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Camel Sequences and Their Applications

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ABSTRACT

Given an even cyclic (0,1) sequence $s = (s_1, ..., s_{2n})$ which consists of n ones and n zeros, let us compute $i + j(mod2n)$ for all n^2 (0,1)-pairs (s_i, s_j) and distribute obtained n^2 numbers between $2n$ "boxes" $1, 2, ..., 2n-1, 2n = 0(mod2n)$. Obviously, the average cardinality of a box is $\frac{n^2}{2n} = \frac{n}{2}$. The following five sequences have quite remarcable box-distributions, "almost average everywhere with two big humps":

$n = 3,\ s = (110100),\ B = (112113) = (1^2\,2\,1^2\,3);$

$n = 7,\ s = (11101000110100),\ B = (33333363333337) = (3^6\,6\,3^6\,7);$

$n = 11,\ s = (1110001001011100010110),\ B = (5^{10}\,10\,5^{10}\,11);$

$n = 5,\ s = (1110010100),\ B = (3333133330) = (3^4\,1\,3^4\,0);$

$n = 17,\ s = (1111101000110001011011010001100010),\ B = (9^{16}\,1\,9^{16}\,0).$

In general, given an *odd* n, box-distributions $(\lfloor\frac{n}{2}\rfloor^{n-1}(n-1)\lfloor\frac{n}{2}\rfloor^{n-1}n)$ and $(\lceil\frac{n}{2}\rceil^{n-1}\ 1\lceil\frac{n}{2}\rceil^{n-1}\ 0)$ as well as the generating sequences will be called the *camel distributions* and the *camel sequences*, respectively *up-camel* and *down-camel*. We conjecture that there are infinitely many of them, both types, though only five are known. The first three sequences above are up-camel, and the last two are down-camel. We consider some applications of camel sequences in extremal graph theory.

Key Words: *(0,1)-sequence, camel sequence, maximin, minimax, extremal graph theory, vertex-enumerated graphs*

1 Introduction. Vertex enumerations optimizing the number of pairwise parallel edges of a graph.

Given a graph $G =< V, E >$ without loops and multiple edges, where $|V| = n, |E| = m$, and an integer $N \geq n$, let us enumerate the vertices by different numbers from $[N] = \{1, ..., N\}$, in other words, let us consider an injective mapping $f : V \to [N]$. Given an f, we will call two different edges $e_1 = (v_1, v_1'), e_2 = (v_2, v_2') \in E$ *parallel* if $f(v_1) + f(v_1') = f(v_2) + f(v_2') (mod\ N)$ (Obviously, all four vertices must be different.) There are N sets (bunches) of parallel edges enumerated by $0, 1, ..., N - 1$. Let us denote by $B(G, N, f) = B = (b_1, ..., b_N)$ the numbers of edges in these bunches. The sum of these N numbers is m, thus the average is $\frac{m}{N}$.

Remark 1 *A geometrical interpretation may be convenient. Let us consider an injective mapping of vertices of G into vertices of a perfect N-gon, the edges being the diagonals or edges of the N-gon which connect the corresponding pairs of vertices. Clearly, the above notion of parallel edges is in agreement with standard Euclidian concept. Some "non-Euclidian" concepts also can be considered. Counting of the corresponding "weakly parallel" edges would give rise to some interesting combinatorial problems, too. (Ron Aharoni, private communication.) However, we will not consider this approach here.*

We will study the following two functions:

$$\alpha(G, N) = \max_{B \in \mathcal{B}} \min_{b \in B} b, \quad \beta(G, N) = \min_{B \in \mathcal{B}} \max_{b \in B} b, \tag{1}$$

where $\mathcal{B} = \mathcal{B}(G, N)$ is the set of the distributions $B = B(G, N, f)$ generated by all the injective mappings $f : V \to [N]$. Obviously,

$$\alpha(G, N) \leq \frac{m}{N} \leq \beta(G, N) \tag{2}$$

because "minimum $\leq$ average $\leq$ maximum" for every distribution.

In this paper we restrict ourselves by the case $N = 2n$ and $G = K_{n,n}$ being the complete bipartite graph with equal parts. Only in section 5 we have a look at the case $G = C_n$.

Let us consider functions

$$\alpha(n) = \alpha(K_{n,n}, 2n), \quad \beta(n) = \beta(K_{n,n}, 2n). \tag{3}$$

Even this very special case is already difficult. Inequalities (2) can be rewritten as

$$\alpha(n) \leq \frac{n}{2} \leq \beta(n) \ \forall n. \tag{4}$$

We conjecture that both functions $\alpha(n)$ and $\beta(n)$ are asympthotically equal to $\frac{n}{2}$, that is $\alpha(n) = \frac{n}{2} - o(n),\ \beta(n) = \frac{n}{2} + o(n)$. However, we will only prove that

$$LIM(\frac{\alpha(n)}{n}, n = 1, 2, ...) \subseteq [\frac{5}{11}; \frac{1}{2}],\ LIM(\frac{\beta(n)}{n}, n = 1, 2, ...) \subseteq [\frac{1}{2}; \frac{9}{17}], \tag{5}$$

where $LIM(X)$ is the set of all the limit points of a sequence X. The lower and upper bounds $\frac{1}{2}$ result from (4) and bounds $\frac{5}{11}$ and $\frac{9}{17}$ provided by the camel sequences given in the abstract. If we could construct *infinite* sets of up- and down-camel sequences then it would result in equalities $sup\, LIM(\frac{\alpha(n)}{n}) = \frac{1}{2}$ and $inf\, LIM(\frac{\beta(n)}{n}) = \frac{1}{2}$ respectively.

Given N=2n and a box-distribution $B = (b_1, ..., b_{2n})$, let us split it into two: odd-box-distribution $B_o = (b_1, ..., b_{2n-1})$ and even-box distribution $B_e = (b_2, ..., b_{2n})$. In section 3 we will introduce the corresponding functions $\alpha_e(n), \alpha_o(n), \beta_e(n), \beta_o(n)$ and prove that all four functions are asympthotically equal to $\frac{n}{2}$. In section 4 we introduce range functions. In section 5 we consider the case of a cycle, $G = C_n$.

2 Functions α and β.

Lemma 1 *For any positive integers k and n functions α and β satisfy the inequalities*

$$k\alpha(n) \leq \alpha(kn), \;\; gb(kn) \geq \beta(kn). \tag{6}$$

Proof. Let us suppose that value $\alpha(n)$ or $\beta(n)$ is realized by a (0,1)-sequence $(s_1, ..., s_{2n})$ which generates a box-distribution $B(s) = (b_1, ..., b_{2n})$. Let us consider the sequence $s^k = (s_1, ..., s_{2n})^k$. It generates the box-distribution $B(s^k) = (kB(s))^k = (kb_1, ..., kb_{2n})^k$ (As usual, x^k means x repeated k times.) Thus the maximal and minimal cardinalities of $B(s^k)$ are respectively k times the maximal and minimal cardinalities of $B(s)$. This and the definition of functions α, β by (1,3) prove (6). $\square$

Inequalities (4) and (6) together imply the chain

$$k\alpha(n) \leq \alpha(kn) \leq \frac{n}{2} \leq \beta(kn) \leq k\beta(n). \tag{7}$$

Lemma 2 *Given an odd n, if there exists an up-camel sequence then $\alpha(n) = \lfloor \frac{n}{2} \rfloor$, if there exists a down-camel sequence then $\beta(n) = \lceil \frac{n}{2} \rceil$.*

Proof. Inequalities (4) hold, $\alpha(n)$ and $\beta(n)$ are integers, and n is odd. Thus $\alpha(n)$ and $\beta(n)$ can not be equal to $\frac{n}{2}$, so they must be equal to $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ respectively. $\square$

First few values of α and β are given in the following table.

n	1	2	3	4	5	6	7	8	9	10	11						
$\alpha(n)$	0	0	1	1	1	2	3	2	3	4	5						
$a(n)$	0	0	1	1	1	2	2	2	3	3	3						
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\beta(n)$	1	2	3	3	3	4	5	6	6	6	7	?	?	?	?	?	9
$b(n)$	1	2	3	3	3	4	5	6	6	6	7	8	9	9	9	10	11

Table 1. First few values taken by functions α and β.

This table also contains the values of functions

$$a(n) = \frac{(n - n(mod\,3))}{3} = \frac{n}{3} + o(n), \quad b(n) = \lceil \frac{(3n + n(mod\,5)/2)}{5} \rceil = \frac{3n}{5} + o(n). \quad (8)$$

We assume here that $n\,mod(k)$ takes values $0, 1, ..., k-1$.

Remark 2 *¿From Table 1 we can see that function α is not monotone increasing, indeed $3 = \alpha(7) > \alpha(8) = 2$. And probably β is not monotone increasing either (though we can not see it from the table); it may happen that already $10 = \beta(16) > \beta(17) = 9$.*

Proposition 1 *For any n the following inequalities hold:*

$$a(n) \leq \alpha(n) \leq \frac{n}{2} \leq \beta(n) \leq b(n) \quad (9)$$

Proof. (sketch) Let us prove the last inequality in (9) for the case $n = 0(mod5)$, that is $n = 5k$. For $n = 5$ there exists the down-camel sequence (1100011010) with box-distribution $(3^4 1 3^4 0)$. According to Lemma 1, the sequence $(1100011010)^k$ generates the box-distribution $((3k)^4 k (3k)^4 0)^k$. Thus $\beta(5k) \leq 3k$ for any k. For other cases the following sequences work:

$(10)(1100011010)^k$ for $n = 5k + 1$,

$(1010)(1100011010)^k$ for $n = 5k + 2$,

$(101010)(1100011010)^k$ for $n = 5k + 3$,

$(11000110)(1100011010)^k$ for $n = 5k + 4$.

Analogously, it is easy to prove the first inequality in (9) for the case $n = 0(mod3)$, that is $n = 3k$. For $n = 3$ there exists the up-camel sequence (110010) with box-distribution (112113). According to Lemma 1, the sequence $(110010)^k$ has box-distribution $(k, k, 2k, k, k, 3k)^k$. Thus $\alpha(3k) \geq k$ for any k. For other cases the following sequences work:

$(10)(110010)^k$ for $n = 3k + 1$,

$(1010)(110010)^k$ for $n = 3k + 2$. $\square$

Remark 3 *The first up-camel sequence (n=3) provides a lower bound for α and the first down-camel sequence (n=5) provides an upper bound for β The larger camel sequences could give us better bounds which however are more difficult to prove.*

Conjecture 1 *There exist infinitely many up-camel sequences and infinitely many down-camel ones.*

This conjecture, if true, would imply that $\frac{1}{2}$ is a limit point of both sequences $\{\frac{\alpha(n)}{n}; n = 1, 2, ...\}$ and $\{\frac{\beta(n)}{n}; n = 1, 2, ...\}$.

Conjecture 2 *For every up-camel (respectively, down-camel) $(2n)$-sequence s (or at least for infinitely many such sequences), for every $i = 1, ..., n-1$, there exists a $(2i)$-sequence s_0 such that $b \geq k\lfloor\frac{n}{2}\rfloor - i$ (respectively $b \leq k\lceil\frac{n}{2}\rceil + i$) for every non-negative integer k and for every number b from the box-distribution $B(s_0 s^k)$.*

The above two conjectures, if true, would imply that both sequences $\{\frac{\alpha(n)}{n}; n = 1, 2, ...\}$ and $\{\frac{\beta(n)}{n}; n = 1, 2, ...\}$ converge and $\frac{1}{2}$ is their common limit; in other words, both functions $\alpha(n)$ and $\beta(n)$ are asympthotically equal to $\frac{n}{2}$. Proposition 1 shows that Conjecture 2 holds for camel sequences (110010) and (1100011010).

3 Functions $\alpha_e, \beta_e, \alpha_o, \beta_o$.

Given graph $G =< V, E >, |V| = n, |E| = m$, an even number $N \geq n$ and an injective mapping $f : V \to [N] = \{1, ..., N\}$, we have defined a box-distribution $B(G, N, f) = B = (b_1, ..., b_N)$. Let us split it into two: odd-box-distribution $B_o(G, N, f) = B_o = (b_1, ...b_{N-1})$ and even-box-distribution $B_e = B_e(G, N, f) = (b_2, ..., b_N)$ and define four functions

$$
\begin{aligned}
\alpha_e(G,N) &= \max_{B\in\mathcal{B}_e}\min_{b\in B} b,\\
\beta_e(G,N) &= \min_{B\in\mathcal{B}_e}\max_{b\in B} b,\\
\alpha_o(G,N) &= \max_{B\in\mathcal{B}_o}\min_{b\in B} b,\\
\alpha_e(G,N) &= \min_{B\in\mathcal{B}_o}\max_{b\in B} b.
\end{aligned}
\tag{10}
$$

where $\mathcal{B}_e = \mathcal{B}_e(G,N)$ and $\mathcal{B}_o = \mathcal{B}_o(G,N)$ are the sets of the distributions $B_e = B_e(G,N,f)$ and respectively $B_e = B_e(G,N,f)$ generated by all the injective mappings $f : V \to [N]$.

Again we restrict ourselves by the case $N = 2n$ and $G = K_{n,n}$ and consider the functions

$$
\begin{aligned}
\alpha_e(n) &= \alpha_e(K_{n,n},2n),\\
\beta_e(n) &= \beta_e(K_{n,n},2n),\\
\alpha_o(n) &= \alpha_o(K_{n,n},2n),\\
\beta_o(n) &= \beta_o(K_{n,n},2n).
\end{aligned}
\tag{11}
$$

Lemma 3 *For any positive integers k and n the functions $\alpha_e, \beta_e, \alpha_o, \beta_o$ satisfy inequalities*

$$
\begin{aligned}
\alpha_e(kn) &\geq k\alpha_e(n),\\
\alpha_o(kn) &\geq k\alpha_e(n),\\
\beta_e(kn) &\leq k\beta_e(n),\\
\beta_o(kn) &\leq k\beta_o(n).
\end{aligned}
\tag{12}
$$

Proof. It is analogous to that of Lemma 1. □

A pair $i, j \in [2n] = \{1, ..., 2n\}$ is called *even* if $i + j$ is even and *odd* if $i + j$ is odd.

Lemma 4 *For any (0,1)-sequence* $s = (s_1, ..., s_{2n})$ *the number of odd (0,1)-pairs is not less than the number of even (0,1)-pairs.*

Proof. Let us denote by a, b, c, d respectively the numbers of even ones, even zeros, odd ones and odd zeros in s. Obviously, $a+b = a+c = b+d = c+d = n$, thus $a = c$ and $b = d$. The number of odd (0,1)-pairs equals $ad+bc = a^2+(n-a)^2$, the number of even (0,1)-pairs equals $ab+cd = 2a(n-a)$. Obviously, $a^2+(n-a)^2 \geq \frac{n^2}{2} \geq 2a(n-a)$ for any a between 0 and n. □

This lemma immediately results in inequalities

$$\alpha_e(n) \leq \frac{n}{2} \leq \beta_o(n) \ \forall n \tag{13}$$

Functions α_o, β_e and α_e are easy to compute.

Proposition 2 $\alpha_o(n) = n, \ \beta_e(n) = 0 \ \forall n.$

Proof. Sequence $s = (01)^n$ generates the box-distribution $B = (n, 0)^n$ which implies both equalities. □

Proposition 3 $\alpha_e(n) = \lfloor \frac{n}{2} \rfloor \ \forall n.$

Proof. If n is even then sequence $(1100)^{\frac{n}{2}}$ generates the even-box-distribution $B_e = (\frac{n}{2})^n$. If n is odd then sequence $(10)(1100)^{\frac{(n-1)}{2}}$ generates the even-box-distribution $B_e = ((\frac{(n-1)}{2}, \frac{(n+1)}{2})^{\frac{(n-1)}{2}} \frac{(n-1)}{2})$. In both cases all even cardinalities are at least $\lfloor \frac{n}{2} \rfloor$. □

Function β_o is more difficult to compute. However, the following inequality holds.

Proposition 4 $\beta_o(n) \leq \frac{(n+n(mod4))}{2} \ \forall n$*, where* $n(mod4)$ *takes values 1,2,3 and 0. In case* $n = 0(mod4)$ *the equality holds.*

Proof. (sketch). Let $n = 4k + i$ where $k \geq 0$ and $i = 1, 2, 3$ or 0. Then $\frac{(n+n(mod4))}{2} = 2k+i$. The sequence $s = (11100100)^k(10)^i$ generates an odd-box-distribution B_o such that all its cardinalities $b_1, ..., b_{2n-1}$ are at most $2k + i$ which proves the inequality. In case $i = 0$ we get $\beta_o(n) \leq \frac{n}{2}$, and according to (13), $\beta_o(n) \geq \frac{n}{2}$, thus $\beta_o(n) = \frac{n}{2}$. □

Remark 4 *Except for the case $n = 0(mod4)$, equality takes place for small values of n too, at least for $n \leq 8$. This implies that function β_e is not monotone increasing.*

Remark 5 *The above inequality of Proposition 4 together with $\beta_o(n) \geq \frac{n}{2}$ imply that $\beta_o(n) = \frac{n}{2} + o(n)$.*

4 Range functions.

Given a box-distribution B, let us define its *range* $r = r(B)$ as the difference between the maximal and the minimal numbers of B.

Given a graph $G =< V, E >, |V| = n, |E| = m$, an even number $N \geq n$ and an injective mapping $f : V \to [N] = \{1, ..., N\}$, we had defined a box-distribution $B(G, N, f) = B = (b_1, ..., b_N)$. The *range* of B is defined as $r(B) = \max_{b \in B} - \min_{b \in B} \ b$. Let us introduce two functions which characterize respectively the minimal and the maximal possible ranges:

$$\delta(G, N) = \min_{B \in \mathcal{B}} r(B), \quad \Delta(G, N) = \max_{B \in \mathcal{B}} r(B) \tag{14}$$

where $\mathcal{B} = \mathcal{B}(G, N)$ is the set of the distributions $B = B(G, N, f)$ generated by all the injective mappings $f : V \to [N]$, as before.

And again we introduce the "even and odd" range functions

$$\begin{aligned} \delta_e(G, N) = \min_{B \in \mathcal{B}_e} r(B), \quad \Delta_e(G, N) = \max_{B \in \mathcal{B}_e} r(B), \\ \delta_o(G, N) = \min_{B \in \mathcal{B}_o} r(B), \quad \Delta_o(G, N) = \max_{B \in \mathcal{B}_o} r(B). \end{aligned} \tag{15}$$

And again we restrict ourselves by the case $N = 2n, \ G = K_{n,n}$ and consider only

$$\delta(n) = \delta(K_{n,n}, 2n), \qquad \Delta(n) = \Delta(K_{n,n}, 2n);$$

$$\begin{aligned} \delta_e(n) &= \delta_e(K_{n,n}, 2n), \qquad \Delta_e(n) = \Delta_e(K_{n,n}, 2n), \\ \delta_o(n) &= \delta_o(K_{n,n}, 2n), \qquad \Delta_o(n) = \Delta_o(K_{n,n}, 2n). \end{aligned} \tag{16}$$

Except for δ, the other five functions are easy to compute.

Proposition 5 *$\delta_o(n) = \delta_e(n) = 0,\ \Delta(n) = n\ \forall n.$*

Proof. According to Lemma 1, the sequence $s = (01)^n$ generates the box-distribution $B(s) = (n, 0)^n$, which proves all three equalities. $\square$

Proposition 6 *$\Delta_o(n) = 2\lfloor \frac{n}{2} \rfloor$, that is $n-1$ if n is odd and n if even.*

Proof. Sequence $(1^n 0^n)$ generates odd-distribution $B_o = (n, n-2, ..., 4, 2, 0, 2, 4, ..., n-2)$ if n is even and $B_o = (n, n-2, ..., 3, 1, 1, 3, ..., n-2)$ if n is odd. $\square$

Proposition 7 *$\Delta_e(n) = n - 1$ if $n = 1, 3$ or $0 (mod\, 4)$ and $\Delta_e(n) = n - 2$ if $n = 2 (mod\, 4)$.*

Proof. (sketch). If $n = 1, 2$ or $3 (mod\, 4)$ then the critical sequence is $s = (1^n 0^n)$, if $n = 0 (mod 4)$ then $s = (11100100)^{\frac{n}{4}}$. $\square$

And we know almost nothing about function δ. Its first few values are given in the Table 2.

n	1	2	3	4	5	6	7	8	9	10
δ	1	2	2	3	3	3	3	4	4	3

Table 2. First few values of $\delta(n)$

Remark 6 *We can see that δ is not monotone but the order of its magnitude is unclear. In this case the camel distributions are not effective because though they are constant almost everywhere, yet they have large ranges because of the humps.*

5 The case of a cycle, $G = C_n$.

Above we considered only the case $G = K_{n,n}$ and $N = 2n$. Now let us consider the case of a simple cycle, $G = C_n$; $N \geq n \geq 3$. Our standard four functions

$$\begin{aligned} \alpha &= \alpha(n, N) = \alpha(C_n, N), \\ \beta &= \beta(n, N) = \beta(C_n, N), \\ \delta &= \delta(n, N) = \delta(C_n, N), \\ \Delta &= \Delta(n, N) = \Delta(C_n, N) \end{aligned} \tag{17}$$

are defined by formulae (1,14), where now $f : [n] \to [N]$ is an injective mapping of n vertices $[n] = \{0, 1, ..., n-1\}$ of C_n into N vertices $[N] = \{0, 1, ..., N-1\}$ of a perfect N-gon P_N. We will specify such a mapping by the sequence $f = (f(0), f(1), ..., f(n-1))$; $f(i) \in [N]$). And we always assume that $N \geq n \geq 3$. Let us recall that, given a mapping f, two different edges $e_1 = (i_1, i'_1), e_2 = (i_2, i'_2)$ of C_n are parallel iff $f(i_1) + f(i'_1) = f(i_2) + f(i'_2)$ $(mod\ N)$. Obviously, all four vertices must be different. Function Δ is very easy to compute.

Proposition 8 $\Delta = \lfloor (n/2) \rfloor$ *for any* $N \geq n \geq 3$.

Proof. Two successive edges of the cycle can not be parallel. And any pairwise disjoint edges can be all parallel. $\square$

Let us note that function β can take only two values: 1 and 2. Indeed, the identical mapping $f = (f(0), f(1), ..., f(n-1) = (0, 1, ..., n-1)$ guarantees that no 3 edges of C_n are parallel. By the definition, $\beta = 1$ iff there exists an f such that there are no parallel edges at all. Obviously, in this case $\alpha = 1,\ \delta = 0$ if $N = n$ and $\alpha = 0,\ \delta = 1$ if $N > n$. If $\beta = 2$ then, obviously, $\alpha = 0$ and $\delta = 2$ for any n and $N \geq n \geq 3$. Thus all we need to determine is is β equal to 1 or 2, or in other words, does a mapping f without parallel edges exist.

Thus we have reduced the computation of functions α, β, δ to the following elementary 'geometrical' problem. Given a perfect N-gon P_N, let us chose n vertices

$f_0, ..., f_{n-1}$ and consider n-cycle $(f_0, f_1), ..., (f_{n-1}, f_0)$. We call such a cycle *normal* if it contains no parallel edges. For which pairs (N, n) normal cycles exist ? Boros and Gurvich (1999) have proven that they fail to exist only for $\{(n, N) = (2i, 2i); i = 2, 3, ...\}$, and for any other pair a cycle without parallel edges is constructed. For completess we repeat this construction here.

5.1 Normal stars for odd n and $N \geq n$.

If n is odd then the following sequences generate no parallel edges whenever $N \geq n \geq 3$:

$n = 3,\ f = (0, 1, 2),\ N \geq 3;$

$n = 5,\ f = (0, 2, 4, 1, 3),\ N \geq 5;$

$n = 7,\ f = (0, 3, 6, 2, 5, 1, 4),\ N \geq 7;$

$n = 9,\ f = (0, 4, 8, 3, 7, 2, 6, 1, 5)\ N \geq 9$; etc.

In fact these sequences form the 'stars', where not only there are no parallel edges but moreover, every two edges have an intersection in the polygon P_N: for successive edges this intersection is a vertex of P_N, otherwise it is a point inside P_N. However, for even n 'stars' fail to exist.

5.2 Normal pseudo-stars for even n and $N > n$.

The following sequences, which we will call *pseudo-stars* generate no parallel edges, whenever $N \geq 3n/2 - 1 \geq 5$:

$n = 4,\ f = (0, 1, 3, 2),\ N \geq 5;$

$n = 6,\ f = (0, 1, 4, 2, 5, 3),\ N \geq 8;$

$n = 8,\ f = (0, 1, 5, 2, 6, 3, 7, 4),\ N \geq 11;$

$n = 10,\ f = (0, 1, 6, 2, 7, 3, 8, 4, 9, 5),\ N \geq 14;$

$n = 12,\ f = (0, 1, 7, 2, 8, 3, 9, 4, 10, 5, 11, 6),\ N \geq 17$, etc.

But now we have a gap: $n \leq N < 3n/2 - 1$. Another family of pseudo-star sequences make this gap much smaller, namely, the following cycles are normal in all the cases, except for $\{(n, N) = (2i, 2i);\ i = 2, 3, ...\}$ and $\{(n, N) = (4j+2, 4j+3);\ j = 1, 2, ...\}$:

$n = 4,\ f = (0, 2, 3, 1),\ N \geq 5;$

$n = 6,\ f = (0, 3, 5, 2, 4, 1),\ N \geq 8;$

$n = 8,\ f = (0, 4, 7, 3, 6,\ 2, 1,\ 5),\ N \geq 9;$

$n = 10,\ f = (0, 5, 9, 4, 8, 3, 7,\ 2, 1,\ 6),\ N \geq 12;$

$n = 12,\ f = (0, 6, 11, 5, 10, 4, 9,\ 3, 2,\ 8, 1, 7),\ N \geq 13;$

$n = 14,\ f = (0, 7, 13, 6, 12, 5, 11, 4, 10,\ 3, 2,\ 9, 1, 8),\ N \geq 16;$

$n = 16,\ f = (0, 8, 15, 7, 14, 6, 13, 5, 12,\ 4, 3,\ 11, 2, 10, 1, 9),\ N \geq 17;$

$n = 18,\ f = (0, 9, 17, 8, 16, 7, 15, 6, 14, 5, 13,\ 4, 3,\ 12, 2, 11, 1, 10),\ N \geq 20;$

$n = 20,\ f = (0, 10, 19, 9, 18, 8, 17, 7, 16, 6, 15,\ 5, 4,\ 14, 3, 13, 2, 12, 1, 11),\ N \geq 21;$ etc.

Let us note that these new 'pseudo-stars' and the old ones generate geometrically the same cycles in P_N, but the numerations of vertices are different because of a shift of the origin.

5.3 Normal cycles do not exist in case $N = n = 2i$.

If $N = n$ then f is a bijection. Let us assume that $N = n$ is even and demonstrate that for every mapping f at least one pair of parallel edges exists in C_n. Let us assume the opposite, that is for an $f = (f(0), f(1), ..., f(n-1))$ there are no parallel edges, in other words, all n sums $f(0) + f(1), f(1) + f(2), ..., f(n-1) + f(0)$ are pairwise different. Then clearly these sums take all the values $(0, 1, ..., n-1)$ and thus $(f(0) + f(1)) + (f(1) + f(2)) + ... + (f(n-1) + f(0)) = 2\sum_{i=0}^{n-1} f(i) = n(n -$

$1)/2(mod\ n)$. But from the other hand, $(f(0), f(1), ..., f(n-1))$ also takes all the values $(0, 1, ..., n-1)$ because $N = n$ and f is a bijection. Thus we get an equality $2\sum_{i=0}^{n-1} f(i) = \sum_{i=0}^{n-1} f(i)\ (mod\ n)$ which holds only if n is odd. If n is even then $0 = 2 \times \sum_{i=0}^{n-1} f(i) \neq \sum_{i=0}^{n-1} = n/2\ (mod\ n)$. For example, $2(0+1+2) = 0+1+2 = 0(mod\ 3)$ but $0 = 2(0+1+2+3) \neq 0+1+2+3 = 2(mod\ 4)$.

5.4 Normal cycles for $n = 2j,\ N = 2j+1$.

Let us demonstrate that parallel edges may not exist in the remaining cases, namely for $(N = 2j+1, n = 2j;\ j \geq 2)$. Indeed, the foolowing cycles are normal:

$n = 4,\ f = (1, 2,\ 4, 3);\ N \geq 5;$

$n = 6,\ f = (1, 3, 2,\ 6, 4, 5);\ N \geq 7;$

$n = 8,\ f = (1, 3, 4, 2,\ 8, 6, 5, 7);\ N \geq 9, N \neq 10;$

$n = 10,\ f = (1, 3, 5, 4, 2,\ 10, 8, 6, 7, 9);\ N \geq 11, N \neq 12, 14;$

$n = 12,\ f = (1, 3, 5, 6, 4, 2,\ 12, 10, 8, 7, 9, 11);\ N \geq 13, N \neq 14, 16, 18;$

$n = 14,\ f = (1, 3, 5, 7, 6, 4, 2,\ 14, 12, 10, 9, 11, 13);\ N \geq 15, N \neq 16, 18, 20, 22;$ etc.

Let us note that in case $(N = 2j+1, n = 2j;\ j \geq 2)$. in all these normal cycles vertex 0 is missing and sums $f(i) + f(i+1)(modN)$ take all the values $1, ..., N-1$ but not 0. This holds in general. Namely, if $N - 1 = n = 2j$ and vertex a is missing in a normal cycle C_n then the sums $f(i) + f(i+1)(modN)$ take all the values $0, 1, ..., N-1$, except for $b = 2a(mod\ N)$. Indeed, $F = \sum_{i=0}^{N-1} i = N(N-1)/2 = 0(mod\ N)$, because N is odd. Further, $f(0) + f(1) + ... + f(n-1) = F - a$ and $(f(0) + f(1)) + (f(1) + f(2)) + ... + (f(n-1) + f(0)) = 2F - 2a = F - b(mod\ N)$. Thus $F = 2a - b = 0(mod\ N)$ and $b = 2a(mod\ N)$.

Let us finally note that the above cycles are obtained by a shift of the origin from the following simpler family of normal cycles, which however leave a gap $n < N < 2n - 3$:

$n = 4,\ f = (1, 3,\ 4, 2);\ N \geq 5;$

$n = 6,\ f = (1, 3, 5,\ 6, 4, 2);\ N \geq 9;$

$n = 8,\ f = (1, 3, 5, 7,\ 8, 6, 4, 2);\ N \geq 13;$

$n = 10,\ f = (1, 3, 5, 7, 9,\ 10, 8, 6, 4, 2);\ N \geq 17;$ etc.

Our case analysis shows that normal cycles exist (i.e. parallel edges may not exist) for all the pairs (N, n) such that $N \geq n \geq 3$, except for the case $(N = n = 2i; i \geq 2)$.

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References

[1] E.Boros and V.Gurvich, On parallel edges in cycles, Dimacs Technical Report 99-28, Rutgers University.