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A short Note on Linear Autarkies, q-Horn Formulas and the Complexity Index

by

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ABSTRACT

It is shown that the tractable class of CNF formulas solvable by Linear Autarkies properly contains the class of q-Horn formulas and that it is incomparable with SLUR.

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1 Introduction

The reader is supposed to be familiar with the basic terminology and facts regarding the Satisfiability Problem (see [7] for a survey).

Some well known tractable classes of Satisfiability Problems are known as Horn [5], renamable Horn [9], extended Horn [3], balanced [4, 12], SLUR (Single Lookahead Unit Resolution) [10], the latter class *including* the preceding ones, and q-Horn [2, 1, 11]. The class of q-Horn formulas includes (renamable) Horn and 2-SAT.

Franco [6] has shown that SLUR and q-Horn are incomparable extensions of Horn.

In this note we prove that another tractable class of formulas which was introduced by Kullmann [8], the class of formulas solvable through *Linear Autarkies* (LinAut for short), properly contains q-Horn but is incomparable with SLUR. As corollaries to our main lemma we also include a few results on the Complexity Index [1].

2 q-Horn formulas

We shall use the characterization of q-Horn formulas by means of the Complexity Index of [1]. However, we choose our formulations in such away that they apply to Boolean values -1 and +1 of the propositional variables, rather than the more customary 0 and 1. The concept of Linear Autarky is easier to understand in this way, because the description of the underlying linear algebra is more transparent in doing so.

The *clause-variable matrix* A associated to a CNF formula Φ is the matrix defined by

$$A_{i,j} = \begin{cases} 1 & \text{if the } i^{th} \text{ clause of } \Phi \text{ contains the } j^{th} \text{ variable with sign 1 (non negated)} \\ -1 & \text{if the } i^t h \text{ clause of } \Phi \text{ contains the } j^{th} \text{ variable with sign } -1 \text{ (negated)} \\ 0 & \text{otherwise.} \end{cases}$$

Thus if Φ is a CNF formula with propositional variables $x_1, \ldots, x_n$ the Satisfiability Problem for Φ reads as the $-1, 1$ Feasibility Problem

$$\begin{cases} Ax \geq -L + 2e \\ x \in \{-1, 1\}^n \end{cases}$$

In the above, L is the *length vector*, having the length of clause i as its i^{th} entry, and e is the all one vector of appropriate dimension.

The *Complexity Index* of a formula with clause-variable matrix A is the solution to the *Linear Program*

$$\begin{cases} \min Z \\ Ax \geq L - 2Ze \\ x \in [-1, 1]^n \end{cases}$$

(after substituting $\alpha_j = \frac{1}{2}(1 - x_j)$ in the definition of [1]).

A formula is q-Horn if its Complexity Index $Z \leq 1$.

3 Linear Autarkies

After Kullmann [8] a formula with clause-variable matrix A has a Linear Autarky $x \epsilon Q^n$ if

$$\begin{cases} Ax \geq 0 \\ x \neq 0. \end{cases}$$

The above concept generalizes an earlier version of Warners and van Maaren [13] which provides a decomposition of a formula in case the cernel of its clause-variable matrix is non zero.
A so called *monotone variable* is the most well known example of a Linear Autarky: If variable x_j appears only positive (negative) in the formula, the vector $x = e_j$ ($x = -e_j$), (e_j is the j^{th} unit vector), is a Linear Autarky.
In general, a Linear Autarky x leads to an autark partial assignment of the formula involved by rounding. To see this, let the partial assignment **sign** be

$$\mathbf{sign}(x_j) = \begin{cases} 1 & \text{if } x_j > 0 \\ -1 & \text{if } x_j < 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

and substitute all defined **sign** (x_j) into the formula. If a clause i is affected by this substitution, that is, if a variable with defined **sign** occurs in clause i, this clause is obviously satisfied by the partial assignment since

$$\sum_j A_{ij} x_j = \sum_{x_j \neq 0} A_{ij} x_j \geq 0$$

and hence not all $A_{ij} x_j$ with defined **sign** (x_j) can be negative.

The above means that a Linear Autarky leads to a non trivial decomposition of the formula involved. One part is satisfied by **sign** and the other part contains only variables with undefined **sign**. Clearly, the latter part is Satisfiability Equivalent to the original formula: it is satisfiable if and only if the original formula is. Kullmann [8] shows, among other things, the following :

(i) Linear Programming guarantees a polynomial time search for Linear Autarkies.

(ii) By repeatedly applying the above decomposition, a formula is Satisfiable Equivalent to a Linear Autarky-free subformula. This subformula is unique and computable in polynomial time.

(iii) A (renamable) Horn formula without unit-clauses is completely solved by (ii), that is, the resulting subformula is empty.

(iv) Every *satisfiable* 2-SAT formula is completely solved by (ii) (notice that in this case a satisfying assignment is in fact a Linear Autarky). In other words, if the resulting Linear Autarky-free subformula is non empty, the formula is unsatisfiable.

In the following, a formula is said to be in the class LinAut if the procedure described in (ii) above results in a complete assignment (the satisfiable case), or in a Linear Autarky -free 2-SAT subformula (the unsatisfiable case).
It is also assumed that CNF formulas have *no* unit-clauses.

4 The main Lemma and its Corollaries

Lemma: Suppose a CNF formula has Complexity Index Z and no Linear Autarkies. Then either

(a) For some clause i (with length L_j) we have $Z > \frac{1}{2}L_i$, or

(b) All clauses have the same length $L = 2Z$.

Proof: If (a) is not true we clearly have $L - 2Ze \geq 0$. Hence by definition of the Complexity Index we have $Ax \geq L - 2Ze \geq 0$ for some $x \in [-1, 1]^n$. Since no Linear Autarkies exist we must have $x = 0$ which yields in turn $L = 2Ze$. This proves the lemma.

Corollary: A q-Horn formula without unit-clauses is in the class LinAut.

Proof: A q-Horn formula has Complexity Index $Z \leq 1$. The unique Linear Autarky-free subformula of such a formula has also Complexity Index $Z' \leq 1$. Since no unit-clauses are present, case (a) of the lemma clearly not applies. Consequently, we see that the subformula is either empty, in which case the original formula has been solved completely and is satisfiable, or this subformula is in fact an unsatisfiable 2-SAT formula.

Corollary: LinAut properly includes q-Horn.

Proof: The formula with clause-variable matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$$

is in LinAut since it has a Linear Autarky $x = (1, -1, 0)$ which solves the formula at hand. It is clearly not q-Horn because

$$\begin{cases} x_1 + x_2 + x_3 & \geq 3 - 2 \\ -x_1 - x_2 - x_3 & \geq 3 - 2 \end{cases}$$

are contradictory.

Franco [6] gives an example (example 2.11 of that paper) of a formula which is q-Horn but not SLUR. In [10] the formula with clause-variable matrix

$$B = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

is used to show that SLUR properly contains extended Horn. The same example shows that SLUR and LinAut are incomparable, since $Bx \geq 0$ implies $x = 0$.
The following corollaries easily follow from the main Lemma.

Corollary: A k-SAT formula without Linear Autarkies has Complexity Index $Z = \frac{1}{2}k$.

Proof: If all clauses have length k the Complexity Index Z clearly satisfies $Z \leq \frac{1}{2}k$ since $A0 \geq (k - 2 \cdot \frac{1}{2}k)e$. By the lemma now it follows that $Z = \frac{1}{2}k$.

Corollary: Let k be the length of the shortest clause of a CNF formula. Then if $Z < \frac{1}{2}k$ the formula is satisfiable.

Proof: Consider the unique Linear Autarky-free subformula. Since Z decreases and k increases considering subformulas, the same assumptions hold for this subformula. By the Lemma, all clauses must have length $2Z$, whence $Z = \frac{1}{2}k$ according to the previous corollary. Thus the subformula at hand is empty.

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