$ form a partition $\mathcal{P}(v_i)$.

Let us recall that partitionability of $\mathcal{C}$ implies the partitionability of $\mathcal{S}$. Thus $XY = J - I$ iff $YX = J - I$. Then let us note that matrix $J - I$ is symmetric. This implies $XY = J - I$ iff $Y^t X^t = J - I$, where t means matrix transposition. Thus the following four matrix products: $XY, YX, Y^t X^t, X^t Y^t$ can be equal to $J - I$ only simultaneously. If pair of matrices (X, Y) generates a partitionable graph G then pair (Y, X) generates the complementary graph G^c, while pair $(X^t Y^t)$ generates dual partitionable graph G^d. Obviously, $G^{cd} = G^{dc}$.

3 Tree-covering families

Let us consider a set C of size ω, and let $\mathcal{A}$ be a family of subsets of C (more precisely, a *multi-family*, i.e. sets in $\mathcal{A}$ may have a multiplicity > 1.) Let us call $\mathcal{A}$ a *tree-covering family*, if

$$A \in \mathcal{A} \Longrightarrow \overline{A} = C \setminus A \in \mathcal{A}, \tag{C1}$$

and if for every point $v \in C$ there is a subfamily $\mathcal{R}_v \subset \mathcal{A}$ which form a partition of $C \setminus \{v\}$, i.e. if

$$\forall \quad v \in C \qquad \exists \quad \mathcal{R}_v \subset \mathcal{A} \text{ such that } C \setminus \{v\} = \biguplus_{A \in \mathcal{R}_v} A, \tag{C2}$$

where $\biguplus$ denotes "disjoint union".

We shall show first that a tree-covering family must have at least $2\omega - 2$ elements. Using the characteristic vectors $\mathbf{x}^A \in \mathbb{B}^C$, $A \in \mathcal{A}$, the vector of all ones $\mathbf{e} \in \mathbb{B}^C$, and the unit vectors $\mathbf{e}_v \in \mathbb{B}^C$ for $v \in C$, conditions (C1) and (C2) can be restated as

$$\forall A \in \mathcal{A} \quad \exists \overline{A} \in \mathcal{A} \quad \text{such that} \quad \mathbf{x}^A + \mathbf{x}^{\overline{A}} = \mathbf{e} \tag{C1*}$$

$$\forall v \in C \quad \exists \mathcal{R}_v \subset \mathcal{A} \quad \text{such that} \quad \sum_{A \in \mathcal{R}_v} \mathbf{x}^A = \mathbf{e} - \mathbf{e}_v \tag{C2*}$$

Lemma 1 *Let $\mathcal{A}$ be a tree-covering family on a finite set C of size ω, and let k denote the number of different sets in $\mathcal{A}$. Then $k \geq 2\omega - 2$.*

Proof. Let us observe first that k is even, since the different sets of $\mathcal{A}$ can be divided into complementary pair by (C1). Let us denote by A_i, $\overline{A}_i$ these complementary pairs, $i = 1, ..., \frac{k}{2}$.

Let us next observe that by (C2*) all vectors of the form $\mathbf{e} - \mathbf{e}_v$ for $v \in C$ can be expressed as linear combinations of the vectors $\mathbf{x}^A$, $A \in \mathcal{A}$. Since $\{\mathbf{e} - \mathbf{e}_v | v \in C\}$ forms a basis of $\mathbb{R}^C$, the set $\{\mathbf{x}^{A_i}, \mathbf{x}^{\overline{A}_i} | i = 1, ..., \frac{k}{2}\}$ must be a generator set of $\mathbb{R}^C$. Let us now consider a subfamily, $B = \{\mathbf{x}^{A_i} | i = 1, ..., \frac{k}{2}\} \cup \{\mathbf{x}^{\overline{A}_1}\}$ consisting of the first complementary pair, and one of the characteristic vectors for all other complementary pairs. According to (C1*), we can obtain all other characteristic vectors by $\mathbf{x}^{\overline{A_i}} = (\mathbf{x}^{A_1} + \mathbf{x}^{\overline{A}_1}) - \mathbf{x}^{A_i}$ for $i > 1$, and hence B is a generating set of $\mathbb{R}^C$, too, implying $|B| \geq \omega$. Since $|B| = 1 + \frac{k}{2}$, the statement of the lemma follows immediately. ■

Let us call a tree-covering family $\mathcal{A}$ on a finite set C of size ω *critical*, if it has the smallest possible size, i.e. if

$$|\mathcal{A}| = 2\omega - 2. \tag{C3}$$

An immediate corollary of Lemma 1 is that all sets of a critical tree-covering family must have a multiplicity of 1. Thus, since in the sequel we shall talk about critical tree-covering families, we do not have to pay special attention to distinguishing families from multi-families.

Let us see first examples for critical tree-covering families: Let us consider an arbitrary spanning tree T on the vertex set $V(T) = C$. The removal of an edge $(u, v) \in E(T)$ divides the set of vertices into two connected components. Let us denote the component containing v but not u by A_{uv} and let A_{vu} be the other component. Finally, let us define a family $\mathcal{A}_T = \{A_{uv}, A_{vu} | (u, v) \in E(T)\}$. Clearly, $\mathcal{A}_T$ has $2\omega - 2$ elements, and $\overline{A}_{uv} = A_{vu}$, i.e. both conditions (C1) and (C3) hold. Furthermore, one can see that for every vertex $u \in C$ the subfamily $\mathcal{R}_u = \{A_{uv} | (u, v) \in E(T)\}$ forms a partition of the vertex set $C \setminus \{u\}$, since T is a spanning tree on C. Thus $\mathcal{A}_T$ is a critical tree-covering family for every spanning tree T.

We shall show next that in fact all critical tree-covering families arise in this way.

Theorem 1 *If $\mathcal{A}$ is a critical tree-covering family on a finite set C, then there exists a spanning tree T on C such that $\mathcal{A} = \mathcal{A}_T$.*

To prove this theorem, we shall need a series of simple lemmas first.

Let us consider a critical tree-covering family $\mathcal{A}$ on the set C ($|C| = \omega$) as in the theorem.

Lemma 2 *If*

$$\mathbf{e} = \sum_{A \in \mathcal{A}} \alpha_A \mathbf{x}^A \tag{16}$$

for some nonnegative real coefficients $\alpha_A \geq 0$ for $A \in \mathcal{A}$, then there exists a complementary pair of sets, $A \in \mathcal{A}$ and $\overline{A} \in \mathcal{A}$, for which both coefficients α_A and $\alpha_{\overline{A}}$ are positive.

Proof. Let us assume indirectly that $\min(\alpha_A, \alpha_{\overline{A}}) = 0$ for all $A \in \mathcal{A}$, and let us choose a subfamily $\mathcal{B} \subset \mathcal{A}$ by defining

$$\mathcal{B} = \{A | \alpha_A > 0\} \cup \{A | \alpha_A = \alpha_{\overline{A}} = 0 \text{ and } v \in A\}$$

where $v \in C$ is a fixed element. Clearly, in this way we chose into $\mathcal{B}$ exactly one set from each complementary pairs in $\mathcal{A}$. The subfamily $\mathcal{B}$ also contains all sets to which the corresponding vector on the right hand side of (16) has a positive coefficient. Using then (C1*) and (16), we can conclude that the vectors $\mathbf{x}^A$, $A \in \mathcal{B}$ must form a generating set, just like in the proof of Lemma 1. This is a contradiction with the fact that $|\mathcal{B}| = \omega - 1$ for a critical tree-covering family, and hence the lemma follows. ■

For a critical tree-covering family $\mathcal{A}$ on the set C, let us choose a subfamily $\mathcal{R}_v$ for every $v \in C$ for which condition (C2) holds.

Lemma 3 *For every set $A \in \mathcal{A}$ there exists a unique vertex $v \in C$ such that $A \in \mathcal{R}_v$.*

Proof. By summing up the equations (C2*), we get

$$\sum_{v \in C} \sum_{A \in \mathcal{R}_v} \mathbf{x}^A = (\omega - 1)\mathbf{e}. \tag{17}$$

Let us denote by m^A the number of points $v \in C$ for which $A \in \mathcal{R}_v$, and let $v \in C$ be a fixed vertex. With this notation (17) can be rewritten as

$$\begin{aligned}(\omega - 1)\mathbf{e} &= \sum_{A \in \mathcal{A}} m^A \mathbf{x}^A \\ &= \sum_{A \in \mathcal{A}, v \in A} \min(m^A, m^{\overline{A}}) \left(\mathbf{x}^A + \mathbf{x}^{\overline{A}}\right) + \sum_{A \in \mathcal{A}} \left(m^A - m^{\overline{A}}\right)_+ \mathbf{x}^A.\end{aligned}$$

where $(a-b)_+ = a-b$ if $a > b$, and $(a-b)_+ = 0$ otherwise. Using (C1*), we obtain finally

$$\left[(\omega - 1) - \sum_{A\in\mathcal{A}, v\in A} \min(m^A, m^{\overline{A}})\right] \mathbf{e} = \sum_{A\in\mathcal{A}} \left(m^A - m^{\overline{A}}\right)_+ \mathbf{x}^A. \tag{18}$$

The right hand side above is a nonnegative combination of nonnegative vectors, hence $\omega-1 \geq \sum_{A\in\mathcal{A},v\in A} \min(m^A, m^{\overline{A}})$ follows. If the left hand side of (18) were in fact non zero, we could obtain from (18) the vector $\mathbf{e}$ as a nonnegative combination of the vectors $\mathbf{x}^A$, $A \in \mathcal{A}$. According to Lemma 2 this would imply that for at least one set $S \in \mathcal{S}$ both $(m^A - m^{\overline{A}})_+$ and $(m^{\overline{A}} - m^A)_+$ are positive, which is impossible, since for any two reals a and b, either $(a-b)_+ = 0$ or $(b-a)_+ = 0$ (or both). This contradiction shows that

$$\omega - 1 = \sum_{A\in\mathcal{A}, v\in A} \min(m^A, m^{\overline{A}}). \tag{19}$$

Thus all the nonnegative coefficients on the right hand side of (18) must also be equal to zero, i.e. $m^A = m^{\overline{A}}$ for all $A \in \mathcal{A}$ follows.

Let us observe next that $m^A > 0$ for all $A \in \mathcal{A}$, since otherwise we have $m^A = m^{\overline{A}} = 0$ for some sets $A \in \mathcal{A}$, implying that the family $\mathcal{A}' = \mathcal{A} \setminus \{A, \overline{A}\}$ is again a tree-covering family of size $|\mathcal{A}| - 2 < 2\omega - 2$, a contradiction to Lemma 1.

Since in the summation of the right hand side of (19) we have $\omega - 1$ terms, and since each of those is a nonnegative integer according to the above, we can conclude from (19) that $m^A = 1$ for all $A \in \mathcal{A}$, hence proving the lemma. ■

The above lemma shows also that in a critical tree-covering family $\mathcal{A}$ on C for every vertex $v \in C$ there is a unique subfamily $\mathcal{R}_v \subset \mathcal{A}$ which forms a partition of the vertices $C \setminus \{v\}$.

Let us now consider a graph T on the vertex set $V(T) = C$ with an edge set defined by

$$E(T) = \{(u,v) | u, v \in C,\ \exists A \in \mathcal{A} \text{ such that } A \in \mathcal{R}_v \text{ and } \overline{A} \in \mathcal{R}_u\}.$$

Since a critical tree-covering family $\mathcal{A}$ consists of $\omega - 1$ complementary pairs, it follows by Lemma 3 that the graph T has exactly $\omega - 1$ edges, one corresponding to each complementary pair of sets of $\mathcal{A}$. For an edge $(u,v) \in E(T)$ let us denote the corresponding complementary sets of $\mathcal{A}$ by A_{uv} and $A_{vu} = \overline{A}_{uv}$ such that $v \in A_{uv}$ and $u \in A_{vu}$.

It is easy to see that Lemma 3 and the above definitions readily imply

Corollary 1 *There are no loops in T, and we have $\mathcal{A} = \{A_{uv}, A_{vu} | (u,v) \in E(T)\}$.* ■

Lemma 4 *For every $v \in C$ we have $\mathcal{R}_u = \{A_{uv} | (u, v) \in E(T)\}$.*

Proof. The relation $\mathcal{R}_u \supseteq \{A_{uv} | (u, v) \in E(T)\}$ follows directly from the definition of the edges of T.

For the converse relation, let $A \in \mathcal{R}_u$ be arbitrary. Then $\overline{A} \in \mathcal{A}$ by (C1), and thus by Lemma 3 there exists a unique vertex $v \in C$ for which $\overline{A} \in \mathcal{R}_v$. Clearly $u \neq v$, since $u \in \overline{A}$ and $\overline{A} \subseteq C \setminus \{v\}$. Therefore, $(u, v) \in E(T)$ and $A = A_{uv}$ follows by the definition of T. ■

Lemma 5 *If $(u, v) \in E(T)$ and $(v, w) \in E(T)$, then $A_{uv} \subset A_{vw}$.*

Proof. According to Lemma 4 we have $A_{vw} \in \mathcal{R}_v$ and $A_{vu} \in \mathcal{R}_v$, thus $A_{vw} \cap A_{vu} = \emptyset$. Since $\overline{A}_{uv} = A_{vu}$, we get $A_{uv} \supseteq A_{vw}$, as a consequence. To see that this is a strict containment relation, it is enough to observe that $v \in A_{uv}$, while $v \notin A_{vw}$. ■

Lemma 6 *There are no circuits in T.*

Proof. Let us assume indirectly that u_1, ..., u_k are vertices from C forming a cycle, i.e. $(u_i, u_{i+1}) \in E(T)$ for $i = 1, ..., k-1$, and $(u_k, u_1) \in E(T)$. Then, by Lemma 5 we would have $A_{u_1u_2} \supset A_{u_2u_3} \supset \cdots \supset A_{u_ku_1} \supset A_{u_1u_2}$, all relations as strict containment, a clear contradiction, proving the lemma. ■

Proof of Theorem 1. The graph T constructed above is a spanning tree on C by Lemma 6, and the equality $\mathcal{A} = \mathcal{A}_T$ follows by Corollary 1 and Lemma 4. ■

4 Reduction and recursive generation of partitionable families.

According to the results of Section 2 we shall be able to represent partitionable (α, ω)-graphs by the (partitionable) family of their ω-cliques.

So let us consider a partitionable (α, ω)-graph G on the vertex set V of n elements, and let $\mathcal{C}$ be the (partitionable) family of its ω-cliques. Let us denote by $\mathcal{S}$ the family of α-stable sets of G, in which we have exactly one vis-a-vis set S_C corresponding to every $C \in \mathcal{C}$, as defined in (11).

Lemma 7 *Every clique $C \in \mathcal{C}$ intersects at least $2\omega - 2$ other cliques from $\mathcal{C}$.*

Proof. Let us denote by $\mathcal{M}_C = \{\tilde{C} \in \mathcal{C} | \tilde{C} \neq C \text{ and } C \cap \tilde{C} \neq \emptyset\}$, and let us start with the following obvious equality:

$$\sum_{\tilde{C} \in \mathcal{M}_C} \sum_{v \in V \setminus (C \cup S_C), \tilde{C} \in \mathcal{P}_v} 1 = \sum_{v \in V \setminus (C \cup S_C)} \sum_{\tilde{C} \in \mathcal{M}_C \cap \mathcal{P}_v} 1.$$

Let us then recall that by (11) we have $\tilde{C} \in \mathcal{P}_v$ iff $v \in S_{\tilde{C}}$, and for sets $\tilde{C} \in \mathcal{M}_C$ we have $S_C \cap S_{\tilde{C}} = \emptyset$ by (14). Thus, the second summation on the left hand side is equal to $|S_{\tilde{C}} \setminus C|$ which is $\alpha - 1$ for all $\tilde{C} \in \mathcal{M}_C$, by (12) and (13). Let us also observe that the second summation on the right hand side of the above equation yields always at least 2, since $C \in \mathcal{P}_v$ only for $v \in S_C$ by (11). Thus, we can rewrite the above equality as

$$|\mathcal{M}_C|(\alpha - 1) = \sum_{v \in V \setminus (C \cup S_C)} \sum_{\tilde{C} \in \mathcal{M}_C \cap \mathcal{P}_v} 1 \geq 2|V \setminus (C \cup S_C)| = 2(\alpha - 1)(\omega - 1),$$

from which we obtain

$$|\mathcal{M}_C| \geq 2(\omega - 1),$$

since $\alpha > 1$ is assumed. ■

An ω-clique $C \in \mathcal{C}$ is called *critical* if it intersects exactly $2\omega - 2$ other ω-cliques of $\mathcal{C}$. Clearly, this can happen only if

$$|\mathcal{M}_C \cap \mathcal{P}_v| = 2 \tag{20}$$

for all $v \in V \setminus (C \cup S_C)$, according to the above proof of Lemma 7. This implies that for a critical clique C, the sets in $\mathcal{M}_C$ can be combined into $\omega - 1$ pairs C^1, C^2, such that $C \subset C^1 \cup C^2$, and C^1 and C^2 belong to the same $\mathcal{P}_v$ partition for some $v \in V \setminus (C \cup S_C)$. Let us denote by E an index set of $\omega - 1$ elements, and let us write $\mathcal{M}_C$ as

$$\mathcal{M}_C = \{C_e^1, C_e^2 | e \in E\},$$

reflecting such a pairing of the elements of $\mathcal{M}_C$. With this notation we have

$$C \subset C_e^1 \cup C_e^2 \text{ and } C_e^1 \cap C_e^2 = \emptyset \text{ for all } e \in E. \tag{21}$$

Furthermore, (20) implies that

$$\forall v \in V \setminus (C \cup S_C) \exists e \in E \text{ such that } C_e^1, C_e^2 \in \mathcal{P}_v. \tag{22}$$

Let us remark that for a critical clique C the sets of the form $C \cap \tilde{C}$ for $\tilde{C} \in \mathcal{M}_C$ are all different, as it is implied by (20).

4.1 Reduction

Given a partitionable family $\mathcal{C}$ of the ω-cliques of a partitionable (α,ω)-graph G on vertex set V, and given a critical clique $C \in \mathcal{C}$, we shall construct another family $\mathcal{C}'$ on the set $V' = V \setminus C$ and show that $\mathcal{C}'$ is partitionable, too, i.e. that $\mathcal{C}'$ is the family of ω-cliques of a partitionable $(\alpha-1,\omega)$-graph G' on the vertex set V'.

Let us consider the family

$$\mathcal{M}_C = \{C_e^1, C_e^2 | e \in E\}$$

as above, and for every $e \in E$ let us define a set

$$C'_e = (C_e^1 \cup C_e^2) \setminus C, \tag{23}$$

and let us define the new family by

$$\mathcal{C}' = (\mathcal{C} \setminus (\mathcal{M}_C \cup \{C\})) \cup \{C'_e | e \in E\}. \tag{24}$$

Theorem 2 *The reduced family $\mathcal{C}'$ is a partitionable family on the set $V' = V \setminus C$.*

Proof. Clearly, all sets in $\mathcal{C}'$ are subsets of V' by the definition, and we have

$$|\mathcal{C}'| = |\mathcal{C}| - (|\mathcal{M}_C| + 1) + |E| = n - (2\omega - 1) + (\omega - 1) = n - \omega = |V \setminus C| = |V'|.$$

Thus, to prove the theorem it is enough to show that for every $v \in V'$ there exists a partition $\mathcal{P}'_v \subset \mathcal{C}'$ partitioning the set $V' \setminus \{v\}$.

Let us consider first the family $\mathcal{P}_v \subset \mathcal{C}$. If $C \in \mathcal{P}_v$, then $\mathcal{P}_v \cap \mathcal{M}_C = \emptyset$, and thus

$$\mathcal{P}'_v = \mathcal{P}_v \setminus \{C\}$$

is a desired partition within $\mathcal{C}'$. On the other hand, if $C \notin \mathcal{P}_v$, then $v \in V \setminus (C \cup \mathcal{S}_C)$, and thus by (20) and (22) there exists a unique $e \in E$ such that

$$\mathcal{P}_v \cap \mathcal{M}_C = \{C_e^1, C_e^2\}.$$

In this case the family

$$\mathcal{P}'_v = \left(\mathcal{P}_v \setminus \{C_e^1, C_e^2\}\right) \cup \{C'_e\}$$

will be a subfamily of $\mathcal{C}'$ partitioning the set $V' \setminus \{v\}$. ■

4.2 Recursion

To be able to find a constructive inverse to the above reduction operation, let us first analyze the structure of the restrictions of the hypergraph $\mathcal{C}$ to the sets C and $V \setminus C$, separately.

Let us observe first that

$$\text{The family } \mathcal{A} = \{C \cap \tilde{C} | \tilde{C} \in \mathcal{M}_C\} \text{ is a critical covering family.} \tag{R1}$$

Clearly, conditions (C1) and (C3) hold by (21) and by the criticality of C. To see (C2), let us define

$$\mathcal{R}_v = \{C \cap \tilde{C} | \tilde{C} \in \mathcal{M}_C \cap \mathcal{P}_v\}$$

for every $v \in C$. Then, $\mathcal{R}_v \subset \mathcal{A}$, and its members form a partition of the set $C \setminus \{v\}$ by the definition, and hence (R1) follows.

Let us remark that according to (R1) and the results in Section 3, $\mathcal{A} = \mathcal{A}_T$ for a (unique) spanning tree T on the vertex set C. On the other hand, Sebő (1996) showed that in a critical clique of a partitionable graph, the critical edges from a spanning tree. One can show easily, using (20) and (22) that these two trees in fact are identical – no surprises.

Let us draw some conclusions about such a tree $T = T(C)$ which can arise as the tree of the critical edges in a critical clique C. Let d_v denote the degree of vertex $v \in C$ in T, or in other words, $d_v = |\mathcal{R}_v|$, for $v \in C$.

Lemma 8 *For every critical clique C of an (α, ω)-graph G, and for all vertices $v \in C$ we have*

$$d_v \leq \alpha.$$

Proof. Let us consider the cliques $\tilde{C} \in \mathcal{M}_C \cap \mathcal{P}_v$ for a vertex $v \in C$. Since all these belong to the same partition, they are pairwise disjoint, and thus we have

$$d_v\omega = \sum_{\tilde{C} \in \mathcal{M}_C \cap \mathcal{P}_v} |\tilde{C}| \leq |V \setminus \{v\}| = \alpha\omega,$$

implying hence the statement. ■

In fact, a stronger inequality holds. Let us denote by $L(T)$ the set of all the leaves of tree $T = T(C)$

Lemma 9 *For every critical clique C of an (α, ω)-graph G we have*

$$|L(T)| \leq \alpha.$$

Proof. Since every leaf node $v \in C$ of T is incident with exactly one tree edge, there exists a unique vertex $u_v \notin C$ corresponding to each leaf node V, for which the set $\tilde{C}_v = \{u_v\} \cup C \setminus \{v\}$ is a clique of G belonging to $\mathcal{M}_C$, according to our analysis above. Since $\tilde{C}_v$ has only one point, namely u_v, outside of C, that vertex hence must belong to the vis-a-vis stable set S_C, because all cliques (other than C) must intersect S_C. Let us also note that such a vertex u_v is adjacent to all vertices of C other than v. This latter implies, in particular that the vertices u_v and u_w corresponding to two different leaf nodes v and w must be different, since otherwise $\{v, u_v\} \subset \tilde{C}_{v_w}$ would imply that $(v, u_v) \in E(G)$, i.e. the set $C \cup \{u_v\}$ would be an $(\omega + 1)$-clique of G. Thus, $|\{u_v | v \in L(T)\}| = |L(T)|$ and $\{u_v | v \in L(T)\} \subseteq S_C$ both hold, implying hence the claim. ■

Let us note next that the family $\mathcal{B} = \{C'_e | e \in E\}$ is a subfamily of $\mathcal{C}'$ of cardinality $\omega - 1$ such that

$$|\mathcal{B} \cap \mathcal{P}'_v| \leq 1 \text{ for all } v \in V', \tag{R2}$$

which follows immediately from the proof of Theorem 2.

Let us note also that sets in $\mathcal{B}$ are split into two by the sets $\tilde{C} \setminus C$ for $\tilde{C} \in \mathcal{M}_C$ such that

$$\forall v \in C \text{ the set } \left(V' \setminus \bigcup_{\tilde{C} \in \mathcal{P}_v \cap \mathcal{M}_C} (\tilde{C} \setminus C) \right) \text{ is partitioned by } \mathcal{C}'. \tag{R3}$$

Indeed, the sets in $\mathcal{P}_v \cap \mathcal{C}'$ for $v \in C$ provide such a partition.

Remark 3 *Condition (R1) can be restated, due to the results in Section 3, as $\mathcal{A} = \mathcal{A}_T$ for some spanning tree T on the vertex set C.*

Remark 4 *Condition (R2) can also be stated in a more convenient way, by (11), saying that the vis-a-vis stable sets $S_{C'_e}$ for $e \in E$ are pairwise disjoint.*

In particular, (R2) holds if all $\omega - 1$ sets $\{C'_e | e \in E(T)\}$ have a vertex in common, according to (14). In this case the resulting partitionable graph has a small transversal. It follows from Theorems 2 and 3 by Sebő (1996).

Remark 5 *Condition (R3) holds automatically if vertex $v \in C$ is a leaf of T. This condition can be translated in terms of the vis-a-vis sets S_C, as well as (R2). Also both these conditions can be translated in terms of the dual partitionable graph G^d.*

We are now ready to show that the above conditions (R1), (R2) and (R3) are essentially the necessary and sufficient conditions one needs to inverse the reduction.

However, we should strengthen (R3) slightly. Let us now assume that we are given a partitionable family $\mathcal{C}'$ of ω-sets on the vertex set V', corresponding to a partitionable

(α,ω)-graph G'. Let C be a set of size ω, disjoint from V', and let T be a spanning tree on C with edge set $E = E(T)$. Let us denote by T_{uv} and T_{vu} the vertex sets of the connected components obtained by removing the edge $(u,v) \in E(T)$ from the tree T, such that $v \in T_{uv}$ and $u \in T_{vu}$. Let finally Γ_v denote the set of neighbors of v in T, i.e. $\Gamma_v = \{u|(u,v) \in E(T)\}$.

Let us further assume that there is a subfamily $\mathcal{B} = \{C'_{uv}|(u,v) \in E(T)\} \subset \mathcal{C}'$ satisfying condition (R2), the cliques of which can be split into two parts $C'_{uv} = B_{uv} \cup B_{vu}$ for $(u,v) \in E(T)$ in such a way that $B_{uv} \cap B_{vu} = \emptyset$, $|B_{uv}| = |T_{uv}|$ (and thus $|B_{vu}| = |T_{vu}|$), and such that

$$\forall v \in C \quad \begin{array}{c} \text{the sets } B_{uv} \text{ for } u \in \Gamma_v \text{ are pairwise disjoint, and} \\ \exists \mathcal{H}_v \subset \mathcal{C}' \setminus \mathcal{B} \text{ partitioning } V' \setminus \bigcup_{u \in \Gamma_v} B_{uv}. \end{array} \tag{R3*}$$

Let us then define

$$\mathcal{C} = (\mathcal{C}' \setminus \mathcal{B}) \cup \{T_{uv} \cup B_{vu}, T_{vu} \cup B_{uv}|(u,v) \in E(T)\} \cup \{C\}. \tag{25}$$

Theorem 3 *The family $\mathcal{C}$ is a partitionable family of ω-cliques of a partitionable $(\alpha+1,\omega)$-graph G on the vertex set $V = V' \cup C$. Furthermore, $C \in \mathcal{C}$ is a critical clique, for which if we apply the reduction, we obtain $\mathcal{C}'$ back.*

Proof. Clearly, $\mathcal{C}$ is a family of size

$$|\mathcal{C}| = |\mathcal{C}'| - |\mathcal{B}| + 2|E(T)| + 1 = |\mathcal{C}'| + \omega = |V'| + |C| = |V|.$$

Thus, to prove the first half of the theorem, we need to show that for every $v \in V$ there exists a subfamily of $\mathcal{C}$ partitioning the set $V \setminus \{v\}$.

Let us consider first points $v \in V'$. If $\mathcal{P}'_v \cap \mathcal{B} = \emptyset$, then

$$\mathcal{P}_v = \mathcal{P}'_v \cup \{C\}$$

is an appropriate partitioning subfamily of $\mathcal{C}$. If $\mathcal{P}'_v \cap \mathcal{B} \neq \emptyset$ then, by our assumptions, there is a unique set C'_{uv} of $\mathcal{B}$ which belongs to $\mathcal{P}'_v$. In this case the family

$$\mathcal{P}_v = (\mathcal{P}'_v \setminus \{C'_{uv}\}) \cup \{T_{uv} \cup B_{vu}, T_{vu} \cup B_{uv}\}$$

is a subfamily of $\mathcal{C}$ partitioning the set $V \setminus \{v\}$.

Let us finally consider the points $v \in C$, and define

$$\mathcal{P}_v = \mathcal{H}_v \cup \{B_{uv} \cup T_{vu}|u \in \Gamma_v\}.$$

Clearly $\mathcal{P}_v \subset \mathcal{C}$ by our definition, and the sets in $\mathcal{H}_v$ cover with no overlap the points $V' \setminus \bigcup_{u\in\Gamma_v} B_{uv}$ by (R3*), while the sets $B_{uv} \cup T_{vu}$ for $u \in \Gamma_v$ cover, without any overlap by (R3*), the rest of V' and $C \setminus \{v\}$. Thus, $\mathcal{P}_v$ is a partition of $V \setminus \{v\}$ for every $v \in C$.

Since the only sets of $\mathcal{C}$ intersecting C in a nontrivial way, are those of the form $B_{uv} \cup T_{vu}$ and $B_{vu} \cup T_{uv}$ for $(u,v) \in E(T)$, there are exactly $2\omega - 2$ such sets, and hence C is a critical clique of the family $\mathcal{C}$. It is now a straightforward verification that the conditions (R1), (R2) and (R3*) hold, and the reduction starting with $\mathcal{C}$ and $C \in \mathcal{C}$ will yield $\mathcal{C}'$. ∎

5 Substituting spiders in webs

¿From practical point of view, condition (R1) is well characterized in Section 3, hence equivalently we always can start with a spanning tree on the ω-set C. However, finding $\omega - 1$ cliques in $\mathcal{C}'$ satisfying (R2), and finding a split of each of these cliques so that (R3*) satisfied, is far not trivial.

Given a partitionable (α, ω)-graph $G' = (V', E')$, and a disjoint ω-set C, let us try to construct a partitionable $(\alpha + 1, \omega)$-graph on the vertex set $V' \cup C$, following the recursion described in the previous section. As we have shown, we must choose first a spanning tree T with $V(T) = C$, and use the critical family defined by its edges in our construction. (And therefore condition (R1) will automatically be satisfied.)

An immediate question arise: can we pick any spanning tree T on the set C? Applying Lemmas 8 and 9 we can conclude that the maximum degree of the vertices in T and even the number of leaves certainly cannot exceed $\alpha + 1$. We also know that a simple path can surely arise, since this is the case with a web, in which all cliques are critical.

In this section we show that in fact there is an infinite family of trees (larger than the family of paths but still very restricted) which can arise as spanning trees in critical cliques, by applying the recursive construction described in the previous section. For this we shall consider (α, ω)-webs and apply the recursion to them starting with a special family of spanning trees.

The (α, ω)-*web*, is the graph $G' = (V', E')$, in which the vertices can be identified with the integers modulo $n = \alpha\omega + 1$, i.e. $V' = \mathbb{Z}_n$, and in which the ω-cliques correspond to consecutive (modulo n) sequences of integers in $\mathbb{Z}_n$.

Let us introduce the notations $\Omega = \{0, 1, ..., \omega - 1\} = \mathbb{Z}_\omega$, $\Lambda = \{1, ..., \alpha\} = \mathbb{Z}_\alpha$, and let us have the convention that arithmetical operations with elements of $\mathbb{Z}_n$ will always be meant modulo n. Furthermore, for a subset $S \subseteq \mathbb{Z}_n$ and an integer $a \in \mathbb{Z}_n$ let us define $a + S = \{a + i | i \in S\}$. The family of ω-cliques of the (α, ω)-web G' then can, more precisely, be described as

$$\mathcal{C}' = \{C'_i = i + \Omega | i \in \mathbb{Z}_n\} \tag{26}$$

while its α-stable sets are

$$\mathcal{S}' = \{S'_i = i + \omega * \Lambda | i \in \mathbb{Z}_n\}. \tag{27}$$

With these definitions, C'_i and S'_i are vis-a-vis for all $i \in \mathbb{Z}_n$.

Let us next define a *spider*. A *spider* is a rooted tree, in which only the root vertex can have degree higher than 2. In particular, a path is a spider, whichever its vertex as chosen as the root. The paths, connecting vertices of degree 1 (leaves) of a spider to its root are called its *legs*.

Theorem 4 *Let us consider an (α, ω)-web $G' = (V', E')$ (on $n = \alpha\omega + 1$ vertices), and a spanning spider $T = (C, E)$ rooted at $r \in C$, where C is an ω-set, disjoint from V', and* *let us assume that for the degree of the root vertex of T we have $d_r \leq \alpha + 1$. Then, the recursion of the previous section can be applied, and an $(\alpha+1,\omega)$-partitionable graph G can be constructed on the vertex set $V' \cup C$, such that C becomes a critical clique of G, and T will be the tree of its critical edges.*

Proof. Let us first identify the vertices of G' with $\mathbb{Z}_n$, as above, and let us introduce coordinates for the vertices of T. Let us number the legs first from 1 to d_r, and then let us associate the pair (k,i) to the vertex $v \in C$, if v belongs to the k-th leg, and is the i-th vertex counted from the leaf on that leg, i.e. $(k,1)$, for $k=1,...,d_r$ are the leaves of T. Let us note that formally all the pairs (k, n_k+1) for $k=1,2,...,d_r$ are corresponding to the root of the tree, where n_k denotes the number of vertices on the k-th leg (not counting the root). With these notations, we have

$$\sum_{k=1}^{d_r} n_k = \omega - 1 \tag{28}$$

and that

$$C = \{r\} \cup \{(k,i) | 1 \leq i \leq n_k,\ 1 \leq k \leq d_r\}. \tag{29}$$

To simplify notations, let us also introduce subintervals of $\mathbb{Z}_n$ by defining

$$[a,b) = \{a + j | j = 0,1,...,(b-a) \mod n\}.$$

For instance for $n=11$ we have $[4,8) = \{4,5,6,7\}$ and $[10,2) = \{10,0,1\}$.

To describe our construction, we need to specify $\omega - 1$ cliques of G' corresponding to the edges of T, and an appropriate split of each of them into two subsets.

With our notation, all the edges of T are of the form $[(k,i),(k,i+1)]$ for some indices $1 \leq k \leq d_r$ and $1 \leq i \leq n_k + 1$. In particular, the edge $[(k,n_k),(k,n_k+1)]$ is the edge of the k-th leg, incident with the root. Then the sets corresponding to the partitions of C induced by these edges are

$$\begin{aligned} T_{[(k,i),(k,i+1)]} &= \{(l,j) | l \neq k\} \cup \{(k,j) | j \geq i+1\}, \text{ while} \\ T_{[(k,i+1),(k,i)]} &= \{(k,j) | j \leq i\}, \end{aligned} \tag{30}$$

for $i=1,...,n_k$, and $k=1,...,d_r$. Clearly, $|T_{[(k,i+1),(k,i)]}| = i$ and $|T_{[(k,i),(k,i+1)]}| = \omega - i$ for all $1 \leq i \leq n_k$ and $1 \leq k \leq d_r$.

Let us now define the associated ω-cliques of G' by

$$\begin{aligned} C'_{[(k,i),(k,i+1)]} &= [k\omega - (n_1 + \cdots n_{k-1} + i), (k+1)\omega - (n_1 + \cdots n_{k-1} + i)) \\ &= C'_{k\omega - (n_1 + \cdots n_{k-1} + i)} \end{aligned} \tag{31}$$

using our notation of (26), for $i=1,2,...,n_k$ and for $k=1,...,d_r$. Let us split each of these cliques into two subintervals given by

$$\begin{aligned} B_{[(k,i),(k,i+1)]} &= [k\omega - (n_1 + \cdots n_{k-1}), (k+1)\omega - (n_1 + \cdots n_{k-1} + i)) \text{ and} \\ B_{[(k,i+1),(k,i)]} &= [k\omega - (n_1 + \cdots n_{k-1} + i), k\omega - (n_1 + \cdots n_{k-1})), \end{aligned} \tag{32}$$

We claim that with these definitions, the clique family $\mathcal{C}$, given as in (25), will indeed define an $(\alpha+1,\omega)$-partitionable graph on the vertex set $V' \cup C$. In order to see this, according to Theorem 3, we have to verify that conditions (R1), (R2) and (R3*) are all satisfied by our construction.

The first condition (R1), as we noted earlier, follows directly from the fact that T is a spanning tree, and the splits $T_{[(k,i),(k,i+1)]}$ and $T_{[(k,i+1),(k,i)]}$ are defined by the edges of this tree. Hence, by Theorem 1, they form indeed a critical tree-covering family on C.

To verify condition (R2), we have to show that the cliques $C'_{k\omega-(n_1+\cdots n_{k-1}+i)}$ for $i = 1,2,...,n_k$ and for $k=1,...,d_r$ all belong to different partitions $\mathcal{P}'_v$ of the (α,ω)-web G'. To this end, let us observe first that, due to the special structure of a web, two cliques C'_i and C'_j $(i<j)$, as defined by (26), belong to the same partition if and only if $j-i \geq \omega$ and $j-i = 0$ or $1 \mod \omega$, i.e. if they do not overlap, and one of the gaps between these two subintervals of the circular $\mathbb{Z}_n$ can be tiled by ω-intervals. Let us now consider two cliques of the form $C'_{k\omega-(n_1+\cdots n_{k-1}+i)}$ and $C'_{k'\omega-(n_1+\cdots n_{k'-1}+i')}$, as in (31). Let us observe that if $k=k'$, then these cliques overlap, and thus cannot belong to the same partition, while for $k>k'$ we have

$$\begin{aligned}(k\omega-(n_1+\cdots n_{k-1}+i)) &- (k'\omega-(n_1+\cdots n_{k'-1}+i')) \\ &= (k-k')\omega-(n_{k'}+\cdots n_{k-1}+i-i').\end{aligned}$$

Since $n_{k'}-i' \geq 0$, $i \geq 1$ and $k>k'$, the sum $n_{k'}+\cdots n_{k-1}+i-i'$ is always positive, and it takes its maximum, if $k'=1$, $k=d_r$, $i=n_{d_r}$ and $i'=1$, when it is $\omega-2$, by (28). Thus

$$1 \leq n_{k'}+\cdots n_{k-1}+i-i' \leq n_1+\cdots+n_{d_r}-1 = \omega-2$$

follows, implying that the quantity $((k-k')\omega-(n_{k'}+\cdots n_{k-1}+i-i'))$, is never 0 or 1 modulo ω.

To verify (R3*) let us note first that the sets, B_{uv} for $u \in \Gamma_v$, as defined in (32) are pairwise disjoint, and consecutive, i.e. form an interval of length

$$\sum_{u\in\Gamma_v} |B_{uv}| = \sum_{u\in\Gamma_v} (\omega-|T_{vu}|) = d_v\omega - |V' \setminus \{v\}| = (d_r-1)\omega+1,$$

for all $v \in C$, and hence the complementary set $V' \setminus \bigcup_{u\in\Gamma_v} B_{uv}$ has its cardinality as a multiple of ω (since $n=\alpha\omega+1$). Thus it can be tiled by ω-cliques of the web G'. Therefore, to verify (R3*), we need to show first that the above hold with the definitions in (32), and second that to tile the sets $V' \setminus \bigcup_{u\in\Gamma_v} B_{uv}$ for $v \in C$ by ω-cliques of G' one does not need the cliques defined in (31).

To see the first part is easy just by looking at the definitions (32). Namely, for leaf vertices there is nothing to check. For the root of T we have the sets

$$B_{[(k,n_k),(k,n_k+1)]} = [k\omega-(n_1+\cdots n_{k-1}),(k+1)\omega-(n_1+\cdots n_{k-1}+n_k)) \tag{33}$$

for $k = 1, 2, ..., d_r$, and these obviously are consecutive, in this order, with no overlap. For an interior vertex (k, i) of a leg (i.e. with $1 < i < n_k$) we have the two sets

$$\begin{aligned} B_{[(k,i+1),(k,i)]} &= [k\omega - (n_1 + \cdots n_{k-1} + i), k\omega - (n_1 + \cdots n_{k-1})) \text{ and} \\ B_{[(k,i-1),(k,i)]} &= [k\omega - (n_1 + \cdots n_{k-1}), (k+1)\omega - (n_1 + \cdots n_{k-1} + i - 1)) \end{aligned} \tag{34}$$

and again these sets are always consecutive without any overlap.

For the second part, let us first have a look again at the sets (33), and let us observe that the complement of their union can be partitioned by the cliques $\mathcal{H}_r = \{C'_{(d_r+j)\omega+1} | j = 0, 1, ..., \alpha - d_r\}$. Since for the cliques of the form $C'_{k\omega-(n_1+\cdots n_{k-1}+i)}$ for $1 \le i \le n_k$ for $1 \le k \le d_r$ (see (31)), we have

$$\omega - 1 \le k\omega - (n_1 + \cdots n_{k-1} + i) \le (d_r - 1)\omega + 1$$

therefore, $\mathcal{H}_r$ indeed does not contain any of these. For the two sets finally in (34), we can see that their complement is partitioned by the cliques

$$\mathcal{H}_{(k,i)} = \{C_{(k+j)\omega-(n_1+\cdots n_{k-1}+i-1)} | j = 1, ..., \alpha - 1\}$$

and again these are all different from those in (31). ■

As an illustration, let us consider the $(2, 5)$-web (anti-hole) on 11 vertices, and the spider on figure 1. In this example we have $\alpha = 2$, $\omega = 5$, (and hence $n = 11$), and, as shown in

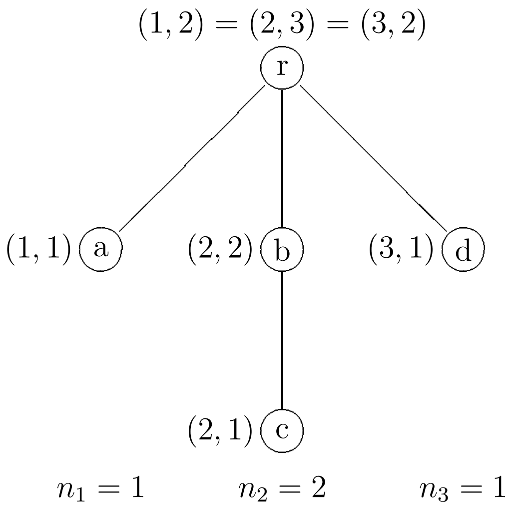

Figure 1: A coordinatized spider on 5 vertices.

figure 1, $r = (1, 2) = (2, 3) = (3, 2)$, $a = (1, 1)$, $b = (2, 2)$, $c = (2, 1)$, and $d = (3, 1)$. Then the sets by (31) and (32) are as follows

$$\begin{array}{lll} C'_{ar} = [4,9) & B_{ra} = [4,5) & B_{ar} = [5,9) \\ C'_{br} = [7,1) & B_{rb} = [7,9) & B_{br} = [9,1) \\ C'_{dr} = [0,5) & B_{rd} = [0,1) & B_{dr} = [1,5) \\ C'_{bc} = [8,2) & B_{bc} = [8,9) & B_{cb} = [9,2) \end{array}$$

The eight sets $[4,5) \cup \{r,b,c,d\}$, $[5,9) \cup \{a\}$, $[7,9) \cup \{r,a,d\}$, $[9,1) \cup \{b,c\}$, $[0,1) \cup \{r,a,b,c\}$, $[1,5) \cup \{d\}$, $[8,9) \cup \{r,a,b,d\}$, and $[9,2) \cup \{c\}$ together with $C = \{r,a,b,c,d\}$ and the seven of the original cliques of the $(2,5)$-web, namely $[1,6)$, $[2,7)$, $[3,8)$, $[5,10)$, $[6,0)$, $[9,3)$ and $[10,4)$ form the clique family of a $(3,5)$-partitionable graph on the 16 vertices of $\mathbb{Z}_{11} \cup C$.

Remark 6 *Even though for $\omega = 3$ all spiders are a simple path of two edges, still, depending on where the root is, we get different results. E.g. starting from the $(2,3)$-web, and the spider $\{r,a,b\}$ forming a 2 edge path with the root at the end, we obtain a $(3,3)$-web. While if we use the spider $\{a,r,b\}$ forming a 2 edge path again, but now having the root in the middle, we get a non-web $(3,3)$-partitionable graph, appearing in [2].*

Remark 7 *For every labeled spider on ω points with $d_r \leq \alpha + 1$, by the above result we can generate an $(\alpha+1,\omega)$-partitionable graphs from an (α,ω)-web, though some of these graphs might be isomorphic.*

Remark 8 *Obviously, $|L(T)| \geq d_v$ for every vertex $v \in T$, and there exists a vertex v in T such that $|L(T)| = d_v$ if and only if T is a spider. Thus for spiders and only for them inequalities of Lemmas 8 and 9 are equivalent.*

6 $(\alpha, 3)$-partitionable families and other experimental results.

For $\omega = 3$ we have the following characterization of critical cliques:

Lemma 10 *A clique is critical if and only if it is in the middle of a gem.*

Proof. There is a unique tree with 3 vertices, let us say b, c, d. There is a unique tree-covering family: $(b,c),(d)$, $(b),(c,d)$. Thus there should be cliques (a,b,c) and (c,d,e). Vertices a and e are different, otherwise we would get a K_4. Vertices a,b,c,d,e form a gem with critical clique (b,c,d) in the middle. ■

We conjecture that for $\omega = 3$ every partitionable graph has a critical clique. The following experimental results support this conjecture. We have verified, that for $\omega = 3$ there exists a gem (and therefore a critical clique) in all partitionable graphs up to $\alpha = 9$. The existence of a diamond was verified for partitionable graphs up to $\alpha = 10$.

In Table 1 we list some additional experimental results. We have generated all the partitionable graphs for $\omega = 3$ and $\alpha = 2, \dots 7$ and for $\omega = 4$ and $\alpha = 4$ and 5. For $\omega = 3$ all graphs have critical cliques, while for $\omega = 4$ this is no longer true.

The column "ST" counts the number of graphs which have a *small transversal*, that is a subset of the vertices of size $\alpha + \omega - 1$ that intersects all ω-cliques and all α-stable sets. The column "C_5" lists the number of partitionable graphs without C_5. Both these values turn out to be very useful parameters in case one is interested to generate all partitionable graphs that are reasonable candidates to be counterexamples to the Strong Perfect Graph Conjecture. It is well known that such graphs have neither a small transversal nor a C_5.

Table 1: The number of partitionable graphs without indifferent edges. (Numbers in bold were not known before)

n	ω	α	# total	# of graphs without crit. clique	ST	C_5	# of graphs constructable by CGPW	our construction
10	3	3	2	0	0	0	2	**2**
13	3	4	5	0	0	1	4	**5**
16	3	5	21	0	0	2	**18**	**21**
19	3	6	154	0	0	7	**138**	**154**
22	3	7	**1488**	**0**	**0**	**22**	**1332**	**1488**
17	4	4	132	6	1	1	**22**	**126**
21	4	5	**8340**	**1431**	0	**4**	**1189**	**6909**
25	4	6	?	?	**0**	?	?	?

Remark 9 *Our computations show that a counterexample to the Strong Perfect Graph Conjecture must have at least 26 vertices. This slightly improves the previous bound 25 given by Gurvich and Udalov (1992). These two bounds are obtained due to a computer analysis of the (4,6)- and (4,5)-graphs, respectively. It was shown that all these graphs have small transversals and thus cannot be counterexamples to the Berge Conjecture. To reach the next bound 29 the case of (5,5)-graphs has to be considered.*

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