**DIMACS Technical Report 99-28**
**May 1999**

# On Parallel Edges in Cycles[1]

by

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[1]The authors gratefully acknowledge the partial support by the Office of Naval Research (Grants N00014-92-J-1375), and by DIMACS, the National Science Foundation Center for Discrete Mathematics and Theoretical Computer Science.
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DIMACS is a partnership of Rutgers University, Princeton University, AT&T Labs-Research, Bell Labs, Telcordia Technologies (formerly Bellcore) and NEC Research Institute.

DIMACS is an NSF Science and Technology Center, funded under contract STC–91–19999; and also receives support from the New Jersey Commission on Science and Technology.

**ABSTRACT**

Let $n \leq N$ be positive integers. We consider the problem of finding an $n$-cycle with no parallel edges in a perfect $N$-gon in the Euclidian plane. We prove that there exist no such $n$-cycle if and only if $N = n$ and even. And we show by construction that for every other pair $(N, n)$, $N \geq n \geq 3$, such an $n$-cycle exists.

**Key Words**: *cycle, normal cycle, perfect $N$-gon, parallel edges*

# 1 Introduction

Let $N$ be a positive integers, and let $[N] = \{0, 1, ..., N-1\}$ denote the vertices of a perfect $N$-gon $P_N$ in the Euclidian plane. Two different chords $(i, i')$ and $(j, j')$ of $P_N$ are *parallel* if $(i + i') = (j + j') \pmod N$. (We interpret the chord $(i, i)$ as the line through vertex $i$ tangent to the circumscribed circle of $P_N$.)

Let us consider a sequence $f(0)$, $f(1)$, ..., $f(n-1)$ of $n$ different vertices of $P_N$, $n \le N$ and consider the cycle $C = C_f$ formed by the edges $(f(i), f(i+1))$ for $i = 0, ..., n-2$ and $(f(n-1), f(0))$. Let us call $C$ a *normal cycle* if it does not contain parallel edges.

We prove that there exist no normal cycles in $P_N$ if and only if $N$ is even and $n = N$. For every other pair $(N, n)$ of integers $N \ge n \ge 3$ we construct a normal $n$-cycle in $P_N$.

# 2 Normal stars and pseudo-stars

If $n$ is odd then the following sequences generate no parallel edges whenever $N \ge n \ge 3$:

$n = 3,\ f = (0, 1, 2),\ N \ge 3;$
$n = 5,\ f = (0, 2, 4, 1, 3),\ N \ge 5;$
$n = 7,\ f = (0, 3, 6, 2, 5, 1, 4),\ N \ge 7;$
$n = 9,\ f = (0, 4, 8, 3, 7, 2, 6, 1, 5)\ N \ge 9$, etc.

In fact these sequences form the *stars*, where not only there are no parallel edges but moreover, every two edges have an intersection in the polygon $P_N$. For successive edges this intersection is a vertex of $P_N$, otherwise it is a point inside $P_N$. However, for even values of $n$ *stars* fail to exist. In these cases, the following *pseudo-star* sequences generate no parallel edges, whenever $N \ge \frac{3n-2}{2} \ge 5$:

$n = 4,\ f = (0, 1, 3, 2),\ N \ge 5;$
$n = 6,\ f = (0, 1, 4, 2, 5, 3),\ N \ge 8;$
$n = 8,\ f = (0, 1, 5, 2, 6, 3, 7, 4),\ N \ge 11;$
$n = 10,\ f = (0, 1, 6, 2, 7, 3, 8, 4, 9, 5),\ N \ge 14;$
$n = 12,\ f = (0, 1, 7, 2, 8, 3, 9, 4, 10, 5, 11, 6),\ N \ge 17$, etc.

For even values of $n$, and for $n < N < \frac{3n-2}{2}$, the following *pseudo-star* sequences provide normal cycles, whenever $(n, N) \ne (4j + 2, 4j + 3)$, for some $j \ge 1$.

$n = 4,\ f = (0, 2, 3, 1),\ N \ge 5;$
$n = 6,\ f = (0, 3, 5, 2, 4, 1),\ N \ge 8;$
$n = 8,\ f = (0, 4, 7, 3, 6,\ 2, 1,\ 5),\ N \ge 9;$
$n = 10,\ f = (0, 5, 9, 4, 8, 3, 7,\ 2, 1,\ 6),\ N \ge 12;$
$n = 12,\ f = (0, 6, 11, 5, 10, 4, 9,\ 3, 2,\ 8, 1, 7),\ N \ge 13;$
$n = 14,\ f = (0, 7, 13, 6, 12, 5, 11, 4, 10,\ 3, 2,\ 9, 1, 8),\ N \ge 16;$
$n = 16,\ f = (0, 8, 15, 7, 14, 6, 13, 5, 12,\ 4, 3,\ 11, 2, 10, 1, 9),\ N \ge 17;$
$n = 18,\ f = (0, 9, 17, 8, 16, 7, 15, 6, 14, 5, 13,\ 4, 3,\ 12, 2, 11, 1, 10),\ N \ge 20;$
$n = 20,\ f = (0, 10, 19, 9, 18, 8, 17, 7, 16, 6, 15,\ 5, 4,\ 14, 3, 13, 2, 12, 1, 11),\ N \ge 21$; etc.

Let us note that these pseudo-stars and the previous ones generate geometrically the same cycles in $P_N$ but the numerations are different because of a shift of the origin.

# 3 Normal cycles do not exist if $N = n = 2i$

If $N = n$ then $f$ is a bijection. Let us assume that $N = n$ is even and let us show that for every mapping $f$ at least one pair of parallel edges exists in $C_n$.

Let us assume indirectly that there is a sequence $f = (f(0), f(1), ..., f(n-1))$ with no parallel edgesl. In other words, all $n$ sums $f(0)+f(1)$, $f(1)+f(2)$, ..., and $f(n-1)+f(0)$ are pairwise different. Then these sums must take all the possible values $\pmod n$, and thus

$$(f(0)+f(1)) + (f(1)+f(2)) + ... + (f(n-1)+f(0)) = 2\sum_{i=0}^{n-1} f(i) = \frac{n(n-1)}{2} \pmod n.$$

On the other hand, the sequence $f(0)$, $f(1)$, ..., $f(n-1)$ also takes all the values $0, 1, ..., n-1$ because $N = n$ and $f$ is a bijection. Thus we obtain the equality

$$2\sum_{i=0}^{n-1} f(i) = \sum_{i=0}^{n-1} f(i) \pmod n,$$

which can hold only if $n$ is odd. Namely, if $n$ is even then

$$0 = 2\sum_{i=0}^{n-1} f(i) \neq \sum_{i=0}^{n-1} = \frac{n}{2} \pmod n.$$

For example, $2(0+1+2) = 0+1+2 = 0 \pmod 3$ but $0 = 2(0+1+2+3) \neq 0+1+2+3 = 2 \pmod 4$.

# 4 Normal cycles for the case $N = 2j+1$ and $n = 2j$, $j \geq 2$

In these cases the following cycles are normal:

$n = 4, N = 5,\ f = (1, 2,\ 4, 3)$;

$n = 6, N = 7,\ f = (1, 3, 2,\ 6, 4, 5)$;

$n = 8, N = 9,\ f = (1, 3, 4, 2,\ 8, 6, 5, 7)$; etc.

$n = 10, N = 11,\ f = (1, 3, 5, 4, 2,\ 10, 8, 6, 7, 9)$; etc.

Let us note that in all these normal cycles vertex 0 is missing and the sums $f(i)+f(i+1) \pmod N$ take all the values $1, ..., N-1$ except 0.

Namely, if $N-1=n=2j$ and vertex $a$ is the missing vertex of $P_N$ in a normal cycle $C_n$, then the sums $f(i)+f(i+1) \pmod N$ take all the values $0,1,...,N-1$, except $b=2a \pmod N$. Indeed,

$$F=\sum_{i=0}^{N-1} i = \frac{N(N-1)}{2} = 0 \pmod N,$$

because $N$ is odd. Furthermore, $f(0)+f(1)+...+f(n-1)=F-a$ and

$$(f(0)+f(1))+(f(1)+f(2))+...+(f(n-1)+f(0)) = 2F-2a = F-b \pmod N.$$

Thus $F=2a-b=0 \pmod N$ and hence $b=2a \pmod N$ follows.

# 5 Conclusions

Our case analysis shows that normal cycles exist for all the pairs $(N,n)$ such that $N \geq n \geq 3$, except if $N=n=2i$ for $i \geq 2$.