DIMACS Technical Report 99-30
June 1999

Fully Consistent Extensions of Partially Defined Boolean Functions with Missing Bits

by

Endre Boros[1,2]   Toshihide Ibaraki[3,4]   Kazuhisa Makino[5]

[1]RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway, NJ 08854-8003, USA (`boros@rutcor.rutgers.edu`). Permanent Member.
[2]This research was partially supported by ONR (Grant N00014-92-J-1375) and DARPA (Contract Number N66001-97-C-8537). The visit to Kyoto University was made possible by the grant (09044160) of the Ministry of Education, Science, Sports and Culture of Japan.
[3]Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan (`ibaraki@i.kyoto-u.ac.jp`).
[4]This research was partially supported by the Scientific Grants in Aid by the Ministry of Education, Science, Sports and Culture of Japan.
[5]Department of Systems and Human Science, Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka 560-8531, Japan (`makino@sys.es.osaka-u.ac.jp`).

ABSTRACT

In this paper we consider four different definitions for an extension of a partially defined Boolean function in which the input contains some missing bits. We show that, for many general and reasonable families of function classes, three of these extensions are mathematically equivalent. However we also demonstrate that such an equivalence does not hold for all classes.

1 Introduction and definitions

A *Boolean function*, or a *function* in short, is a mapping $f : \mathbb{B}^n \mapsto \mathbb{B}$, where $\mathbb{B} = \{0, 1\}$. Given a function f, a *Boolean vector* $x \in \mathbb{B}^n$ is called its *true vector*, if $f(x) = 1$, and its *false vector*, if $f(x) = 0$. Let us denote the set of true vectors of f by $T(f)$, and let $F(f) = \mathbb{B}^n \setminus T(f)$ denote the set of its false vectors.

Let us denote by $\mathcal{C}_{all}$ the family of all Boolean functions $f : \mathbb{B}^n \mapsto \mathbb{B}$, and let us call any subfamily of $\mathcal{C}_{all}$ a *class*. We shall consider various classes of Boolean functions in the sequel, defined in many different ways.

A *partially defined Boolean function* (a *pdBf* in short) is defined by a pair of sets (T, F) such that $T, F \subseteq \mathbb{B}^n$. A Boolean function f is called an *extension* of the pdBf (T, F) if $T \subseteq T(f)$ and $F \subseteq F(f)$ hold, that is, if such an f *correctly classifies* all the vectors $a \in T$ and $b \in F$. Let us denote by $\mathcal{E}(T, F)$ the family of extensions of the pdBf (T, F). Evidently, the disjointness of the sets T and F is a necessary and sufficient condition for the existence of an extension $\mathcal{E}(T, F) \neq \emptyset$. It may not be evident, however, to find out if a given pdBf has an extension belonging to a particular class $\mathcal{C}$ of Boolean functions, or not. This problem has been studied in various fields such as learning theory, knowledge discovery, data mining and logical analysis of data [1, 5, 6, 10, 11, 13].

In practical cases, the fact that $\mathcal{C} \cap \mathcal{E}(T, F) = \emptyset$ might be due to some classification errors in the input. To correct this type of errors, provided that they are not in a large number, one can consider the optimization problem of finding the largest subsets $T^* \subseteq T$ and $F^* \subseteq F$ for which $\mathcal{E}(T^*, F^*) \cap \mathcal{C} \neq \emptyset$ holds. These problems have extensively been studied (e.g., in [10, 13]) for a large variety of classes.

In this paper we shall consider another type of error in the input, the case in which some data vectors are "incomplete" in the sense that some of their components are not available at the time of reading the input. Such missing information may either be due to some measurement errors at some earlier stages of data generation, or they are the results of data entry errors, or such lack of information might be due to the high cost of obtaining those.

To model such situations, let us consider the set $\mathbb{M} = \{0, 1, *\}$, and let us interpret the asterisk components $*$ of a vector $v \in \mathbb{M}^n$ as missing bits. Then, a *pdBf with missing bits* (or in short a *pBmb*) can be defined as a pair $(\tilde{T}, \tilde{F})$, where $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$. Given a pBmb, it is possible to consider more than one notion of extensions f, depending on how to interpret $*$'s in the extensions; in this paper, we give four different definitions, two of which have already been discussed in [8, 7, 9, 11]. We then prove for many important classes of functions that three of these definitions are equivalent. However, it is also demonstrated that this equivalence does not hold all classes.

2 Extensions of pBmbs

For a vector $v \in \mathbb{M}^n$, let us introduce the notations $ON(v) = \{j \,|\, v_j = 1,\ j = 1, 2, \ldots, n\}$ and $OFF(v) = \{j \,|\, v_j = 0,\ j = 1, 2, \ldots, n\}$. For a subset $\tilde{A} \subseteq \mathbb{M}^n$, let $S(\tilde{A}) = \{(v, j) | v \in \tilde{A},\ v_j = *\}$ be the collection of all missing bits of the vectors in $\tilde{A}$. If $\tilde{A}$ is a singleton $\{v\}$, we shall also write $S(v)$ instead of $S(\{v\})$. Clearly, $\mathbb{B}^n \subseteq \mathbb{M}^n$, and $v \in \mathbb{B}^n$ holds if and only if $S(v) = \emptyset$. Let us consider a binary assignment $\alpha \in \mathbb{B}^Q$ to a subset $Q \subseteq S(\tilde{A})$ of the missing bits. Then v^α denotes the vector obtained from $v \in \tilde{A}$ by replacing the $*$ components which belong to Q by the binary values assigned by α:

$$v_j^\alpha = \begin{cases} v_j & \text{if } (v,j) \notin Q \\ \alpha(v,j) & \text{if } (v,j) \in Q. \end{cases}$$

Let $\tilde{A}^\alpha$ denote the set $\{v^\alpha \mid v \in \tilde{A}\}$. For example, for the set $\tilde{A} = \{u = (1,*,0,1), v = (0,1,*,*), w = (1,1,*,0)\} \subset \mathbb{M}^4$ we have $S(\tilde{A}) = \{(u,2), (v,3), (v,4), (w,3)\}$. If $Q = \{(u,2),(v,4)\}$, an assignment $(\alpha(u,2),\alpha(v,4)) = (1,0) \in \mathbb{B}^Q$ yields $\tilde{A}^\alpha = \{u^\alpha = (1,1,0,1), v^\alpha = (0,1,*,0), w^\alpha = (1,1,*,0)\}$.

For vectors $v, w \in \mathbb{M}^n$, we shall write $v \gtrapprox w$ if there exists an assignment $\alpha \in \mathbb{B}^{S(\{v,w\})}$ for which $v^\alpha \geq w^\alpha$ holds (componentwise), and we say that v is *potentially greater* than w. If both $v \gtrapprox w$ and $w \gtrapprox v$ hold then we write $v \approx w$, and say that v is *potentially identical* with w. Note that $v \approx w$ holds if and only if there is an assignment $\alpha \in \mathbb{B}^{S(\{v,w\})}$ such that $v^\alpha = w^\alpha$. For example, if $v = (1,0,*,1,*)$ and $w = (0,0,0,1,*)$ then $v \gtrapprox w$ holds, but they are not potentially equivalent due to the difference of their first digits, while for the pair of vectors $v' = (*,0,1,*,1)$ and $w' = (1,0,*,*,1)$ we have $v' \approx w'$.

To a pBmb $(\tilde{T},\tilde{F})$ we shall always associate the set $S = S(\tilde{T} \cup \tilde{F})$ of its missing bits. For a pBmb $(\tilde{T},\tilde{F})$ and an assignment $\alpha \in \mathbb{B}^S$, let $(\tilde{T}^\alpha,\tilde{F}^\alpha)$ be the pdBf defined by $\tilde{T}^\alpha = \{a^\alpha \mid a \in \tilde{T}\}$ and $\tilde{F}^\alpha = \{b^\alpha \mid b \in \tilde{F}\}$.

Unlike in case of pdBfs the notion of an extension is not uniquely and well-defined for pBmbs. Given a pBmb $(\tilde{T},\tilde{F})$, let us consider now various possibilities of defining its extendibility. Let us call a pBmb $(\tilde{T},\tilde{F})$ *consistent* with respect to a class $\mathcal{C}$ of Boolean functions, if there exists an assignment $\alpha \in \mathbb{B}^S$ for which the pdBf $(\tilde{T}^\alpha,\tilde{F}^\alpha)$ has an extension in $\mathcal{C}$. A Boolean functions $f \in \mathcal{E}(\tilde{T}^\alpha,\tilde{F}^\alpha) \cap \mathcal{C}$ will be called a *consistent extension* of $(\tilde{T},\tilde{F})$ in the class $\mathcal{C}$.

Problem CE($\mathcal{C}$)	
Input:	A pBmb $(\tilde{T},\tilde{F})$, where $\tilde{T},\tilde{F} \subseteq \mathbb{M}^n$.
Question:	Does $(\tilde{T},\tilde{F})$ have a consistent extension in class $\mathcal{C}$?

In case of YES, an assignment $\alpha \in \mathbb{B}^S$, for which the pdBf $(\tilde{T}^\alpha,\tilde{F}^\alpha)$ has an extension in $\mathcal{C}$, together with such an extension $f \in \mathcal{E}(\tilde{T}^\alpha,\tilde{F}^\alpha) \cap \mathcal{C}$, must also be provided.

Let us note that, in case $(\tilde{T},\tilde{F})$ has a consistent extension, the output of CE($\mathcal{C}$) might not be unique, and at an important extreme end, it may occur that for every possible interpretations of the missing bits the obtained pdBf has an extensions belonging to $\mathcal{C}$. Let us call a pBmb $(\tilde{T},\tilde{F})$ *fully consistent* with the class $\mathcal{C}$ if this occurs, i.e., if the pdBf $(\tilde{T}^\alpha,\tilde{F}^\alpha)$ has an extension in $\mathcal{C}$ for every $\alpha \in \mathbb{B}^S$ (the corresponding extensions may differ for different αs.)

	PROBLEM FC($\mathcal{C}$)
Input:	A pBmb $(\tilde{T}, \tilde{F})$, where $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$.
Question:	Is $(\tilde{T}, \tilde{F})$ fully consistent with the class $\mathcal{C}$?

Let us remark that, unlike for problem CE($\mathcal{C}$), confirming a YES for problem FC($\mathcal{C}$) might become a computational burden because one may have to provide $2^{|S|}$ different extensions, for each possible assignment to the missing bits of $(\tilde{T}, \tilde{F})$. For this reason, we shall consider a special case in which in fact all these extensions coincide. Let us call a Boolean function f a *robust extension* of a given pBmb $(\tilde{T}, \tilde{F})$ if

$$f(a^\alpha) = 1 \text{ and } f(b^\alpha) = 0 \text{ for all } a \in \tilde{T},\ b \in \tilde{F} \text{ and for all } \alpha \in \mathbb{B}^S.$$

The corresponding decision problem of deciding the existence of a robust extensions for a given pBmb $(\tilde{T}, \tilde{F})$ and a specified class $\mathcal{C}$ can be stated as follows.

	PROBLEM RE($\mathcal{C}$)
Input:	A pBmb $(\tilde{T}, \tilde{F})$, where $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$.
Question:	Does $(\tilde{T}, \tilde{F})$ have a robust extension in class $\mathcal{C}$?

In case of YES, a robust extension $f \in \mathcal{C}$ must also be provided. Let us denote by $\mathcal{E}(\tilde{T}, \tilde{F})$ the family of all robust extensions of a given pBmb $(\tilde{T}, \tilde{F})$.

Let us remark now that even in this special case, the verification of a YES computationally may not be an easy problem. Consider for instance the case when the output function f is represented by a DNF. Then, verifying that $f(a^\alpha) = 1$ holds for a vector $a \in \tilde{T}$ and for all $\alpha \in \mathbb{B}^{S(a)}$ might be as difficult as the satisfiability problem, in general. As it is well-known, the satisfiability problem is NP-complete, even if its input is restricted to cubic DNFs, for example (see [14]).

We shall therefore consider a further special case, when such difficulties will not arise. Consider an elementary conjunction (i.e., term)

$$t(\mathbf{x}) = \bigwedge_{j \in P} x_j \bigwedge_{j \in N} \overline{x}_j$$

for some subsets $P, N \subseteq \{1, 2, \ldots, n\}$ with $P \cap N = \emptyset$. We shall call t a *robust term* with respect to a pBmb $(\tilde{T}, \tilde{F})$ and a vector $a \in \tilde{T}$, if $t(a^\alpha) = 1$ for all $\alpha \in \mathbb{B}^{S(a)}$ and $t(b^\beta) = 0$ for all $b \in \tilde{F}$ and $\beta \in \mathbb{B}^{S(b)}$. Let us note that a term t is robust with respect to $(\tilde{T}, \tilde{F})$ and $a \in \tilde{T}$ if and only if $S(a) \cap (P \cup N) = \emptyset$, and $(ON(b) \cap N) \cup (OFF(b) \cap P) \neq \emptyset$ for all $b \in \tilde{F}$, conditions which are all easy to check. Let us then call a Boolean function f a *very robust extension* of $(\tilde{T}, \tilde{F})$, if it is a robust extension which can be represented by a disjunction of robust terms.

Problem VR($\mathcal{C}$)	
Input:	A pBmb $(\tilde{T}, \tilde{F})$, where $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^n$.
Question:	Does $(\tilde{T}, \tilde{F})$ have a very robust extension in class $\mathcal{C}$?

In case of YES, a very robust term for each $a \in \tilde{T}$ must also be provided. Let us denote by $\mathcal{E}^*(\tilde{T}, \tilde{F})$ the family of all very robust extensions of the pBmb $(\tilde{T}, \tilde{F})$.

Problems CE($\mathcal{C}$) and RE($\mathcal{C}$) and some related optimization problems have been considered extensively for various classes in [7, 8, 9, 11]. In this paper we concentrate on the relations between FC($\mathcal{C}$), RE($\mathcal{C}$) and VR($\mathcal{C}$).

It is quite immediate to see from the above definitions that very robust extensions are robust as well, and that pBmbs which have robust extensions in a given class $\mathcal{C}$ are also fully consistent with that class.

Somewhat surprisingly, we can show that for many very general families of classes these three problems are in fact equivalent. We can also demonstrate that such an equivalence does not hold for all classes, for instance by showing that for certain classes $\mathcal{C}$ problem RE($\mathcal{C}$) is polynomially solvable, while FC($\mathcal{C}$) is co-NP-complete.

3 Classes of Boolean functions

We shall assume in the sequel that Boolean functions are represented either by an explicit algebraic form, or by an oracle. In either case, it is possible to compute the values of such a function for given input vectors. In each case in the sequel, we shall make clear what is the representation of the considered family of functions.

Let us call an elementary conjunction of literals a *term*, and an elementary disjunction of literals a *clause*. The most common representation we shall consider for a Boolean function f will be either a *disjunctive normal form* (or DNF in short), which is a disjunction of terms, or a *conjunctive normal form* (or CNF in short), which is a conjunction of clauses.

For two Boolean functions f and g, we shall write $f \leq g$, and say that f is a *minorant* of g, or equivalently that g is a *majorant* of f, if $g(x) = 1$ is implied whenever $f(x) = 1$ holds. For a Boolean expression A, let us denote by $\overline{A} = 1 - A$ its *negation*. The components of the (unknown) vector $x = (x_1, ..., x_n)$ will be called Boolean *variables*, while variables and their complements together are called *literals*.

Terms which are minorants of a Boolean function f are called its *implicants*, and those implicants which are maximal minorants of f are called its *prime implicants*. It is well-known that every Boolean function can be represented by the DNF formed by the disjunction of all of its prime implicants. It is also well-known that in general, there are many other DNFs representing the same function.

Clauses majorizing a Boolean function f are called its *implicates*, and those implicates which are minimal majorants of f are called its *prime implicates*. Analogously to the above, the CNF obtained by taking the conjunction of all prime implicates of a Boolean function f

is also a representation of f, and in most cases it is also not unique; most Boolean functions can be represented by many different CNFs.

We shall consider many different classes of Boolean functions, whose definitions will be either via some representation independent functional properties, or by properties of some of the DNF (or CNF) representations, or via some other representations.

A large family of classes of the first type are the *transitive* or *generalized monotone* classes. Let us consider a partial order $\preceq$ of the Boolean vectors $\mathbb{B}^n$, and let us say that a Boolean function f is *$\preceq$-monotone*, if $f(x) \leq f(y)$ holds whenever $x \preceq y$. For a given partial order $\preceq$ on $\mathbb{B}^n$, let us denote by $\mathcal{C}_\preceq$ the family of all $\preceq$-monotone Boolean functions. Then, we shall call a class $\mathcal{C}$ *transitive*, if there exists a partial order $\preceq$ on $\mathbb{B}^n$ for which $\mathcal{C} = \mathcal{C}_\preceq$.

Most notable examples for transitive classes are the family of positive Boolean functions, $\mathcal{C}_+ = \mathcal{C}_\geq$, where $f \in \mathcal{C}_+$ if $f(x) \geq f(y)$ holds whenever $x \geq y$ holds (componentwise), and the family $\mathcal{C}_{reg}$ of *regular* Boolean functions, where $\mathcal{C}_{reg} = \mathcal{C}_\succcurlyeq$ holds for the relation $\succcurlyeq$, which is defined by $x \succcurlyeq y$ if and only if $\sum_{j=1}^k x_j \geq \sum_{j=1}^k y_j$ for all $k = 1, 2, \ldots, n$.

Another frequently used partial order on the Boolean cube is a "tilted" monotone order. To an arbitrary vector $b \in \mathbb{B}^n$, we can associate a partial order $\geq_b$ of the Boolean cube $\mathbb{B}^n$ by defining that $v \geq_b w$ holds if and only if $v \oplus b \geq w \oplus b$ holds, where $\oplus$ denotes the exclusive-or operation (the componentwise mod 2 addition, e.g., $(1100) \oplus (0110) = (1010)$). In other words, $\geq_b$ is like the regular monotone order $\geq$ in which b plays the role of the zero-vector $(0, 0, ..., 0)$, and $\overline{b}$ is the maximum vector. The family of $\geq_b$-monotone functions will be denoted by $\mathcal{C}_{\geq_b}$. Thus, in particular $\mathcal{C}_+ = \mathcal{C}_{\geq_0}$ holds.

Let us finally remark that the family $\mathcal{C}_{all}$ of all Boolean functions itself is a transitive class, corresponding to the "empty" partial order on $\mathbb{B}^n$.

Some other non-transitive classes, defined via a representation independent property can be obtained by taking the union of various transitive classes. For instance, a function f is called *unate* if it is $\geq_b$-monotone for some vector $b \in \mathbb{B}^n$. The family of unate functions, hence is the union of all the $\geq_b$-monotone classes, $\mathcal{C}_{unate} = \bigcup_{b \in \mathbb{B}^n} \mathcal{C}_{\geq_b}$.

Other examples for classes defined via a representation independent property include the family of *self-dual* functions $\mathcal{C}_{SD}$, consisting of functions f for which $f = f^d$, where the *dual* f^d of a Boolean function f is defined by $f(x_1, ..., x_n) = \overline{f}\,(\overline{x}_1, ..., \overline{x}_n)$. Similarly, the family of *dual-minor* functions $\mathcal{C}_{D\text{-}minor}$ consists of the functions satisfying the inequality $f \leq f^d$, while the class of *dual-major* functions $\mathcal{C}_{D\text{-}major}$ is formed by the functions satisfying $f \geq f^d$.

We shall also mention briefly decomposable functions (see e.g. [4]). For given subsets $S_0, S_1 \subseteq \{1, 2, ..., n\}$ a Boolean function f is called *$F(S_0, F_1(S_1))$-decomposable* if there exists Boolean functions $h : \mathbb{B}^{S_1} \mapsto \mathbb{B}$ and $g : \mathbb{B}^{S_0 \cup \{h\}} \mapsto \mathbb{B}$ such that $f(x_1, ..., x_n) = g((x_i : i \in S_0), h(x_j : j \in S_1))$ holds for all $x = (x_1, ..., x_n) \in \mathbb{B}^n$. Decomposable functions play an important role in VLSI synthesis. Let $\mathcal{C}_{F(S_0, F_1(S_1))}$ denote this class of Boolean functions.

The above functional classes we can also denote, for clarification, by explicitly writing the defining relation, e.g. $\mathcal{C}_+ = \mathcal{C}_{f(x) \geq f(y) \text{ if } x \geq y}$, $\mathcal{C}_{SD} = \mathcal{C}_{f = f^d}$, $\mathcal{C}_{D\text{-}minor} = \mathcal{C}_{f \leq f^d}$, etc.

Another large family of classes, the so called *DNF-classes* are defined via their DNF representation. Let us consider a family of terms $\mathbb{T}$, and let us define the corresponding

DNF-class $\mathcal{C}_{\mathbb{T}}$ by $\mathcal{C}_{\mathbb{T}} = \{f(x) = \bigvee_{t\in\mathcal{S}} t(x) \,|\, \mathcal{S} \subseteq \mathbb{T}\}$ as the collection of all Boolean functions formed by the disjunction of a subset of terms from $\mathbb{T}$.

For example, if $\mathbb{T}$ consists of all terms of degree at most k (elementary conjunctions involving at most k literals), then the corresponding DNF-class is the family of the so-called k-DNFs, $\mathcal{C}_{k\text{-}DNF}$. The special cases of linear functions ($\mathcal{C}_{1\text{-}DNF}$) and quadratic functions ($\mathcal{C}_{2\text{-}DNF}$) should, in particular be mentioned.

Another notable example for a DNF-class is the family of *Horn* functions, $\mathcal{C}_{Horn}$. A Boolean function f is called *Horn*, if it can be represented by a DNF in which every term involves at most one negative variable. In other words, if $\mathbb{T}$ is the family of terms involving at most one negative variable, then Horn functions form the corresponding DNF-class, $\mathcal{C}_{\mathbb{T}} = \mathcal{C}_{Horn}$.

Let us remark that DNF-classes $\mathcal{C}_{\mathbb{T}}$ for which $\mathbb{T}$ is closed under consensus (for definition see Section 5) will play a special role due to the property that, for such a DNF, all its prime implicants must also belong to $\mathbb{T}$. Among consensus closed classes we should mention 2-DNFs, Horn functions, and $\geq_b$-monotone functions.

Let us add that many of the DNF-classes can also be defined via a representation independent relation. For instance, the family of Horn functions is characterized in [16] as exactly those functions f for which the set $F(f)$ of their false vectors is closed under the componentwise conjunction. Thus, if we define $x \wedge y = (x_1 \wedge y_1, ..., x_n \wedge y_n)$, we can write $\mathcal{C}_{Horn} = \mathcal{C}_{f(x\wedge y)\leq f(x)\vee f(y) \text{ for all } x \text{ and } y}$. It is also true that some of the classes defined "naturally" via a representation independent property, could also be defined as a DNF-class. For example, if $\mathbb{T}$ consists of all terms with no negative literals involved, then the corresponding DNF-class $\mathcal{C}_{\mathbb{T}}$ is exactly the class of positive Boolean functions, $\mathcal{C}_{+}$.

Given the Boolean functions f and g, we shall call g a *minor* of f, and will denote it by $g \sqsubseteq f$, if g can be represented by a disjunction of some of the prime implicants of f. Let us then call a class $\mathcal{C}$ *minor closed* if $f \in \mathcal{C}$ and $g \sqsubseteq f$ imply $g \in \mathcal{C}$. Minor closed classes include, in particular, all consensus closed DNF classes, and unions of those, such as renamable Horn functions, unate functions, q-Horn functions (see e.g., [3]), etc.

One other important class of functions, the class $\mathcal{C}_{TH}$ of *threshold* functions, is defined usually by a different representation. A Boolean function f is called *threshold* if there exist real numbers w_1, ..., w_n and w_0 such that $f(x) = 1$ if and only if the inequality $\sum_{j=1}^{n} w_j x_j \geq w_0$ holds. In other words, f is threshold exactly when the sets $T(f)$ and $F(f)$, viewed as point sets in the Euclidean space $\mathbb{R}^n$, are linearly separable. Of course, threshold functions could also be represented by DNFs (or CNFs), but for most threshold functions such a representation would be much less efficient computationally.

4 Equivalencies between RE($\mathcal{C}$) and FC($\mathcal{C}$)

In this section we shall show a series of results claiming, somewhat surprisingly, the mathematical equivalence of problems RE($\mathcal{C}$) and FC($\mathcal{C}$), under some widely applicable conditions.

Let us first consider those classes $\mathcal{C}$ of Boolean functions which are closed under conjunction and disjunction; i.e., $f \wedge g \in \mathcal{C}$ and $f \vee g \in \mathcal{C}$ for all $f, g \in \mathcal{C}$.

Theorem 1 *Let us assume that the class $\mathcal{C}$ of Boolean functions is closed under conjunction and disjunction, and that the pBmb $(\tilde{T}, \tilde{F})$ is fully consistent with $\mathcal{C}$. Then, $(\tilde{T}, \tilde{F})$ also has a robust extension.*

Proof. By the assumption of full consistency, for every pair $\alpha \in \mathbb{B}^{S(\tilde{T})}$ and $\beta \in \mathbb{B}^{S(\tilde{F})}$, there exists an extension $f_{\alpha,\beta} \in \mathcal{C} \cap \mathcal{E}(\tilde{T}^{\alpha}, \tilde{F}^{\beta})$. Let us then consider the Boolean function f defined by

$$f = \bigvee_{\alpha \in \mathbb{B}^{S(\tilde{T})}} \Big(\bigwedge_{\beta \in \mathbb{B}^{S(\tilde{F})}} f_{\alpha,\beta} \Big).$$

We claim that f is a robust extension of $(\tilde{T}, \tilde{F})$ in the class $\mathcal{C}$. First, $f \in \mathcal{C}$ follows from our assumption that $\mathcal{C}$ is closed under conjunction and disjunction.

To see that f is a robust extension of $(\tilde{T}, \tilde{F})$, let us first consider a vector $a \in \tilde{T}$ and an arbitrary assignment $\alpha^* \in \mathbb{B}^{S(\tilde{T})}$. Since $a^{\alpha^*} \in \tilde{T}^{\alpha^*}$, we have $\bigwedge_{\beta \in \mathbb{B}^{S(\tilde{F})}} f_{\alpha^*,\beta}(a^{\alpha^*}) = 1$ by the fact that $f_{\alpha^*,\beta} \in \mathcal{E}(\tilde{T}^{\alpha^*}, \tilde{F}^{\beta})$ for all $\beta \in \mathbb{B}^{S(\tilde{F})}$. Thus $f(a^{\alpha^*}) = 1$ is implied, for all $\alpha^* \in \mathbb{B}^{S(\tilde{T})}$.

Analogously, for a vector $b \in \tilde{F}$ and an assignment $\beta^* \in \mathbb{B}^{S(\tilde{F})}$ we can observe first that $f_{\alpha,\beta^*}(b^{\beta^*}) = 0$ holds for all $\alpha \in \mathbb{B}^{S(\tilde{T})}$, implied again by $f_{\alpha,\beta^*} \in \mathcal{E}(\tilde{T}^{\alpha}, \tilde{F}^{\beta^*})$. Thus, in this case $\bigwedge_{\beta \in \mathbb{B}^{S(\tilde{F})}} f_{\alpha,\beta}(b^{\beta^*}) = 0$ follows for all $\alpha \in \mathbb{B}^{S(\tilde{T})}$, implying hence $f(b^{\beta^*}) = 0$, for all $\beta^* \in \mathbb{B}^{S(\tilde{F})}$.

These two observations then show that f is indeed a robust extension of $(\tilde{T}, \tilde{F})$. $\square$

Since the existence of a robust extension always implies the full consistency, the above theorem, together with the result in [2], establishes the following corollary.

Corollary 1 *Problems* RE($\mathcal{C}$) *and* FC($\mathcal{C}$) *are mathematically equivalent, whenever the class $\mathcal{C}$ is closed under disjunction and conjunction. This applies, in particular, to all transitive classes, including $\mathcal{C}_{all}$, $\mathcal{C}_+$ and $\mathcal{C}_{regular}$.*

Let us consider next certain lattice like transitive relations. We shall say that a partial order $\succeq$ on $\mathbb{B}^n$ is *cube-lattice like* if there is a unique $\succeq$-maximum and a unique $\succeq$-minimum in any subcube of $\mathbb{B}^n$, or equivalently, if for every term t, there are unique vectors $u, v \in T(t)$ such that $u \succeq w \succeq v$ holds for all $w \in T(t)$. Let us note that all partial orders mentioned in the previous section (e.g., $\geq_b$ for $b \in \mathbb{B}^n$, $\succcurlyeq$, etc.) are cube-lattice like, and there are many others. For instance, an arbitrary permutation of the 2^n vertices of $\mathbb{B}^n$, viewed as a linear order, is cube-lattice like.

To every vector $a \in \mathbb{M}^n$ we can associate the subcube $B(a) = \{u \in \mathbb{B}^n | u \approx a\} = \{a^{\alpha} | \alpha \in \mathbb{B}^{S(a)}\}$ of $\mathbb{B}^n$ consisting of all Boolean vectors one can obtain from a by assigning binary values to its missing bits. Given a vector $a \in \mathbb{M}^n$ and a cube-lattice like partial order $\succeq$ on $\mathbb{B}^n$, let us denote by $a^+ \in B(a)$ (resp., $a^- \in B(a)$) the unique $\succeq$-maximal vector (resp., the unique $\succeq$-minimal vector) in the subcube $B(a)$. Furthermore, for any subset $S \subseteq \mathbb{M}^n$ let us introduce the notation $S^+ = \{a^+ | a \in S\}$ and $S^- = \{a^- | a \in S\}$ for the corresponding subsets of Boolean vectors.

With these notations, we can state the following generalization of [8, Lemma 2]:

Lemma 1 *If $\mathcal{C}$ is an arbitrary subfamily of a transitive class $\mathcal{C}_\succeq$ with a cube-lattice like partial order $\succeq$, then a pBmb $(\tilde{T}, \tilde{F})$ has a robust extension in $\mathcal{C}$ if and only if the pdBf $(\tilde{T}^-, \tilde{F}^+)$ has an extension in $\mathcal{C}$.*

Proof. Since $\tilde{T}^- = \tilde{T}^{\alpha^*}$ for some $\alpha^* \in \mathbb{B}^{S(\tilde{T})}$, and $\tilde{F}^+ = \tilde{F}^{\beta^*}$ for some $\beta^* \in \mathbb{B}^{S(\tilde{F})}$, it follows that any robust extension of $(\tilde{T}, \tilde{F})$ will be an extension of $(\tilde{T}^-, \tilde{F}^+)$, by the definition of a robust extension.

To see the reverse direction, let us assume that $f \in \mathcal{E}(\tilde{T}^-, \tilde{F}^+) \cap \mathcal{C}$. Since all functions in $\mathcal{C}$ are $\succeq$-monotone, and since $a^\alpha \succeq a^-$ holds, we have $f(a^\alpha) \geq f(a^-) = 1$ implied for all $a \in \tilde{T}$ and $\alpha \in \mathbb{B}^{S(\tilde{T})}$. Similarly, $b^\beta \leq b^+$ and $f(b^+) = 0$ implies $f(b^\beta) \leq f(b^+) = 0$ for all $b \in \tilde{F}$ and $\beta \in \mathbb{B}^{S(\tilde{F})}$. Hence, this function f is also a robust extension of $(\tilde{T}, \tilde{F})$ in $\mathcal{C}$. $\square$

This lemma immediately implies the following statement.

Theorem 2 *If the class $\mathcal{C}$ is a subfamily of a transitive class $\mathcal{C}_\succeq$ with a cube-lattice like partial order $\succeq$, then the problems* RE($\mathcal{C}$) *and* FC($\mathcal{C}$) *are equivalent.*

Proof. Indeed, if the pBmb $(\tilde{T}, \tilde{F})$ is fully consistent with the class $\mathcal{C}$, then the pdBf $(\tilde{T}^-, \tilde{F}^+)$ has an extension $f \in \mathcal{C} \cap \mathcal{E}(\tilde{T}^-, \tilde{F}^+)$. This f will then be a robust extension of $(\tilde{T}, \tilde{F})$ in $\mathcal{C}$ by Lemma 1. The converse direction is obvious by the definitions. $\square$

Corollary 2 *For any subfamily $\mathcal{C}$ of $\mathcal{C}_+$, $\mathcal{C}_{regular}$ and $\mathcal{C}_{\geq b}$ for any $b \in \mathbb{B}^n$, the problems* RE($\mathcal{C}$) *and* FC($\mathcal{C}$) *are equivalent.*

Let us consider DNF-classes.

Theorem 3 *If the pBmb $(\tilde{T}, \tilde{F})$ is fully consistent with respect to a DNF-class $\mathcal{C}_\mathbb{T}$, then $(\tilde{T}, \tilde{F})$ has a robust extension in $\mathcal{C}_\mathbb{T}$.*

Proof. Let us observe first that given a true vector $a \in \tilde{T}$, and an assignment $\alpha \in \mathbb{B}^{S(a)}$, each false vector $b \in \tilde{F}$ has a unique assignment $\beta = \beta(\alpha) \in \mathbb{B}^{S(b)}$ minimizing the Hamming distance between the Boolean vectors a^α and b^β.

Let us fix an arbitrary vector $a \in \tilde{T}$ and an assignment $\alpha \in \mathbb{B}^{S(\tilde{T})}$, and define $\beta^* \in \mathbb{B}^{S(\tilde{F})}$ as the unique assignment which coincides with $\beta(\alpha) \in \mathbb{B}^{S(b)}$ for all $b \in \tilde{F}$. Such an assignment obviously can be constructed by concatenating the $\beta(\alpha)$ assignments for $b \in \tilde{F}$, since the sets $S(b)$ for $b \in \tilde{F}$ are pairwise disjoint.

Since $(\tilde{T}, \tilde{F})$ is fully consistent with $\mathcal{C}_{\mathbb{T}}$, by our assumption, there exists a Boolean function $g \in \mathcal{C}_{\mathbb{T}} \cap \mathcal{E}(\tilde{T}^{\alpha}, \tilde{F}^{\beta^*})$. Since a^{α} is a true vector of such an extension, g must have a term $t_{a,\alpha} \in \mathbb{T}$ for which $t_{a,\alpha}(a^{\alpha}) = 1$.

We claim that $t_{a,\alpha}(b^{\beta}) = 0$ holds for all $b \in \tilde{F}$ and $\beta \in \mathbb{B}^{S(\tilde{F})}$. To see this, let us observe that for every vector $b \in \tilde{F}$ there must be a literal in $t_{a,\alpha}$ at which b^{β^*} and a^{α} are different, since otherwise $t_{a,\alpha}(b^{\beta^*}) = t_{a,\alpha}(a^{\alpha}) = 1$ would follow, contradicting the fact that g is an extension of the pdBf $(\tilde{F}^{\alpha^*}, \tilde{F}^{\beta^*})$. Then this literal does not correspond to any component of $S(b)$, otherwise we could switch its value in β^* to decrease the Hamming distance to a^{α}. Thus, this literal does not agree with any b^{β} for $\beta \in \mathbb{B}^{S(\tilde{F})}$, and hence the claim follows.

Then, the Boolean function defined by

$$f = \bigvee_{a \in \tilde{T}, \alpha \in \mathbb{B}^{S(\tilde{T})}} t_{a,\alpha}.$$

is a robust extension of $(\tilde{T}, \tilde{F})$ in $\mathcal{C}_{\mathbb{T}}$. Indeed, the equations $f(b^{\beta}) = 0$ hold for all $b \in \tilde{F}$ and $\beta \in \mathbb{B}^{S(\tilde{F})}$ according to the above claim. Furthermore, for a true vector $a \in \tilde{T}$ and an arbitrary assignment $\alpha \in \mathbb{B}^{S(\tilde{T})}$ we have $f(a^{\alpha}) = 1$ implied by $t_{a,\alpha}(a^{\alpha}) = 1$. $\square$

Corollary 3 *The problems* RE($\mathcal{C}$) *and* FC($\mathcal{C}$) *are equivalent for all DNF-classes, thus in particular for $\mathcal{C}_{all}$, $\mathcal{C}_{+}$, $\mathcal{C}_{k\text{-}DNF}$, $\mathcal{C}_{Horn}$, etc.*

Corollaries 1, 2 and 3, together with the complexity results of RE($\mathcal{C}$) in [11], imply the following theorem.

Theorem 4 *Problem* FC($\mathcal{C}$) *is polynomially decidable for $\mathcal{C} = \mathcal{C}_{all}$, $\mathcal{C}_{+}$, $\mathcal{C}_{regular}$. $\mathcal{C}^{+}_{k\text{-}DNF}$ (for a constant k), $\mathcal{C}_{k\text{-}DNF}$ (for $k = 1, 2$), $\mathcal{C}_{Horn}$, $\mathcal{C}^{+}_{F(S_0,F_1(S_1))}$, $\mathcal{C}^{+}_{SD}$, $\mathcal{C}^{+}_{D\text{-}minor}$ and $\mathcal{C}^{+}_{D\text{-}major}$, where $\mathcal{C}^{+}_{X}$ denotes the class of positive functions in $\mathcal{C}_{X}$, while it is co-*NP*-complete for $\mathcal{C} = \mathcal{C}_{k\text{-}DNF}$ (for a constant $k \geq 3$).*

Let us turn our attention to self-dual, dual-minor and dual-major functions. Let us recall (see e.g., [10]) that a pdBf (T, F) (with $T \cap F = \emptyset$) has a self-dual extension if and only if $T \cap \overline{T} = \emptyset$ and $F \cap \overline{F} = \emptyset$, where for a subset $X \subseteq \mathbb{B}^n$ the set $\overline{X}$ is defined by $\overline{X} = \{\overline{x} = (\overline{x}_1, \overline{x}_2, ..., \overline{x}_n) | x \in X\}$. Stating it formally:

$$\mathcal{E}(T, F) \cap \mathcal{C}_{SD} \neq \emptyset \quad \Longleftrightarrow \quad T \cap \overline{T} = \emptyset \text{ and } F \cap \overline{F} = \emptyset.$$

Similarly, necessary and sufficient conditions for the dual-minor and dual-major classes are also known, and can be stated as

$$\mathcal{E}(T, F) \cap \mathcal{C}_{D\text{-}minor} \neq \emptyset \quad \Longleftrightarrow \quad T \cap \overline{T} = \emptyset,$$

and

$$\mathcal{E}(T,F)\cap\mathcal{C}_{D\text{-}major}\neq\emptyset \iff F\cap\overline{F}=\emptyset.$$

Note that each of the right hand side conditions above is easy to check by looking at the vectors in T or in F, one-by-one.

Let us also recall (see e.g., [11]) that by defining $[\tilde{X}]=\{x^\alpha|x\in\tilde{X} \text{ and } \alpha\in\mathbb{B}^{S(x)}\}\subseteq\mathbb{B}^n$ for subsets $\tilde{X}\subseteq\mathbb{M}^n$, the robust extendibility of a pBmb $(\tilde{T},\tilde{F})$ (satisfying $[\tilde{T}]\cap[\tilde{F}]=\emptyset$) can similarly be characterized for these classes as:

$$\mathcal{E}(\tilde{T},\tilde{F})\cap\mathcal{C}_{SD}\neq\emptyset \iff [\tilde{T}]\cap\overline{[\tilde{T}]}=\emptyset \text{ and } [\tilde{F}]\cap\overline{[\tilde{F}]}=\emptyset,$$

$$\mathcal{E}(\tilde{T},\tilde{F})\cap\mathcal{C}_{D\text{-}minor}\neq\emptyset \iff [\tilde{T}]\cap\overline{[\tilde{T}]}=\emptyset,$$

and

$$\mathcal{E}(\tilde{T},\tilde{F})\cap\mathcal{C}_{D\text{-}major}\neq\emptyset \iff [\tilde{F}]\cap\overline{[\tilde{F}]}=\emptyset.$$

Here again, testing the right hand side conditions can be done efficiently, since it requires testing conditions of the type $a\not\approx\overline{b}$ for some vectors $a,b\in\tilde{T}\cup\tilde{F}$, where $\overline{x}$ for a vector $x\in\mathbb{M}^n$ is defined by $\overline{x}_j=1-x_j$ if $x_j\in\{0,1\}$, and $\overline{x}_j=*$ if $x_j=*$.

Theorem 5 *Problems* RE($\mathcal{C}$) *and* FC($\mathcal{C}$) *are equivalent for* $\mathcal{C}=\mathcal{C}_{SD}$ *(resp.,* $\mathcal{C}_{D\text{-}minor}$ *and* $\mathcal{C}_{D\text{-}major}$*), if* $(*,*,\ldots,*)\notin\tilde{T}\cup\tilde{F}$ *(resp.,* $(*,*,\ldots,*)\notin\tilde{T}$ *and* $(*,*,\ldots,*)\notin\tilde{F}$*).*

Proof. Because of all these proofs are similar, we include here the proof only for dual-minor functions. Analogous proofs can easily be obtained for the other cases.

Since the existence of a robust dual-minor extension of a pBmb $(\tilde{T},\tilde{F})$ implies, by the definitions, the full consistency of $(\tilde{T},\tilde{F})$ with the class $\mathcal{C}_{D\text{-}minor}$, we only show here that the full consistency of a pBmb $(\tilde{T},\tilde{F})$ with respect to $\mathcal{C}_{D\text{-}minor}$ implies that $\mathcal{E}(\tilde{T},\tilde{F})\cap\mathcal{C}_{D\text{-}minor}\neq\emptyset$.

Let us assume now that a given pBmb $(\tilde{T},\tilde{F})$ satisfies $(*,*,\ldots,*)\notin\tilde{T}$ and is fully consistent with $\mathcal{C}_{D\text{-}minor}$, which implies that for every $\alpha\in\mathbb{B}^{S(\tilde{T})}$ and $\beta\in\mathbb{B}^{S(\tilde{F})}$ the pdBf $(\tilde{T}^\alpha,\tilde{F}^\beta)$ has a dual-minor extension. According to the above, this means that

$$\tilde{T}^\alpha\cap\tilde{F}^\beta=\emptyset \text{ and } \tilde{T}^\alpha\cap\overline{\tilde{T}^\alpha}=\emptyset \tag{1}$$

holds for all $\alpha\in\mathbb{B}^{S(\tilde{T})}$ and $\beta\in\mathbb{B}^{S(\tilde{F})}$. Thus, $[\tilde{T}]\cap[\tilde{F}]=\emptyset$ follows by the first half of (1).

Let us observe next that the second half of (1) immediately implies that $a_1^{\alpha_1}\neq\overline{a_2^{\alpha_2}}$ holds for all $\alpha_1\in\mathbb{B}^{S(a_1)}$ and $\alpha_2\in\mathbb{B}^{S(a_2)}$, whenever $a_1\neq a_2$, $a_1,a_2\in\tilde{T}$. Since $a^\alpha\neq\overline{a^{\alpha'}}$ also holds for all $a\in\tilde{T}$ and $\alpha,\alpha'\in S(a)$, because $ON(a)\cup OFF(a)\neq\emptyset$ according to the assumption $(*,*,\cdots,*)\notin\tilde{T}$, the relation $[\tilde{T}]\cap\overline{[\tilde{T}]}=\emptyset$ is implied, and therefore $\mathcal{E}(\tilde{T},\tilde{F})\cap\mathcal{C}_{D\text{-}minor}\neq\emptyset$ follows. $\square$

Theorem 6 *Problems* RE($\mathcal{C}$) *and* FC($\mathcal{C}$) *are polynomially decidable for* $\mathcal{C} = \mathcal{C}_{SD}$, $\mathcal{C}_{D\text{-}minor}$ *and* $\mathcal{C}_{D\text{-}major}$.

Proof. Similarly to the previous theorem, we shall only consider the case of dual minor functions. Since RE($\mathcal{C}_{D\text{-}minor}$) is polynomially solvable (see [10]), we need to consider the problem FC($\mathcal{C}_{D\text{-}minor}$) only in the case of $(*, *, \cdots, *) \in \tilde{T}$. Then, by (1), we can easily see that a pBmb $(\tilde{T}, \tilde{F})$ in this case is fully consistent with $\mathcal{C}_{D\text{-}minor}$ if and only if $\tilde{F} = \emptyset$ and $|\tilde{T}| = 1$ (i.e., $\tilde{T} = \{(*, *, \cdots, *)\}$). Since the latter condition can be checked in polynomial time, this completes our proof. □

4.1 Characteristic vector representations

We shall first review the *monotone theory* of Boolean functions, as introduced in [12].

Let us define a Boolean function $\mathcal{M}_b[z]$, the so called *monotone extension* of the vector $z \in \mathbb{B}^n$ with respect to another fixed vector $b \in \mathbb{B}^n$, by defining its set of true vectors as

$$T(\mathcal{M}_b[z]) = \{v \in \mathbb{B}^n \mid v \geq_b z\}. \tag{2}$$

The *monotone extension* of a function f with respect to the vector $b \in \mathbb{B}^n$ is defined as

$$\mathcal{M}_b[f] = \bigvee_{z \in T(f)} \mathcal{M}_b[z]. \tag{3}$$

It is easy to see that for any vector $b \in \mathbb{B}^n$, the equations (2) and (3) imply that the Boolean operator $\mathcal{M}_b[f]$ is monotone over the set of Boolean functions

$$\mathcal{M}_b[f] \geq \mathcal{M}_b[g] \text{ whenever } f \geq g;$$

that $\mathcal{M}_b[f]$ is distributive over the disjunction,

$$\mathcal{M}_b[f \vee g] = \mathcal{M}_b[f] \vee \mathcal{M}_b[g];$$

and finally that the inequality

$$\mathcal{M}_b[f \wedge g] \leq \mathcal{M}_b[f] \wedge \mathcal{M}_b[g]$$

must hold for all functions f and g. Furthermore, the inequalities

$$f \leq \mathcal{M}_b[f] \text{ for all } b \in \mathbb{B}^n \tag{5a}$$

follow directly from (3), since $z \in T(\mathcal{M}_b[z])$ for all $z \in \mathbb{B}^n$ by (2). Also the property

$$b \in F(\mathcal{M}_b[f]) \text{ for all } b \in F(f), \tag{5b}$$

follows from (3), since $b \notin \mathcal{M}_b[z]$ holds for all $z \neq b$ by (2).

From properties (5a) and (5b), we obtain the inequalities

$$\bigwedge_{b \in F(f)} \mathcal{M}_b[f] \leq f \leq \bigwedge_{b \in \mathbb{B}^n} \mathcal{M}_b[f] \leq \bigwedge_{b \in F(f)} \mathcal{M}_b[f], \tag{6}$$

which then must all be equations. Thus, an arbitrary function f can be characterized as

$$f = \bigwedge_{b \in \mathbb{B}^n} \mathcal{M}_b[f] = \bigwedge_{b \in F(f)} \mathcal{M}_b[f]. \tag{7}$$

Let us note that on the right hand side of (7), one may not need all the vectors $b \in F(f)$ to represent f, that is, $f = \bigwedge_{b \in B} \mathcal{M}_b[f]$ may hold for some subset $B \subseteq F(f)$. This leads to the following definition (see e.g., [12]). A set of vectors $B \subseteq \mathbb{B}^n$ is called a *basis for a function* f, if $f = \bigwedge_{b \in B} \mathcal{M}_b[f]$ holds. Furthermore, a subset $B \subseteq \mathbb{B}^n$ is called a *basis for a class of functions* $\mathcal{C}$, if it is a basis for all the functions in $\mathcal{C}$. Clearly, $\mathbb{B}^n$ and $F(f)$ are bases for any function f, according to (7), and $\mathbb{B}^n$ is a basis for any class $\mathcal{C}$ of Boolean functions.

Bases are known for various classes of Boolean functions. For instance, it is known that for the class $\mathcal{C}_{CNF\text{-}Horn} = \{\overline{f} \,|\, f \in \mathcal{C}_{Horn}\}$ of Horn CNFs (functions that can be represented by a CNF in which every clause contains at most one positive literal), the set

$$B_{CNF\text{-}Horn} = \{b \,|\, \sum_{j=1}^{n} b_j \geq n-1\} \tag{8}$$

is a basis (see e.g., [15]).

Let us now consider a class $\mathcal{C}$ of functions, and assume that $B_{\mathcal{C}}$ is a non-empty basis for $\mathcal{C}$ such that $f \neq \bigwedge_{b \in B_{\mathcal{C}}} \mathcal{M}_b[f]$ holds for any function $f \notin \mathcal{C}$. In other words, assume that

$$f \in \mathcal{C} \;\text{ if and only if }\; f = \bigwedge_{b \in B_{\mathcal{C}}} \mathcal{M}_b[f]. \tag{9}$$

Note that not all classes of Boolean functions have such a basis. For example, any class $\mathcal{C}$ not containing the constant 1 function does not satisfy (9), since $\mathbf{1} = \bigwedge_{b \in B_{\mathcal{C}}} \mathcal{M}_b[\mathbf{1}]$ holds for every non-empty basis $B_{\mathcal{C}}$. In other words, $\mathbf{1} \in \mathcal{C}$ is a necessary condition for a class $\mathcal{C}$ to have a basis B satisfying (9).

On the other hand, condition (9) is known to be satisfied for several CNF-classes. For example, for the families of clauses $\mathbb{C}_{k1} = \{$*clauses containing at most k positive literals*$\}$, and $\mathbb{C}_{k0} = \{$*clauses containing at most k negative literals*$\}$, the corresponding CNF-classes $\mathcal{C}_{\mathbb{C}_{k0}}$ and $\mathcal{C}_{\mathbb{C}_{k1}}$ do have bases satisfying condition (9) (see e.g., [15].) These classes, the so called *k-quasi Horn* and *k-quasi reverse Horn* CNFs, contain, as special cases corresponding to $k = 1$, the better known families of Horn and reverse Horn CNFs.

Theorem 7 *If $\mathcal{C}$ is a class of functions which has a non-empty basis $B_{\mathcal{C}}$ satisfying* (9)*, then the problems* RE($\mathcal{C}$) *and* FC($\mathcal{C}$) *are equivalent.*

Proof. Since the existence of a robust extension implies full consistency by the definitions, we shall prove only the non-trivial implication.

Let us assume that $\mathcal{C}$ is a class which has a non-empty basis $B_{\mathcal{C}}$ satisfying (9), and let us consider a pBmb $(\tilde{T}, \tilde{F})$ which has no robust extension in the class $\mathcal{C}$. We shall show that in this case $(\tilde{T}, \tilde{F})$ is not fully consistent with $\mathcal{C}$, either. This then will imply the theorem.

For this end, let us note that the pBmb $(\tilde{T}, \emptyset)$ has a robust extension in $\mathcal{C}$, since the constant **1** function is an obvious robust extension of it, which belongs to $\mathcal{C}$, as we observed above. We claim that the pBmb $(\tilde{T}, \emptyset)$ has a unique minimal robust extension $f \in \mathcal{C}$, defined by

$$f = \bigwedge_{b \in B_{\mathcal{C}}} \Big(\bigvee_{a \in \tilde{T},\, \alpha \in \mathbb{B}^{S(a)}} \mathcal{M}_b[a^\alpha] \Big). \tag{10}$$

It is clear that f is a robust extension of $(\tilde{T}, \emptyset)$. Moreover, for any robust extension $g \in \mathcal{C}$ of $(\tilde{T}, \emptyset)$, $g \geq \mathcal{M}_b[a^\alpha]$ must hold for all $a \in \tilde{T}$, $\alpha \in \mathbb{B}^{S(a)}$ and $b \in B_{\mathcal{C}}$, because of (3) and the definitions of a robust extension and of a basis for the class $\mathcal{C}$. Thus, $g \geq f$ follows. Let us finally show $f \in \mathcal{C}$, which completes our claim.

$$\mathcal{M}_b[f] = \bigvee_{a \in \tilde{T},\, \alpha \in \mathbb{B}^{S(a)}} \mathcal{M}_b[a^\alpha] \tag{11}$$

follows by (3) and (10) for any $b \in B_{\mathcal{C}}$, and thus we have

$$f = \bigwedge_{b \in B_{\mathcal{C}}} \mathcal{M}_b[f]$$

by (10). Assumption (9) therefore implies that $f \in \mathcal{C}$.

For an arbitrary vector $b \in \mathbb{B}^n$, let us denote by $\alpha_b \in \mathbb{B}^{S(\tilde{T})}$ the unique assignment for which $a^{\alpha_b} \leq_b a^\alpha$ holds for all $a \in \tilde{T}$ and $\alpha \in \mathbb{B}^{S(\tilde{T})}$. Then we have $\mathcal{M}_b[f] = \bigvee_{a \in \tilde{T}} \mathcal{M}_b[a^{\alpha_b}]$, implied by (11) and the selection of the assignments α_b for $b \in B_{\mathcal{C}}$. Thus, we obtain

$$f = \bigwedge_{b \in B_{\mathcal{C}}} \big(\bigvee_{a \in \tilde{T}} \mathcal{M}_b[a^{\alpha_b}] \big) \tag{12}$$

implied by the fact that $B_{\mathcal{C}}$ is a basis for $f \in \mathcal{C}$.

Since, by our assumption, this function f cannot be a robust extension of $(\tilde{T}, \tilde{F})$, there must exist $d \in \tilde{F}$ and $\delta \in \mathbb{B}^{S(d)}$ such that $f(d^\delta) = 1$. Thus, by (12), for every $b \in B_{\mathcal{C}}$, there exists a vector $a_b \in \tilde{T}$ such that $\mathcal{M}_b[a_b^{\alpha_b}](d^\delta) = 1$, i.e.,

$$d^\delta \geq_b a_b^{\alpha_b}. \tag{13}$$

Let us then consider arbitrary assignments $\alpha^* \in \mathbb{B}^{S(\tilde{T})}$ and $\beta^* \in \mathbb{B}^{S(\tilde{F})}$ such that β^* agrees with δ on $S(d)$, and $\alpha^*(v, j) = d_j^\delta$ whenever $(v, j) \in S(a_b)$ for all $b \in B_{\mathcal{C}}$. Then, by (13) the inequalities

$$d^{\beta^*} \geq_b a_b^{\alpha^*} \geq_b a_b^{\alpha_b} \tag{14}$$

hold for all $b \in B_{\mathcal{C}}$.

With this definition, we claim that the pdBf $(\tilde{T}^{\alpha^*}, \tilde{F}^{\beta^*})$ does not have an extension in $\mathcal{C}$. This then implies that $(\tilde{T}, \tilde{F})$ is not fully consistent with respect to the class $\mathcal{C}$.

To see this claim, let us observe that an arbitrary function $g \in \mathcal{C}$, for which $T(g) \supseteq \tilde{T}^{\alpha^*}$ holds, satisfies the relations

$$\begin{aligned} g = \bigwedge_{b \in B_{\mathcal{C}}} \mathcal{M}_b[g] &= \bigwedge_{b \in B_{\mathcal{C}}} \Big(\bigvee_{z \in T(g)} \mathcal{M}_b[z] \Big) \\ &\geq \bigwedge_{b \in B_{\mathcal{C}}} \Big(\bigvee_{a \in \tilde{T}^{\alpha^*}} \mathcal{M}_b[a] \Big) \geq \bigwedge_{b \in B_{\mathcal{C}}} \mathcal{M}_b[a_b^{\alpha^*}] \geq \bigwedge_{b \in B_{\mathcal{C}}} \mathcal{M}_b[a_b^{\alpha_b}] \end{aligned}$$

and thus $g(d^{\beta^*}) = 1$ follows by (14). This proves that g is indeed not an extension of $(\tilde{T}^{\alpha^*}, \tilde{F}^{\beta^*})$. □

5 Very robust extensions

Very robust extensions play a computationally important role, since when they exists, they usually can efficiently be constructed. For instance, we shall show below that for most DNF-classes $\mathcal{C}$, if RE($\mathcal{C}$) can be solved in polynomial time, then a very robust extension can also be provided at the same time.

Let us recall that a term

$$t(x_1, \ldots, x_n) = \bigwedge_{j \in P} x_j \bigwedge_{j \in N} \overline{x}_j$$

is called a robust term with respect to $a \in \tilde{T}$ for a pBmb $(\tilde{T}, \tilde{F})$, if $t(a^\alpha) = 1$ for all $\alpha \in \mathbb{B}^{S(a)}$, and $t(b^\beta) = 0$ for all $b \in \tilde{F}$ and $\beta \in \mathbb{B}^{S(b)}$. In other words, if and only if

$$P \subseteq ON(a) \text{ and } N \subseteq OFF(a), \text{ and} \tag{15}$$

$$P \cap OFF(b) \neq \emptyset \text{ or } N \cap ON(b) \neq \emptyset \text{ for every vector } b \in \tilde{F}. \tag{16}$$

Since both of these conditions are independent of the assignments to missing bits of $(\tilde{T}, \tilde{F})$, checking these conditions is quite straightforward. Therefore, verifying that a given DNF is a very robust extension of a pBmb $(\tilde{T}, \tilde{F})$ can be done in linear time in the size of $(\tilde{T}, \tilde{F})$. It is also clear from the definition that in a very robust extension one never needs more than $|\tilde{T}|$ terms.

Furthermore, looking at conditions (15) and (16), it is easy to see that finding a robust term for a given pBmb $(\tilde{T}, \tilde{F})$ and vector $a \in \tilde{T}$ reduces to a feasibility question in an associated setcovering problem, and hence it is computationally tractable in most cases. The above immediately imply for instance the following statement.

Corollary 4 *Problem* VR($\mathcal{C}$) *can be solved in polynomial time for* $\mathcal{C} = \mathcal{C}_{all}$*, for* $\mathcal{C}_{\geq_b}$ *for with* $b \in \mathbb{B}^n$ *(thus in particular for* $\mathcal{C} = \mathcal{C}_+$*), for* $\mathcal{C} = \mathcal{C}_{Horn}$ *(and for all related classes, such as k-quasi Horn and k-quasi reverse Horn for any fixed k), and for* $\mathcal{C} = \mathcal{C}_{k\text{-}DNF}$ *with k fixed.*

To discuss properties of very robust extensions, we shall recall the consensus method and some of its properties (see e.g., [17, 18]). Given two terms $t = \bigwedge_{j\in P} x_j \bigwedge_{j\in N} \overline{x}_j$ and $t' = \bigwedge_{j\in P'} x_j \bigwedge_{j\in N'} \overline{x}_j$, we say that they are in *conflict* at variable x_j if $j \in (P\cap N')\cup(N\cap P')$ (i.e., if x_j appears in one and $\overline{x}_j$ appears in the other). If t and t' are in conflict at exactly one of the variables, then their *consensus* is a term $t'' = [t, t']$ defined by

$$t'' = \bigwedge_{j\in(P\setminus N')\cup(P'\setminus N)} x_j \bigwedge_{j\in(N\setminus P')\cup(N'\setminus P)} \overline{x}_j.$$

In other words, the consensus of t and t' is the conjunction of all the literals appearing in these terms, except the two, corresponding to the conflicting variable. It is easy to see that the inequality $t'' \leq t \vee t'$ holds. This implies, in particular that if t and t' are implicants of the Boolean function f, then their consensus $t'' = [t, t']$ (when exists) is also an implicant of f. The *consensus method* is the algorithm, in which consensuses of implicants of a given DNF of f are formed as long as new implicants are generated. It is well-known (see e.g., [17]) that this method is complete in the sense that all prime implicants of f will be obtained in this way, starting from any DNF representation of f. Of course, all these notions and results can straightforwardly be translated for CNF representations using De Morgan's laws. The corresponding operation between clauses is known as *resolution*.

Returning to robust terms, we are now ready to prove the following statement.

Lemma 2 *If* f *is a robust extension of the pBmb* $(\tilde{T}, \tilde{F})$*, then for every vector* $a \in \tilde{T}$*,* f *has a prime implicant* $t_a \leq f$*, which is a robust term with respect to* a*.*

Proof. Let us consider an (arbitrary) DNF representation of f:

$$f = \bigvee_{i=1}^{m} t_i, \tag{17}$$

where

$$t_i = \bigwedge_{j\in P_i} x_j \bigwedge_{j\in N_i} \overline{x}_j. \tag{18}$$

Given a vector $a \in \tilde{T}$, we substitute $x_j = 1$ for variables with $j \in ON(a)$, and $x_j = 0$ for variables with $j \in OFF(a)$ into (17). Let $I \subseteq \{1, 2, ..., m\}$ denote the set of indices of those terms of (17) which do not vanish after this substitution. For terms t_i for $i \in I$, we have

$$P_i \cap OFF(a) = \emptyset \text{ and } N_i \cap ON(a) = \emptyset. \tag{19}$$

Let us denote the resulting DNF by $f' = \bigvee_{i \in I} t'_i$, where

$$t'_i = \bigwedge_{j \in P_i \setminus ON(a)} x_j \bigwedge_{j \in N_i \setminus OFF(a)} \overline{x}_j. \tag{20}$$

Since $1 = f(a^\alpha) = f'(a^\alpha)$ holds for all $\alpha \in \mathbb{B}^{S(a)}$, it follows that f' is the constant $\mathbf{1}$ function, and thus the only prime implicant $\mathbf{1}$ of f' can be obtained by a chain of consensuses, starting with the terms of f'. Let us note that if the terms t'_i and t'_k for some $i \neq k$, $i, k \in I$ have a consensus, then so do the terms t_i and t_k. Furthermore, the variables x_j, $j \in ON(a)$, appear only positively, and the variables x_j, $j \in OFF(a)$, appear only negatively in the resulting consensus $[t_i, t_k]$. Applying this observation recursively, we can repeat the same chain of consensuses which produced $\mathbf{1}$ from t'_i, $i \in I$, with the corresponding terms t_i, $i \in I$, yielding an implicant t'_a of f.

Clearly, t'_a involves only literals x_j for some $j \in ON(a)$ and $\overline{x}_j$ for some $j \in OFF(a)$, and thus t'_a satisfies condition (15). Therefore, by deleting some literals from t'_a if needed, we can obtain a prime implicant t_a of f still satisfying (15).

Let us note finally that any (prime) implicant of f must satisfy conditions (16) simply because f is a robust extension of $(\tilde{T}, \tilde{F})$. $\square$

This lemma implies the equivalence of RE($\mathcal{C}$) and VR($\mathcal{C}$) for minor closed classes. Let us recall that a class $\mathcal{C}$ is minor closed, if $f \in \mathcal{C}$ and $g \sqsubseteq f$ imply $g \in \mathcal{C}$.

Theorem 8 *If $\mathcal{C}$ is a minor closed class, then problems* VR($\mathcal{C}$) *and* RE($\mathcal{C}$) *are equivalent.*

Proof. Since a very robust extension is also a robust extension, let us prove only the non-trivial direction of the stated equivalence.

Let us assume that $(\tilde{T}, \tilde{F})$ is a pBmb which has a robust extension $f \in \mathcal{E}(\tilde{T}, \tilde{F}) \cap \mathcal{C}$. According to Lemma 2, for every $a \in \tilde{T}$, f has a prime implicant $t_a \leq f$ which is a robust term of $(\tilde{T}, \tilde{F})$. We claim that the Boolean function

$$g = \bigvee_{a \in \tilde{T}} t_a \tag{21}$$

is a very robust extension of $(\tilde{T}, \tilde{F})$ in the class $\mathcal{C}$.

Clearly, $g \leq f$ is a minor of f by its definition, hence $g \in \mathcal{C}$ is implied by the facts that $\mathcal{C}$ is minor closed and $f \in \mathcal{C}$. Since $f \in \mathcal{E}(\tilde{T}, \tilde{F})$, the inequality $g \leq f$ also implies that $g(b^\beta) \leq f(b^\beta) = 0$ for all $b \in \tilde{F}$ and $\beta \in \mathbb{B}^{S(b)}$. Also, since g contains a robust term t_a for every $a \in \tilde{T}$, it follows that $g(a^\alpha) = 1$ holds for all $a \in \tilde{T}$ and $\alpha \in \mathbb{B}^{S(a)}$. This implies that g is a robust extension of the pBmb $(\tilde{T}, \tilde{F})$. Finally, since g contains only robust terms, it is a very robust extension of $(\tilde{T}, \tilde{F})$. $\square$

Corollary 5 *For minor closed classes, such as $\mathcal{C} = \mathcal{C}_{r\text{-}Horn}$, $\mathcal{C}_{unate}$, $\mathcal{C}_{q\text{-}Horn}$ and $\mathcal{C}_{D\text{-}minor}$, problems* RE($\mathcal{C}$) *and* VR($\mathcal{C}$) *are equivalent.*

This corollary and the complexity results of RE($\mathcal{C}$) in [11] imply the following theorem.

Theorem 9 *Problem* VR($\mathcal{C}$) *is polynomially solvable for* $\mathcal{C} = \mathcal{C}_{D\text{-}minor}$*, while it is* NP*-hard for* $\mathcal{C} = \mathcal{C}_{r\text{-}Horn}$ *and* $\mathcal{C}_{unate}$*.*

Let $\mathbb{T}$ be said to be *closed under consensus* if $[t, t'] \in \mathbb{T}$ holds for every pair of terms $t, t' \in \mathbb{T}$ such that they are in conflict at exactly one of the variables (their consensus $[t, t']$ exists). Moreover, we say that DNF-classes $\mathcal{C}_{\mathbb{T}}$ is *consensus closed* if $\mathbb{T}$ is closed under consensus.

Since consensus closed classes are also minor closed, and since problems RE($\mathcal{C}$) and FC($\mathcal{C}$) are equivalent for all DNF-classes by Corollary 3, the above theorem has the following immediate corollary.

Corollary 6 *For the DNF-classes* $\mathcal{C} = \mathcal{C}_{all}$*,* $\mathcal{C}_{Horn}$*,* $\mathcal{C}_{+}$*, and* $\mathcal{C}_{2\text{-}DNF}$*, problems* VR($\mathcal{C}$)*,* RE($\mathcal{C}$) *and* FC($\mathcal{C}$) *are all equivalent.*

We can proceed even further. For a pBmb $(\tilde{T}, \tilde{F})$, a robust extension f is called *minimal* with respect to $\mathcal{C}$ if there is no robust extension g in $\mathcal{C}$ such that $g < f$.

Corollary 7 *For a DNF-class* $\mathcal{C}_{\mathbb{T}}$ *for which* $\mathbb{T}$ *is closed under consensus, any minimal robust extension of pBmb* $(\tilde{T}, \tilde{F})$ *with respect to* $\mathcal{C}_{\mathbb{T}}$ *is very robust and has size* $O(n|\tilde{T}|)$*. In particular, this is true for the classes of* $\mathcal{C}_{all}$*,* $\mathcal{C}_{+}$*,* $\mathcal{C}_{Horn}$ *and* $\mathcal{C}_{2\text{-}DNF}$*.*

Proof. For a pBmb $(\tilde{T}, \tilde{F})$, let f be a minimal robust extension (with respect to $\mathcal{C}_{\mathbb{T}}$). As in the proof of Theorem 8, let us consider the function g in (21) corresponding to f. This g is a very robust extension of $(\tilde{T}, \tilde{F})$, and it is a minor of f; hence, $g \leq f$. Since f is minimal, $g = f$ holds. Thus f is a very robust extension and can be represented by a DNF whose size is $O(n|\tilde{T}|)$. $\square$

Let us remark here that Corollary 7 does not say the complexity of computing a minimal robust extension of a pBmb $(\tilde{T}, \tilde{F})$. In fact, the complexity depends on the class $\mathcal{C}$ which you consider. It is easy to see that, for classes $\mathcal{C} = \mathcal{C}_{all}$ and $\mathcal{C}_{+}$, one can compute in polynomial time a minimal robust extension in $\mathcal{C}$; while the problem is co-NP-hard for classes $\mathcal{C} = \mathcal{C}_{Horn}$ and $\mathcal{C}_{2\text{-}DNF}$.

6 Cases of non-equivalence between FC($\mathcal{C}$), RE($\mathcal{C}$), and VR($\mathcal{C}$)

In this section we shall show that problems FC($\mathcal{C}$), RE($\mathcal{C}$) and VR($\mathcal{C}$) are not always equivalent, despite the many quite general equivalences we have shown in the previous sections. We first give several classes $\mathcal{C}$ for which FC($\mathcal{C}$) and RE($\mathcal{C}$) are not equivalent, followed by a class for which RE($\mathcal{C}$) and VR($\mathcal{C}$) are not equivalent.

First, one might think that Theorem 1 could be generalized to prove the equivalence of RE($\mathcal{C}$) and FC($\mathcal{C}$) for classes closed under conjunction (but not necessarily closed under disjunction). This, however, is not the case, as the following simple example shows. Let us consider the class $\mathcal{C}^*$ consisting of functions f for which $f(v)f(w) = 0$ holds for all pairs of vectors $v, w \in \mathbb{B}^n$ which are at Hamming distance 1. Clearly, this class $\mathcal{C}^*$ is closed under conjunction. Let us now consider the pBmb $(\tilde{T}, \tilde{F})$ given by $\tilde{T} = \{(1, *)\}$ and $\tilde{F} = \emptyset$. Since the equation $f(1, 0) = f(1, 1) = 1$ must hold for any robust extension f of $(\tilde{T}, \tilde{F})$, f does not belong to $\mathcal{C}^*$. Therefore, $(\tilde{T}, \tilde{F})$ has no robust extension in $\mathcal{C}^*$. However, $f = x_1 x_2 \in \mathcal{C}^*$ is an extension of the pdBf $(\{(1, 1)\}, \emptyset)$ and $g = x_1 \bar{x}_2 \in \mathcal{C}^*$ is an extension of the pdBf $(\{(1, 0)\}, \emptyset)$. These imply that $(\tilde{T}, \tilde{F})$ is fully consistent with $\mathcal{C}^*$.

Let us demonstrate next that, for the important class $\mathcal{C} = \mathcal{C}_{unate}$, problems RE($\mathcal{C}$) and FC($\mathcal{C}$) are not equivalent. For this, consider the pBmb $(\tilde{T}, \tilde{F})$ defined in the table below:

$\tilde{T}$	$*$	0	0	0
	1	1	1	1
$\tilde{F}$	1	1	0	0
	1	0	1	0
	0	0	0	1

We claim that this pBmb does not have a robust unate extension. This is, because RE($\mathcal{C}_{unate}$) is equivalent to VR($\mathcal{C}_{unate}$), by Corollary 5, and thus it is enough to show that $(\tilde{T}, \tilde{F})$ does not have a very robust extension. For this, let us observe that the only robust term for $a_{(1)} = (*, 0, 0, 0)$ is $\overline{x}_2 \overline{x}_3 \overline{x}_4$, while all robust terms for $a_{(2)} = (1, 1, 1, 1)$ involve either x_2, or x_3, or x_4. Therefore, any very robust extension of $(\tilde{T}, \tilde{F})$ must include conflicting terms, and hence it is not unate.

On the other hand, $(\tilde{T}, \tilde{F})$ is fully consistent with respect to $\mathcal{C}_{unate}$. This is shown by the following two tables, representing the two pdBfs obtainable from $(\tilde{T}, \tilde{F})$ by considering two assignments in $\mathbb{B}^{S(\tilde{T}, \tilde{F})}$. Underneath the tables we give the corresponding unate extensions, showing that $\mathcal{E}(\tilde{T}^\alpha, \tilde{F}^\alpha) \cap \mathcal{C}_{unate} \neq \emptyset$ for all $\alpha \in \mathbb{B}^{S(\tilde{T}, \tilde{F})}$.

T^0	0	0	0	0
	1	1	1	1
F	1	1	0	0
	1	0	1	0
	0	0	0	1
$\overline{x}_1 \overline{x}_4 \vee x_2 x_3$				

T^1	1	0	0	0
	1	1	1	1
F	1	1	0	0
	1	0	1	0
	0	0	0	1
$x_1 \overline{x}_2 \overline{x}_3 \vee x_1 x_4$				

We also show that problems RE($\mathcal{C}_{r\text{-}Horn}$) and FC($\mathcal{C}_{r\text{-}Horn}$) are not equivalent, either, by the example given in the table below.

$\tilde{T}$	0 0 0 $*$
	1 1 1 1
$\tilde{F}$	0 1 1 1
	1 0 1 1
	1 1 0 1
	1 0 0 0
	0 0 1 1
	0 1 0 1

We claim first that $(\tilde{T}, \tilde{F})$ does not have a robust renamable Horn extension. For this, it is enough to show that $(\tilde{T}, \tilde{F})$ does not have a very robust renamable Horn extension, because problems RE($\mathcal{C}_{r\text{-}Horn}$) and VR($\mathcal{C}_{r\text{-}Horn}$) are equivalent by Corollary 5. To see that $(\tilde{T}, \tilde{F})$ does not have a very robust renamable Horn extension, let us observe that for each of the true vectors in $\tilde{T}$ there exists only one minimal robust term; namely $\overline{x}_1\overline{x}_2\overline{x}_3$ for vector $a_{(1)} = (0, 0, 0, *)$, and $x_1x_2x_3$ for $a_{(2)} = (1, 1, 1, 1)$. Thus, in any very robust extension of $(\tilde{T}, \tilde{F})$ there will be a pair of terms which have three conflicting variables, and hence it cannot be renamed to Horn.

To see that $(\tilde{T}, \tilde{F})$ is fully consistent with $\mathcal{C}_{r\text{-}Horn}$, we give in the tables below the two possible pdBfs, as well as the corresponding renamable Horn extensions.

T^1	0 0 0 1
	1 1 1 1
F	0 1 1 1
	1 0 1 1
	1 1 0 1
	1 0 0 0
	0 0 1 1
	0 1 0 1
$x_1x_2x_3 \vee \overline{x}_2\overline{x}_3x_4$	

T^0	0 0 0 0
	1 1 1 1
F	0 1 1 1
	1 0 1 1
	1 1 0 1
	1 0 0 0
	0 0 1 1
	0 1 0 1
$x_1x_2x_3 \vee \overline{x}_1\overline{x}_2\overline{x}_4$	

In the first case renaming x_2, while in the second case renaming x_2 and x_4 yield Horn DNFs.

Let us consider next the class of decomposable functions, $\mathcal{C}_{F(S_0,F_1(S_1))}$. Recall that problem RE($\mathcal{C}_{F(S_0,F_1(S_1))}$) can be solved with the help of the following associated graph [10, 11]. Given a pBmb $(\tilde{T}, \tilde{F})$ and subsets $S_0, S_1 \subseteq \{1, 2, ..., n\}$ (with $S_0 \cap S_1 = \emptyset$), let us associate a graph $G = (W, E_{blue} \cup E_{yellow})$ defined as follows:

$$W = \{a[S_1] \mid a \in T, \exists b \in F\ a[S_0] \approx b[S_0]\} \cup \{b[S_1] \mid b \in F, \exists a \in T\ b[S_0] \approx a[S_0]\}$$

$$E_{blue} = \{(a[S_1], b[S_1]) \mid a \in T,\ b \in F,\ a[S_0] \approx b[S_0]\},\ \text{and}$$

$$E_{yellow} = \{(u,v) \mid u,v \in W,\ u \approx v\}.$$

Then the pBmb $(\tilde{T},\tilde{F})$ has a robust extension in $\mathcal{C}_{F(S_0,F_1(S_1))}$ if and only if the graph obtained from G by contracting all the edges in E_{yellow}, is bipartite. Let us also note that for the special case of a pdBf (T,F) we have $E_{yellow} = \emptyset$, and hence a pdBf has a $g(S_0,h(S_1))$-decomposable extension if and only if the associated graph $G = (W, E_{blue})$ is bipartite.

Let us then consider the pBmb $(\tilde{T},\tilde{F})$ in Table 1, with $S_0 = \{1,2\}$, $S_1 = \{3,4\}$, $\tilde{T} = \{a_{(1)}, a_{(2)}\}$ and $\tilde{F} = \{b_{(1)}, b_{(2)}\}$. The corresponding graph G is shown in Figure 1.

	S_0		S_1	
$a_{(1)}$	$*$	0	1	1
$a_{(2)}$	0	1	$*$	0
$b_{(1)}$	1	0	0	1
$b_{(2)}$	$*$	1	$*$	1

Table 1: A pBmb $(\tilde{T},\tilde{F})$ with $\tilde{T} = \{a_{(1)}, a_{(2)}\}$ and $\tilde{F} = \{b_{(1)}, b_{(2)}\}$, which has no robust extension in $\mathcal{C}_{F(S_0,F_1(S_1))}$.

It has two blue edges and two yellow edges, and after the contraction of the yellow edges $(a_{(1)}[S_1], b_{(2)}[S_1])$ and $(b_{(1)}[S_1], b_{(2)}[S_1])$, the blue edge $(a_{(1)}[S_1], b_{(1)}[S_1])$ will become a loop. Thus the contracted graph is not bipartite, and hence $(\tilde{T},\tilde{F})$ has no robust extension in $\mathcal{C}_{F(S_0,F_1(S_1))}$.

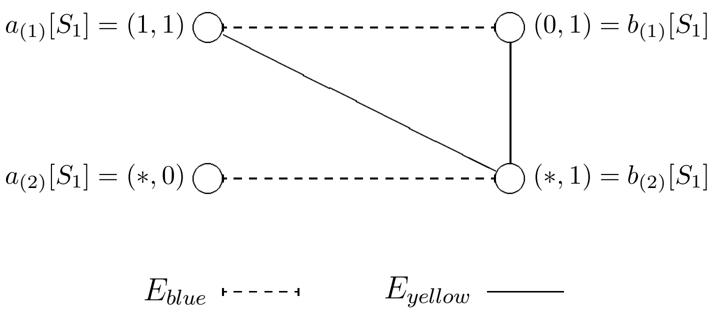

Figure 1: The graph corresponding to the example of Table 1.

On the other hand, we can observe that the two potential identities $a_{(1)}[S_1] \approx b_{(2)}[S_1]$ and $b_{(1)}[S_1] \approx b_{(2)}[S_1]$, yielding the two yelow edges, cannot be realized by the same assignment. This implies that for every assignment $\alpha \in \mathbb{B}^{S(\tilde{T},\tilde{F})}$ either $a_{(1)}[S_1]$ coincides with $b_{(2)}[S_1]$ or $b_{(1)}[S_1]$ coincides with $b_{(2)}[S_1]$, but never both. Thus the graph obtained by contracting yellow edges in the graph associated to $(\tilde{T}^{\alpha}, \tilde{F}^{\beta})$ for every $\alpha \in \mathbb{B}^{S(\tilde{T})}$ and $\beta \in \mathbb{B}^{S(\tilde{F})}$ is a simple path of two edges; i.e., it is bipartite, proving that $\mathcal{E}(\tilde{T}^\alpha, \tilde{F}^\beta) \cap \mathcal{C}_{F(S_0,F_1(S_1))} \neq \emptyset$. This shows that $(\tilde{T}, \tilde{F})$ is fully consistent with the class $\mathcal{C}_{F(S_0,F_1(S_1))}$.

Our final class for which RE($\mathcal{C}$) and FC($\mathcal{C}$) are not equivalent is the class of threshold Boolean functions $\mathcal{C}_{TH}$. Let us recall first that a pdBf (T, F) has a threshold extension, if and only if there exist $n+1$ real numbers $w_1, w_2, \ldots, w_n$ and w_0 such that:

$$\sum_{j=1}^{n} w_j a_j \geq w_0 \text{ for all } a \in T, \text{ and} \tag{22}$$

$$\sum_{j=1}^{n} w_j b_j < w_0 \text{ for all } b \in F. \tag{23}$$

It is well known that this condition is also equivalent to the disjointness of their respective convex hulls,

$$conv(T) \cap conv(F) = \emptyset, \tag{24}$$

where $conv(X)$ denotes the convex hull of the set X in the n-dimensional real space. It is also easy to see by the definitions that a pBmb $(\tilde{T}, \tilde{F})$ has a robust threshold extension if and only if

$$conv(\tilde{T}) \cap conv(\tilde{F}) = \emptyset, \tag{25}$$

where $conv(\tilde{X}) = conv(\cup_{\alpha \in \mathbb{B}^{S(\tilde{X})}} \tilde{X}^\alpha)$ for a subset $\tilde{X} \subseteq \mathbb{M}^n$.

Let us now consider the pBmb $(\tilde{T}, \tilde{F})$ defined by

$\tilde{T}$	$*$	1	1	1
	0	0	0	0
$\tilde{F}$	1	1	1	0
	0	1	0	1
	0	0	1	1

The only one missing bit of $(\tilde{T}, \tilde{F})$ has two possible interpretations, yielding $T^1 = \{(1,1,1,1), (0,0,0,0)\}$ and $T^0 = \{0,1,1,1), (0,0,0,0)\}$. It is easy to verify that the threshold Boolean function defined by $5x_1 - 3x_2 - 3x_3 + 2x_4 \geq 0$ is an extension of the pdBf $(T^1, \tilde{F})$, and that $-5x_1 + 2x_2 + 2x_3 - 3x_4 \geq 0$ defines a threshold extension of $(T^0, \tilde{F})$. Hence, the pBmb $(\tilde{T}, \tilde{F})$ is fully consistent with $\mathcal{C}_{TH}$. However, $(\tilde{T}, \tilde{F})$ has no robust threshold extension by (25), since the fractional vector $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ belongs to the convex hulls of both $\tilde{T}$ and $\tilde{F}$.

In concluding this section, we demonstrate with the example $(\tilde{T}, \tilde{F})$ defined in the table below that RE($\mathcal{C}_{3\text{-}DNF}$) and VR($\mathcal{C}_{3\text{-}DNF}$) are not equivalent.

$\tilde{T}$	1	1	$*$	1	1
$\tilde{F}$	1	1	0	1	0
	1	1	0	0	1
	1	0	1	1	1
	0	1	1	1	1

This pBmb does not have a very robust 3-DNF extension, because the only robust term for $a = (1, 1, *, 1, 1)$ is the quartic term $t = x_1x_2x_4x_5$.

On the other hand, the cubic DNF

$$\phi = x_1x_2x_3 \vee \overline{x}_3x_4x_5$$

is a robust extension of $(\tilde{T}, \tilde{F})$, hence $\mathcal{E}(\tilde{T}, \tilde{F}) \cap \mathcal{C}_{3\text{-}DNF} \neq \emptyset$.

Let us remark that VR($\mathcal{C}_{3\text{-}DNF}$) is always polynomially decidable, while RE($\mathcal{C}_{3\text{-}DNF}$) is co-NP-complete (see e.g., [8]).

7 Complexity of FC($\mathcal{C}_{TH}$)

We have already seen in Section 6 that problems RE($\mathcal{C}_{TH}$) and FC($\mathcal{C}_{TH}$) are not equivalent. It it known that RE($\mathcal{C}_{TH}$) is polynomially solvable [11]. However, problem FC($\mathcal{C}_{TH}$) has a different complexity as shown in the next theorem.

Theorem 10 *Problem* FC($\mathcal{C}_{TH}$) *is co-NP-complete, even if* $|S(a)| \leq 1$ *holds for all* $a \in \tilde{T} \cup \tilde{F}$.

Proof. First we show that FC($\mathcal{C}_{TH}$) belongs to co-NP. By (24), a pBmb $(\tilde{T}, \tilde{F})$ is not fully consistent with the class $\mathcal{C}_{TH}$ if and only if there exists an assignment $\alpha \in \mathbb{B}^S$ such that $conv(\tilde{T}^\alpha) \cap conv(\tilde{F}^\alpha) \neq \emptyset$. Therefore, FC($\mathcal{C}_{TH}$) is in co-NP, since the last condition can be checked in polynomial time (for instance by linear programming).

To prove the completeness, we reduce the following NP-complete problem to our problem (see e.g., [14]).

Problem Exact Cover	
Input:	A hypergraph $\mathcal{H} = (V, H)$ such that $V = \{1, 2, \ldots, n\}$ and $H = \{E_1, E_2, \ldots, E_m\}$, where $E \subseteq V$ for all $E \in H$.
Question:	Is there an $H^* \subseteq H$ which exactly covers V; i.e., for which $E \cap E' = \emptyset$ for all $E \neq E' \in H^*$ and $\bigcup_{E \in H^*} E = V$?

We may assume without loss of generality that any H^* which exactly covers V contains E_1. This does not affect the NP-hardness of the problem, as it can be seen easily, since we always can modify the input by including one more hyperedge, E_1, which is disjoint from all other hyperedges of H.

Let $V_1 = \{n+1, n+2, \ldots, n+m\}$ and $V_2 = \{n+m+1, n+m+2, \ldots, n+2m\}$ and let $W = V \cup V_1 \cup V_2$. We shall denote by $(R; S)$ the vector $v \in \mathbb{M}^W$ for which $ON(v) = R$ and $S(v) = \{(v, j) \,|\, j \in S\}$. (Then $OFF(v) = V \setminus (R \cup S)$; thus in particular, $v = (R; \emptyset)$ denotes a binary vector.) Let us define a pBmb $(\tilde{T}, \tilde{F})$ by the following $\tilde{T}, \tilde{F} \subseteq \mathbb{M}^W$.

$$\begin{aligned}\tilde{T} &= \{a_{(1)} = (V \cup \{n+1\} \cup \{n+m+1\}; \emptyset)\} \\ &\quad \cup \{a_{(i)} = (\{n+m+i\}; \{n+i\}), i = 2, 3, \ldots, m\} \\ \tilde{F} &= \{b_{(0)} = (\emptyset; \emptyset)\} \cup \{b_{(1)} = (E_1 \cup \{n+1\} \cup V_2; \emptyset)\} \\ &\quad \cup \{b_{(i)} = (E_i \cup \{n+i\}; \emptyset), i = 2, 3, \ldots, m\}.\end{aligned}$$

For this pBmb we have $|S(a)| \leq 1$ for all $a \in \tilde{T}$ and $S(\tilde{F}) = \emptyset$. Thus, we write simply $\tilde{F}$ instead of $\tilde{F}^\alpha$, in the sequel.

We claim that this $(\tilde{T}, \tilde{F})$ is not fully consistent with $\mathcal{C}_{TH}$ if and only if an exact cover $H^* \subseteq H$ exists. This will then imply the theorem.

First, we show the "only-if" part of the above claim. Let us assume that for an assignment $\alpha \in \mathbb{B}^{S(\tilde{T})}$ the pdBf $(\tilde{T}^\alpha, \tilde{F})$ has no threshold extension. It follows from (24) that there exist nonnegative real numbers θ_i $(i = 1, 2, \ldots, m)$ and η_i $(i = 0, 1, \ldots m)$ such that

$$\sum_{i=1}^{m} \theta_i = 1, \tag{26}$$

$$\sum_{i=0}^{m} \eta_i = 1, \tag{27}$$

$$\sum_{i=1}^{m} \theta_i a^\alpha_{(i)} = \sum_{i=0}^{m} \eta_i b_{(i)}. \tag{28}$$

By comparing the corresponding components on the two sides of (28), we now derive the exact values of θ_i and η_i. Looking first at the components corresponding to V_2, we find

$$ON(b_{(1)}) \cap V_2 = V_2, \text{ and } ON(b_{(i)}) \cap V_2 = \emptyset \text{ for } i = 0, 2, ..., m, \tag{29}$$

on the right hand side, and

$$ON(a^\alpha_{(i)}) = \{n+m+i\} \text{ for } i = 1, 2, ..., m \tag{30}$$

on the left hand side of (28). Thus, the equalities between these components in (28) imply

$$\eta_1 = \theta_i \text{ for } i = 1, 2, ..., m,$$

from which

$$\eta_1 = \theta_i = \frac{1}{m} \text{ for } i = 1, 2, ..., m \tag{31}$$

follows by (26). Looking next at the components corresponding to V_1, we have

$$ON(b_{(0)}) \cap V_1 = \emptyset \text{ and } ON(b_{(i)}) \cap V_1 = \{n+i\} \text{ for } i = 1, 2, ..., m \tag{32}$$

on the right hand side, while $ON(a_{(1)}) \cap V_1 = \{n+1\}$ and

$$ON(a_{(i)}) \cap V_1 = \begin{cases} \{n+i\} & \text{if } \alpha(a_{(i)}, n+i) = 1, \\ \emptyset & \text{otherwise,} \end{cases} \quad \text{for } i = 2, 3, \ldots, m \tag{33}$$

are standing on the left hand side of (28). From the equalities between these components, and from (31) it follows that

$$\eta_i = \begin{cases} \frac{1}{m} & \text{if } \alpha(a_{(i)}, n+i) = 1, \\ 0 & \text{if } \alpha(a_{(i)}, n+i) = 0. \end{cases} \quad \text{for } i = 2, 3, \ldots, m \tag{34}$$

Then from these and (27) $\eta_0 = 1 - \sum_{i=1}^{m} \eta_i \geq 0$ follows.

Let us define now a family $H^* \subseteq H$ by

$$H^* = \{E_i \,|\, \eta_i = \frac{1}{m},\ i = 1, 2, ..., m\},$$

and let us prove that H^* is an exact cover of H. For this, let us finally compare the components on the two sides of (28), corresponding to the set V. On the left side we find

$$ON(a_{(1)}) \cap V = V \text{ and } ON(a_{(i)}) \cap V = \emptyset \text{ for } i = 2, 3, ..., m, \tag{35}$$

while on the other side we have

$$ON(b_{(0)}) \cap V = \emptyset \text{ and } ON(b_{(i)}) \cap V = E_i \text{ for } i = 1, 2, ..., m. \tag{36}$$

Thus, the equalities between these components, (31) and (34) together imply that the sets in H^* must form an exact cover, which concludes the proof of the "only-if" part of our claim.

For the "if" part, take an arbitrary exact cover $H^* \subseteq H$, and associate an assignment $\alpha \in \tilde{F}$ to it by defining

$$\alpha(a_{(i)}, n+i) = \begin{cases} 1 & \text{if } E_i \in H^* \\ 0 & \text{otherwise.} \end{cases}$$

It is then easy to see that with the nonnegative real numbers

$$\theta_i = \frac{1}{m}, \text{ for } i = 1, 2, ..., m,$$

$$\eta_i = \begin{cases} \frac{1}{m} & \text{if } E_i \in H^*, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } i = 1, 2, ..., m,$$

and $\eta_0 = 1 - \frac{|H^*|}{m}$, all equations in (26), (27) and (28) hold. Hence $\tilde{T}^\alpha$ and $\tilde{F}$ are not linearly separable, which proves that $(\tilde{T}, \tilde{F})$ is not fully consistent with $\mathcal{C}_{TH}$. □

8 Conclusion

In this paper, we have considered the relations between problems FC($\mathcal{C}$), RE($\mathcal{C}$) and VR($\mathcal{C}$). We showed that for many general and reasonable families of classes such as $\mathcal{C} = \mathcal{C}_{all}$, $\mathcal{C}_{Horn}$, $\mathcal{C}_+$, and $\mathcal{C}_{2\text{-}DNF}$, these three problems are equivalent. We also demonstrated that such an equivalence does not hold for all classes $\mathcal{C}$. For instance, we showed that problem RE($\mathcal{C}_{TH}$) is polynomially solvable, while FC($\mathcal{C}_{TH}$) is co-NP-complete.

References

[1] M. Anthony and N. Biggs. *Computational Learning Theory.* Cambridge University Press, 1992.

[2] J. C. Bioch. Personal communication, 1995.

[3] E. Boros, Y. Crama, and P.L. Hammer. Polynomial time inference of all valid implications for horn and related formulae. *Annals of Mathematics and Artificial Intelligence*, 1:21–32, 1990.

[4] E. Boros, V. Gurvich, P.L. Hammer, T. Ibaraki, and A. Kogan. Decompositions of partially defined boolean functions. *Discrete Applied Mathematics*, 62:51–75, 1995.

[5] E. Boros, P.L. Hammer, T. Ibaraki, and A. Kogan. Logical analysis of numerical data. *Mathematical Programming*, 79:163–190, August 1997.

[6] E. Boros, P.L. Hammer, T. Ibaraki, A. Kogan, E. Mayoraz, and I. Muchnik. An implementation of logical analysis of data. *IEEE Transactions on Knowledge and Data Engineering*, to appear.

[7] E. Boros, T. Ibaraki, and K. Makino. Boolean analysis of incomplete examples. In Rolf Karlsson and Andrzej Lingas, editors, *Algorithm Theory – SWAT'96*, volume 1097 of *Lecture Notes in Computer Science*, pages 440–451, Berlin - New York, 1996. Springer Verlag. 5th Scandinavian Workshop on Algorithm Theory, Reykjavik, Iceland, July 3-5, 1996.

[8] E. Boros, T. Ibaraki, and K. Makino. Extensions of partially defined boolean functions with missing data. RUTCOR Research Report 6-96, Rutgers University, 640 Bartholomew Road, Piscataway, NJ 08854-8003, USA, 1996.

[9] E. Boros, T. Ibaraki, and K. Makino. Monotone extensions of boolean data sets. In M. Li and A. Maruoka, editors, *Algorithmic Learning Theory – ALT'97*, volume 1316 of *Lecture Notes in Artificial Intelligence*, pages 161–175, Berlin - New York, 1997. Springer Verlag.

[10] E. Boros, T. Ibaraki, and K. Makino. Error-free and best-fit extensions of partially defined boolean functions. *Information and Computation*, 140:254–283, 1998.

[11] E. Boros, T. Ibaraki, and K. Makino. Logical analysis of binary data with missing bits. *Artificial Intelligence*, 107:219–263, 1999.

[12] N.H. Bshouty. Exact learning boolean functions via the monotone theory. *Information and Computation*, 123:146–153, 1995.

[13] Y. Crama, P.L. Hammer, and T. Ibaraki. Cause-effect relationships and partially defined boolean functions. *Annals of Operations Research*, 16:299–326, 1988.

[14] M.R. Garey and D.S. Johnson. *Computers and Intractability.* Freeman, New York, NY, 1979.

[15] R. Khardon and D. Roth. Reasoning with models. *Artificial Intelligence*, 87:187–213, 1996.

[16] J.C.C. McKinsey. The decision problem for some classes of sentences without classifiers. *Journal of Symbolic Logic*, 8:61–76, 1943.

[17] W. Quine. A way to simplify truth functions. *American Mathematical Monthly*, 62:627–631, 1955.

[18] S. Rudeanu. *Boolean Functions and Equations.* North-Holland, Amsterdam, 1974.