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Characterizations of Consistent Marked Graphs

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ABSTRACT

A *marked graph* is a graph with a + or − sign on each vertex and is called *consistent* if each cycle has an even number of − signs. This concept is motivated by problems of communication networks and social networks. We present some new characterizations/recognition algorithms for consistent marked graphs.

1 Introduction

A graph is called *marked* if every vertex has a sign + or − and a marked graph is called *consistent* if every cycle has an even number of − signs. The concept of consistency is analogous to the concept of balance in signed graphs, graphs with a + or − sign on each *edge*. A signed graph is called *balanced* if every cycle has an even number of − signs. (For a variety of references on balance, see Johnsen [11] and Roberts [13].) In this paper, we give a number of new characterizations of consistency of marked graphs.

Consistent marked graphs were introduced by Beineke and Harary [3]. An analogous concept for marked digraphs was introduced by Beineke and Harary [4]. The concept was motivated by communication networks. If binary messages are sent through a network with vertices having sign − reversing messages and vertices having sign + leaving them unchanged, then a consistent marked graph has the following consistency property: If a message is sent from x to y through two different vertex-disjoint paths and x and y have the same sign, then y will receive the same message no matter which path is followed. If G is a marked graph, let us say that the *sign* of a path or cycle is + if there is an even number of − signs and − otherwise. It should be noted that even for consistent marked graphs, even if x and y have the same sign, there can be two paths from x to y of different signs – the hypothesis of vertex disjointness is needed to conclude that they have the same sign. Consistent marked graphs arise in a similar way in social networks, networks whose vertices are people. If some people always lie and some always tell the truth, a consistent social network has the property that if a message is sent from x to y and they have the same sign, then y will receive the same message independent of the path followed.

The notion of consistency of a marked graph has proven useful in the theory of balance of signed graphs. Harary and Kabell [6,7] were able to describe an efficient algorithm for determining if a given signed graph is balanced by setting up a correspondence between marked graphs and balanced signed graphs. This correspondence has also been useful in solving the problem of counting balanced signed graphs (Harary and Kabell [7]). (The problems of enumerating both balanced signed graphs and marked graphs are also studied by Harary, Palmer, Robinson, and Schwenk [8].) The problem of characterizing consistent marked digraphs was solved by Beineke and Harary [4]. Rao [12] obtained an early characterization of consistent marked graphs and also gave a polynomial algorithm for recognizing them. Acharya [1,2] also gave characterizations. Hoede [10] characterized consistent marked graphs in terms of fundamental cycles of a cycle basis and observed that the characterization gives rise to a polynomial algorithm for determining whether a marked graph is consistent that is considerably simpler than that of Rao. Here, we shall give a more efficient algorithm than Hoede's. Beineke and Harary [4] introduced the problem of determining if an unmarked digraph can be marked using at least one − sign so that the resulting marked digraph is consistent. Roberts [14] studied this problem for marked graphs.

In this paper, we shall use the following notation in a (marked) graph G: V_+ is the set of + vertices, V_- the set of − vertices, n the number of vertices, m the number of edges. If S is a set of vertices or edges of G, then $G[S]$ is the subgraph induced by S. If T is a tree in G, then the unique path in T between vertices x and y is denoted by T_{xy}. A special case of this notation is P_{xy} when the tree is a path P. We say that vertices x and y in G are *k-connected* (*k-edge-connected*) if there are k vertex-disjoint (edge-disjoint) paths between x and y.

Cycle bases play a central role in the characterization of consistent marked graphs. We say that a set of cycles in a graph G is a *cycle basis* if it is a minimal set with the property that every cycle can be expressed as a symmetric difference of cycles in the set. One well known way to construct a cycle basis is to find a spanning tree T of G. A *tree chord* relative to T is an edge not in T. Every tree chord defines a unique cycle of G, called a *fundamental cycle*, whose remaining edges are in T. The set of fundamental cycles is well-known to be a cycle basis for G.

2 Characterizations of consistent marked graphs

The notion of 3-connectedness between a vertex pair plays a central role in studying marked graphs, as is shown in the following theorem due to Beineke and Harary.

Theorem 1.[3] *If a marked graph G is consistent and $u, v \in V(G)$ are 3-connected, then u and v receive the same sign.*

Let G be a marked graph and $x, y \in V(G)$ with the same sign. If they are positive, a path joining x and y is said to be *coherent* if it contains an even number of negative vertices and is said to be *incoherent* otherwise. If they are negative, a path joining x and y is said to be *coherent* if it contains an odd number of negative vertices, and is said to be *incoherent* otherwise.

We have the following two theorems.

Theorem 2. *A marked graph G is consistent if and only if for any two vertices u and v with the same sign, any two vertex disjoint paths between u and v are either both coherent or both incoherent.*

Proof. Necessity is obvious.

For sufficiency, let C be a cycle of G. C has at least three vertices and at least two of them have the same sign, say u and v. u and v separate C into two vertex disjoint paths. By hypothesis, these two paths are both coherent or both incoherent. Therefore C is positive.

Q.E.D.

Theorem 3. *Suppose that G is a 2-connected marked graph that is not a cycle. Then G is consistent if and only if for any 3-connected pair u and v, they receive the same sign and paths between them either are all coherent or are all incoherent.*

Proof. *Necessity.* By Theorem 1, u and v receive the same sign. Let P^1, P^2 and P^3 be three vertex disjoint paths from u to v. Since G is consistent, P^1, P^2 and P^3 are all coherent or are all incoherent. Suppose they are all coherent. The case where they are all incoherent is similar. Suppose there is also a path P from u to v which is incoherent. We choose P such that $|V(P)\backslash V(P^1 \cup P^2 \cup P^3)|$ is the minimum. Clearly because P is incoherent, P must have common vertices with $V(P^1 \cup P^2 \cup P^3)\backslash\{u, v\}$, otherwise G is not consistent.

Suppose P and P^1 have common vertices. Let x and y be two vertices on both P and P^1 such that at least one of x, y is not u or v, and vertices between x and y on P^1 are not on P. We now note that there are three vertex-disjoint paths from x to y, one of which is P^1_{xy}. If at most one of $V(P^2)$ and $V(P^3)$ intersects $V(P_{xy})$, then the other two paths are P_{xy} and the path obtained by P_{xu}, P^2 or P^3 from u to v, then P_{vy}. Suppose both $V(P^2)$ and $V(P^3)$ intersect $V(P_{xy})$ and on P_{xy}, the first vertex on $V(P^2) \cup V(P^3)$ is a and the last is b. Without loss of generality, $a \in V(P^2)$. Now if b is also in $V(P^2)$, then the other two paths are P_{xa}, P^2_{ab}, P_{by} and P_{xu}, P^3_{uv}, P_{vb}. If b is in $V(P^3)$, then the other two paths are P_{xa}, P^2_{av}, P_{vb}, and P_{xu}, P^3_{ub}, P_{by}. Therefore, x and y are 3-connected and hence they receive the same sign.

The paths P_{xy} and P^1_{xy} form a cycle. Since G is consistent, the cycle is positive and the two paths P^1_{xy} and P_{xy} either are both coherent or both incoherent. Therefore if we replace P_{xy} by P^1_{xy} on P to get P', then P' is also incoherent since P is, but $|V(P')\backslash V(P^1 \cup P^2 \cup P^3)| < |V(P)\backslash V(P^1 \cup P^2 \cup P^3)|$, which is a contradiction.

Sufficiency. Let C be a cycle of G. Since G is 2-connected and not a cycle, there are two vertices u and v on C which are 3-connected. Then u and v receive the same sign and two paths on C from u to v either are both coherent or both incoherent, which implies that C is positive.

Q.E.D.

The following theorem is given by Acharya [2] and Rao [12].

Theorem 4.[2,12] *If marked graph G is consistent, then*

1. $G[V_-]$ is bipartite;

2. Between each connected component of $G[V_+]$ and each set in the bipartition of each component of $G[V_-]$, there is at most one edge.

Let G be a graph and $A \subseteq V(G)$. We use the terminology "*shrink A to a single vertex a*" to mean to replace A by a single vertex a to get graph G' and to take $av \in E(G')$ if and only if there is an edge in G between v and some vertex of A. If $G[A]$ is connected and bipartite with two sets C and D, "*shrink A to a single edge cd*" will mean to shrink C and D to two vertices c and d respectively. The terminology "*subdivide an edge*" will mean to insert a vertex in the edge.

As indicated in [3], if a marked graph G satisfies conditions 1 and 2 of Theorem 4, we can shrink each component of $G[V_+]$ into a single vertex to get graph G'. Then G is consistent if and only if G' is. Actually we can also shrink each component of $G[V_-]$, if it is not a vertex, into a single edge to get graph G'. Then G is consistent if and only if G' is.

Theorem 5. *Let G be a marked graph satisfying conditions 1 and 2 of Theorem 4. We shrink each component of $G[V_-]$, if it is not a single vertex, into a single edge to get graph G'. Then G is consistent if and only if G' is.*

Proof. Suppose $u, v \in V_-$ are in the same set in the bipartition of a component of $G[V_-]$, u, v have a common neighbor $x \in V_-$ and G' is obtained from G by shrinking u and v into a vertex w. We prove that there is a negative cycle in G if and only if there is a negative cycle in G'. This will prove the theorem, since by iterating the procedure, we shrink each part of each component of $V[G_-]$ into a single vertex and each component of $V[G_-]$ into a single edge.

Assume that G has a negative cycle C. If C contains at most one of u, v, then clearly G' has a corresponding negative cycle. Suppose both u and v are on C, say $C = u, u_1, ..., u_r, v, v_1, ..., v_s, u$. If either r or $s = 1$, suppose without loss of generality that $r = 1$. Since u and v have no common positive neighbor by condition 2, u_1 is a negative vertex. After shrinking, we get $C' = w, v_1, ..., v_s, w$, with two negative vertices less than C. If s is also 1, then C negative implies that v_1 is positive, which violates condition 2. Therefore, $s > 1$ and so C' is a negative cycle. Now suppose both $r, s \geq 2$. Then one of the two cycles $C_1 = w, u_1, ..., u_r, w$ and $C_2 = w, v_1, ..., v_s, w$ must be negative.

Now assume that G' has a negative cycle C'. If it does not contain the new vertex w, then C' is also a negative cycle in G. The conclusion holds. Suppose C' contains w, say $C' = ww_1...w_tw$. There are two cases. Suppose first that the following condition **P** holds: either both w_tu and uw_1 or both w_tv and vw_1 are in G, then there is a corresponding negative cycle in G. Note that if x is w_1 or w_t, the condition is satisfied. If condition **P** fails, we know that uw_1 or vw_1 and uw_t or vw_t are in G. If uw_1 is in G (which we may assume without loss of generality), then uw_t, w_1v are not in G. If x is not on C' in G', then $C = uw_1...w_tvxu$ is a negative cycle of G. If x is on the cycle C', say $C' = ww_1...w_rxw_{r+2}...w_tw$, then there are two cycles $C_1 = uw_1...w_rxu$ and $C_2 = xw_{r+2}...w_tvx$ in G and one must be negative.

Q.E.D.

Suppose G is a marked graph, $G[V_-]$ is bipartite, and G' is obtained from G by shrinking each component of $G[V_+]$ into a single vertex and shrinking each component of $G[V_-]$, if it is not a single vertex, into a single edge. The structure of G' is fairly simple. $G'[V'_+]$ is an independent set and $G'[V'_-]$ is a matching together with some isolated vertices.

Furthermore, as indicated by Hoede [10], if we subdivide each edge of $G'[V_-]$ and give $+$ to all inserting vertices to get graph G'', then G' is consistent if and only if G'' is consistent.

Noticing that G'' is bipartite with one part of positive vertices and another part of negative vertices, we have the conclusion that G'' is consistent if and only if it only contains cycles with length $4k$. In other words, after polynomial time translating, the problem of checking if a marked graph is consistent reduces to the problem of checking if a bipartite graph only contains cycles with length $4k$.

Another way to see this is as follows. Let G be a marked graph. We insert a positive vertex in any edge incident to two negative vertices and three vertices with signs $-, + -$ in that order in any edge incident to two positive vertices to get graph $T(G) = G''$. Then $T(G)$

is bipartite with one part of positive vertices and another part of negative vertices, and G is consistent if and only if $T(G)$ only has cycles with length 0 mod 4. To prove this, we use induction on $s(G) = m(G) - q(G)$, where $m(G)$ is the number of edges of G and $q(G)$ is the number of edges of G joining vertices of different sign. If $s(G) = 0$, then $T(G) = G$. Since every cycle of G alternates $-$ and $+$ signs, G is consistent if and only if the length of each cycle is congruent to 0 mod 4. By induction, if $s(G) > 0$, let $W(G)$ be obtained from G by adding vertices on one edge according to the procedures in the definition of $T(G)$. Note that G is consistent if and only if $W(G)$ is consistent, $T(W(G)) = T(G)$, and $s(W(G)) < s(G)$. Thus, by inductive hypothesis, G is consistent if and only if $W(G)$ is consistent if and only if $T(W(G))$ has all cycles of length 0 mod 4 if and only if $T(G)$ has all cycles of lengths 0 mod 4.

In fact, it is easy to see that the problem of determining if a marked graph is consistent is equivalent to the problem of determining if a bipartite graph has all cycles of length 0 mod 4. The polynomial reduction from the second problem to the first goes as follows. Let G be a bipartite graph. Assign $+$ signs to vertices of one bipartite class and $-$ signs to vertices of the other class, getting a marked graph H. Then every cycle of G has length 0 mod 4 if and only if every cycle of H has an even number of $-$ signs, i.e., if and only if H is consistent.

To summarize:

Theorem 6. *The problem of determining if a marked graph is consistent is polynomially equivalent to the problem of determining if a bipartite graph has all cycles of length 0 mod 4.*

Another way to characterize a consistent marked graph is due to Hoede.

Theorem 7.[10] *Let G be a marked graph and T be a spanning tree of G. G is consistent if and only if G satisfies the following two conditions.*

1. Each fundamental cycle relative to T is positive.

2. On each common path of a pair of fundamental cycles relative to T, the two end vertices have the same sign.

Theorem 7 provides a polynomial algorithm to check if a marked graph is consistent. There are $m - n + 1$ fundamental cycles relative to T in total, where m is the number of edges of the graph and n the number of vertices. Hence there are $O(m^2)$ pairs of cycles to check for the condition 2. For each pair, the checking time is $O(n)$. The complexity to check condition 1 is $O(mn)$. Therefore the complexity of Hoede's algorithm is $O(m^2 n)$. Algorithm 1 in Section 3 gives an $O(mn)$ algorithm.

The following Theorem is equivalent to Theorem 7.

Theorem 8. *Let G be a marked graph and T be a spanning tree. G is consistent if and only if G satisfies the following two conditions.*

1. Each fundamental cycle relative to T is positive.

2. Each cycle which is symmetric difference of two fundamental cycles relative to T is *positive.*

Proof. We prove that under condition 1, condition 2 in the two theorems are equivalent.

Let C be a symmetric difference of two fundamental cycles C_1, C_2, where C_1 has a negative signs, C_2 has b negative signs, and the common path has c negative signs (exclude end vertices). Then C has $a+b-2c$ negative signs if end vertices have the $+$ sign or $a+b-2c-2$ negative signs otherwise. Therefore a and b are even implies C is positive. The other direction is similar.

Q.E.D.

Because of Theorem 6, the following theorem, due to Conforti and Rao [5], can be thought of as the bipartite version of Theorem 8.

Theorem 9. [5] *Let G be a bipartite graph and T be a spanning tree of G. All cycles in G have length 0 mod 4 if and only if G satisfies the following two conditions.*

1. Each fundamental cycle relative to T has length 0 *mod* 4.

2. Each cycle which is symmetric difference of two fundamental cycles relative to T has length 0 *mod* 4.

Theorem 9 provides an algorithm with complexity $O(m^2n)$ to determine if all cycles in a graph G have length 0 mod 4. In Section 3 we will provide two $O(mn)$ algorithms (Algorithms 2 and 3) to complete the same task.

Theorem 10. *Let G be a marked graph. The following three statements are equivalent.*

1. Let T be a spanning tree of G. Each common path of a pair of fundamental cycles relative to T has end vertices with the same sign.

2. Each 3-connected vertex pair has the same sign.

3. Each 3-edge-connected vertex pair has the same sign.

The proof of Theorem 10 is easily obtained from two lemmas which we state and prove next.

Lemma 1. *Let T be a spanning tree of graph G and the vertex pair u, v be 3-connected in G. Then there exist vertices $v_1, v_2, ..., v_k$ such that for any of the pairs $\{u, v_1\}, \{v_i, v_{i+1}\}, i = 1, ..., k-1, \ldots, \{v_k, v\}$, there are two fundamental cycles relative to T with the pair as the end vertices of their common path.*

Proof. For any two 3-connected vertices u and v, we define $\sigma(u, v)$ to be the smallest number k so that there are three pairwise vertex-disjoint paths in G from u to v with k tree chords on these paths. We now use induction on $\sigma(u, v)$ to prove this theorem.

Notice that if P^1, P^2 and P^3 are three disjoint paths from u to v, at least two of them contain tree chords. If $\sigma(u, v) = 2$, then u, v themselves are end points of common paths of two fundamental cycles. The conclusion holds with $\sigma(u, v) = 2$.

Assume the conclusion holds for all 3-connected vertex pairs u, v with $\sigma(u, v) \leq k$ and we consider a 3-connected vertex pair x, y with $\sigma(x, y) = k + 1$. Let P^1, P^2 and P^3 be three pairwise disjoint paths from x to y such that the total number of tree chords on these paths is $\sigma(x, y) = k + 1$.

There is a unique path P on T joining x and y. Suppose edge xw is on P. We claim xw is on one of P^1, P^2 and P^3. Suppose not. Let z be the first vertex on P from x to y and on one of P^1, P^2 or P^3. Assume z is on P^1. We replace P^1_{xz} by P_{xz} to get three paths with fewer tree chords, which contradicts the definition of P^1, P^2 and P^3.

Suppose without loss of generality that xw is on P^2. Then P^1 and P^3 must contain tree chords. Let the first tree chords on P^1 and P^3 from x to y be e' and e'' respectively.

The removal of edge xw divides T into two parts and the two tree chords e' and e'' join these two parts. Consider two fundamental cycles with tree chords e' and e'' respectively. One end point of their common path is x. Let the other one be u. We claim that u, y are 3-connected and $\sigma(u, y) \leq k$, which will complete the proof.

Case 1. u is on P^2. In the fundamental cycle containing e'', suppose the path from u to P^3 hits P^3 at b. Then there are three paths from u to y, i.e., $P^2_{uy}, P^2_{ux}P^1$ and $T_{ub}P^3_{by}$. Since the numbers of tree chords on P^2 and P^2_{uy} are the same, on P^1 and on $P^2_{ux}P^1$ are the same and on $T_{ub}P^3_{by}$ is less than on P^3, we have $\sigma(u, y) \leq k$. See Figure 1 (a).

Case 2. u is not on P^2. In the common path on the two fundamental cycles, let b be the last vertex on P^2. In the fundamental cycle containing e', suppose the path from u to P^1 hits P^1 at a, and in the fundamental cycle containing e'', suppose the path from u to P^3 hits P^3 at c. Then $T_{ua}P^1_{ay}, T_{ub}P^2_{by}$ and $T_{uc}P^3_{cy}$ are three paths from u to y. By similar reasoning as in Case 1, we have $\sigma(u, y) \leq k$. See Figure 1 (b).

Q.E.D.

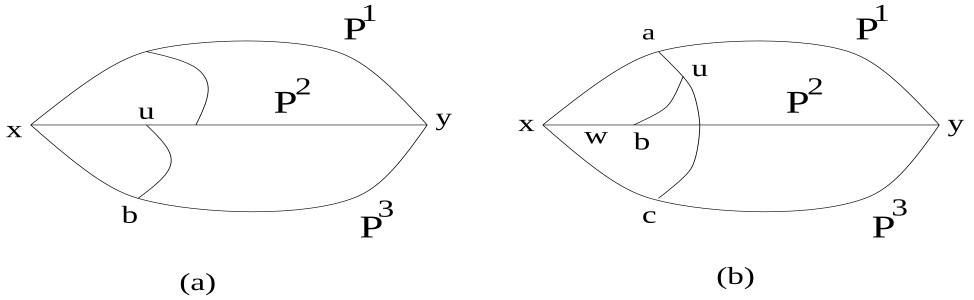

Figure 1: Illustration of Proof of Lemma 1

Similar to Lemma 1, we can get the following lemma.

Lemma 2. *Let G be a graph and u, v be 3-edge-connected. Then there exist vertices $v_1, v_2, ..., v_k$ such that each pair of $\{u, v_1\}, \{v_i, v_{i+1}\}, i = 1, ..., k-1, \{v_k, v\}$ is 3-connected.*

Proof. If u and v are 3-connected, the conclusion obviously holds. So we suppose u and v are not 3-connected.

We select three edge disjoint paths P^1, P^2 and P^3 from u to v such that the total lengths of them are minimum.

Define $X = \{x :$ at least two of P^1, P^2, P^3 contain $x, x \neq u, x \neq v\}$. Then $X \neq \emptyset$. Let $X = \{v_1, ..., v_k\}$. We claim that X has the following property:

If two paths, say P^1 and P^2, each contain vertices $v_i \in X$ and $v_j \in X$, and v_i is before v_j on path P^1 from u to v, then v_i is before v_j on path P^2 from u to v.

Suppose the claim is not true, and v_i is before v_j on P^1 and v_j is before v_i on P^2. Then $\overline{P^1} = P^1_{uv_i}P^2_{v_iv}$, $\overline{P^2} = P^2_{uv_j}P^1_{v_jv}$ and P^3 are three edge disjoint paths with total length less than the original ones, which shows our claim is true.

Let $v_0 := u$, $v_{k+1} := v$ and $X' = X \cup \{u, v\}$. We can arrange vertices in X' in order according to their appearance on three paths from u to v. Suppose the set in order is $X' = \{v_0, v_1, ..., v_k, v_{k+1}\}$. We claim that

Any two vertices v_i and v_{i+1} are 3-connected, $i = 0, ..., k$.

For $i = 0$, if v_1 is on all three paths, we prove that then v_0 and v_1 are 3-connected. Suppose v_1 is on P^1 and P^2 but not on P^3. There is a vertex $v_i, i > 1$, on P^3 and P^1 since v is on P^3 and P^1. Take the smallest i. Then $P^1_{v_0v_1}, P^2_{v_0v_1}$ and $P^3_{v_0v_i}P^1_{v_iv_1}$ are three vertex disjoint paths from v_0 to v_1. The case $i = k$ is analogous.

Consider the case $0 < i < k$. If two vertices v_i and v_{i+1} are both on three paths, they are 3-connected. If one is on three paths, the proof is just as above, so we may assume neither is on three paths. Now suppose both v_i and v_{i+1} are on P^1 and P^2. Let v_s be on P^1 and P^3 with the biggest $s < i$ and v_t be on P^1 and P^3 with the smallest $t > i + 1$. Then $P^1_{v_iv_{i+1}}, P^2_{v_iv_{i+1}}$ and $P^1_{v_iv_s}P^3_{v_sv_t}P^1_{v_tv_{i+1}}$ are three vertex disjoint paths from v_i to v_{i+1}. Suppose v_i is on P^1 and P^2 and v_{i+1} is on P^2 and P^3. Let v_s be on P^1 and P^3 with the biggest $s < i$ and v_t be on P^1 and P^3 with the smallest $t > i+1$. Then $P^2_{v_iv_{i+1}}, P^1_{v_iv_s}P^3_{v_sv_{i+1}}$ and $P^1_{v_iv_t}P^3_{v_tv_{i+1}}$ are three vertex disjoint paths from v_i to v_{i+1}.

Q.E.D.

To complete the proof of Theorem 10, note that Lemma 1 shows that 1 implies 2 and Lemma 2 that 2 implies 3. Since each common path of a pair of fundamental cycles has end vertices that are 3-edge-connected, 3 implies 1.

By Theorem 10, the following theorem is obviously equivalent to Theorem 7. However, we give a direct proof of one direction without using Hoede's conclusion. (The other direction is obvious.)

Theorem 11. *Let T be a spanning tree of a marked graph G. G is consistent if and only if it satisfies the following two conditions.*

1. *Each fundamental cycle relative to T is positive.*
2. *Each 3-connected vertex pair has the same sign.*

Proof. To prove necessity, note that condition 1 is obvious and condition 2 follows from Theorem 1. To prove sufficiency, we prove that every cycle is positive by induction on the number of chords, edges joining two non-consecutive vertices on the cycle. If the cycle only contains one chord, then it is a fundamental cycle and therefore is positive. Suppose all cycles with $\leq k$ chords are positive, and we consider a cycle C with $k+1$ chords. Since any two vertices on C are connected in T, we can find two vertices u, v on C such that T_{uv} only has u, v in common with C and both paths on C from u to v, which are denoted by C^1_{uv} and C^2_{uv}, contain chords. Now both cycles $C^1_{uv}T_{vu}$ and $C^2_{uv}T_{vu}$ have fewer chords than C. By inductive hypothesis, they are both positive. Also u, v are 3-connected and they have the same sign, which implies that C is positive.

Q.E.D.

To generalize Theorem 11, we ask: Can we replace "each fundamental cycle relative to T " by "each cycle in a basis", i.e., replace tree basis by any basis.

Let G be a graph. We can get a basis by repeating the following process until we can no longer do so. Pick an edge e. If there is a cycle containing e, put the cycle in the basis. Delete e. A basis which is obtained by this method is called an *ordering basis.* (In [9], it is called a *fundamental basis.*) Note that a tree basis is an ordering basis. We first note that Theorem 11 holds for ordering bases.

Theorem 12. *Let G be a marked graph and $\mathcal{B}$ be an ordering basis of G. G is consistent if and only if G satisfies the following two conditions.*

1. Each cycle in $\mathcal{B}$ is positive.

2. Each 3-connected vertex pair has the same sign.

Proof. Necessity of conditions 2 follows from Theorem 1. To prove sufficiency, we let cycles in $\mathcal{B}$ be $C_n, ..., C_m$ according to the order they are found in construction of the ordering basis and arrange edges such that $e_i, i = n, ..., m$, is the edge we are using to find C_i and $e_i, i = 1, ..., n-1$, are the other edges.

Let $G_i = G[\{e_1, ..., e_{n-1}\} \cup E(\cup^i_{j=n} C_j], i \geq n$. Note that $G_m = G$. We use induction on i. By condition 1, the conclusion that G_i is consistent holds for $i = n$. Suppose it holds for $i = k-1$ and now we want to prove it for $i = k$.

A *block* in a graph is a maximal 2-connected subgraph. We only have to prove that the block of G_k which contains edge e_k is consistent or we can suppose G_k itself is a block. There is a spanning tree T in G_{k-1} which contains $C_k \backslash e_k$. Since G_{k-1} is consistent, all fundamental cycles of G_{k-1} with respect to T are positive. T is also a spanning tree of G_k and since C_k is positive by condition 1, all fundamental cycles of G_k are positive. This plus condition 2 implies G_k is consistent, by Theorem 11.

Q.E.D.

We now want to generalize the conclusion of Theorems 11 and 12 to any basis. First for a graph G an *Eulerian subgraph* of G is a subgraph where each vertex has degree 2. An Eulerian subgraph is a union of some cycles. We now give a generalization of positive cycles. Let G be a marked graph and G' a subgraph. Denote by $d^-(v)$ the number of negative vertices adjacent to v in G' and $\lambda(G') = \Sigma_{v \in V(G')} d^-(v)$. Suppose G' is an Eulerian subgraph. Then G' is said to be *coherent* if $\lambda(G') \equiv 0 \bmod 4$ and *incoherent* if $\lambda(G') \equiv 2 \bmod 4$. Notice that $\lambda(G')$ is always even if G' is an Eulerian subgraph. We can also generalize the definition to paths. Suppose $P = v_1...v_k$ is a path and v_1 and v_k have the same sign. Then P is said to be *coherent* if $\lambda(P) \equiv 0 \bmod 4$ and *incoherent* if $\lambda(P) \equiv 2 \bmod 4$. Notice that $\lambda(P)$ is even if and only if v_1 and v_k have the same sign. This definition of coherent is the same as that in the beginning of this section.

The following theorem is now easy to prove.

Theorem 13. *A marked graph is consistent if and only if each of its Eulerian subgraphs is coherent.*

Now we can prove the following theorem.

Theorem 14. *Let G be a marked graph and $\mathcal{B}$ be a basis of G. Then G is consistent if and only if G satisfies the following two conditions.*

1. Each cycle in $\mathcal{B}$ is positive.

2. Each 3-connected vertex pair has the same sign.

Proof. Necessity of conditions 1 and 2 follows by the definition of consistency and Theorem 1. To prove sufficiency, we prove that if G and $\mathcal{B}$ satisfy conditions 1 and 2, then each Eulerian subgraph of G is coherent. Consistency of G follows by Theorem 13.

Since an Eulerian subgraph is a union of several cycles and each cycle is a symmetric difference of several cycles in $\mathcal{B}$, each Eulerian subgraph G' is a symmetric difference of several cycles in $\mathcal{B}$. We use induction on the number k of cycles.

$k = 1$. G' is just a cycle in $\mathcal{B}$ and is positive by condition 1.

Suppose that each Eulerian subgraph which is the symmetric difference of at most $k-1$ cycles in $\mathcal{B}$ is coherent. Suppose G' is an Eulerian subgraph which is the symmetric difference of G'' and C, where $C \in \mathcal{B}$ and G'' is an Eulerian subgraph which is the symmetric difference of $k-1$ cycles in $\mathcal{B}$ and hence is coherent by the induction hypothesis.

If $E(C) \cap E(G'') = \emptyset$, the conclusion holds. So we suppose $E(C) \cap E(G'') \neq \emptyset$. We first prove the conclusion for the case G'' is connected. Let $P = v_1, ..., v_k$ be a path of C such that $v_1, v_k \in V(G'')$ and $v_2, ..., v_{k-1} \notin V(G'')$. If v_1 and v_k are in the same block of G'', then v_1 and v_k are 2-connected in the block and 3-connected in $G'' \cup P$. Therefore v_1 and v_k receive the same sign and $\lambda(P)$ is even.

Suppose v_1 and v_k are not in the same block of G''. Pick a path Q from v_1 to v_k in G'' and let $u_1, \ldots, u_s$ be the cut vertices of G'' in order on Q. Then v_1 and u_1, u_i and u_{i+1}, $i = 1, ..., s-1, u_s$ and v_k are pairwise 2-connected in G''. Vertices v_1 and u_1 are 3-connected in G because we can find a third path vertex-disjoint from the first two by using $Q_{u_1v_1}$ followed by P followed by $Q_{v_ku_1}$. The other cases are proved in the same way, using $Q_{u_{i+1}v_1}, P, Q_{v_ku_{i+1}}$ and $Q_{u_sv_1}, P, Q_{v_ku_s}$. Therefore v_1 and v_k receive the same sign and $\lambda(P)$ is even.

C is the union of some paths, which we can partition into two subsets, $C \cap G''$ and $C \backslash G''$. By the above conclusion, $\lambda(C \backslash G'')$ is even. Since $\lambda(C) \equiv 0 \bmod 4$, $\lambda(C \cap G'') \equiv \lambda(C \backslash G'')$ mod 4. G' is obtained from G'' by replacing $C \cap G''$ by $C \backslash G''$. Therefore $\lambda(G') \equiv \lambda(G'')$ mod 4, i.e., G' is coherent.

Now suppose that G'' is disconnected. If $E(C)$ intersects only one of the components of G'', the proof is the same as for the case where G'' is connected. So suppose that there are at least two components in G'', say G_1 and G_2, such that $E(C) \cap E(G_i) \neq \emptyset$ for $i = 1, 2$. Let $C = v_0 v_1, ..., v_k v_{k+1} ...$. Suppose that $P = v_1, ..., v_k$ such that v_1 and v_k are in the same component and v_0 and v_{k+1} are not in this component. Then v_1 and v_k are 3-connected in G' and therefore in G, and receive the same sign by condition 2. We can partition C into two subgraphs F_1 and F_2, where F_2 is a union of paths whose end vertices are in different components of G'' but whose middle vertices are not in G''. By the above discussion, $\lambda(F_1)$ is even and so therefore is $\lambda(F_2)$.

By the proof of the case when G'' is connected, $\lambda(F_1 \backslash G'')$ is even. Hence, so is $\lambda(C \backslash G'')$, since $\lambda(F_2)$ is even and $F_2 \cap G'' = \emptyset$, so

$$\lambda(C \backslash G'') = \lambda((F_1 \cup F_2) \backslash G'') = \lambda(F_1 \backslash G'' \cup F_2).$$

Moreover, $\lambda(C \cap G'') = \lambda(F_1 \cap G'')$ is also even since $\lambda(F_1)$ and $\lambda(F_1 \backslash G'')$ are even. Thus, since $\lambda(C) \equiv 0 \bmod 4$, it follows that $\lambda(C \cap G'') \equiv \lambda(C \backslash G'') \bmod 4$, as before, and then, as before, $\lambda(G') \equiv \lambda(G'') \bmod 4$, i.e., G' is coherent.

Q.E.D.

3 Algorithms

The above theorems provide an algorithm to test the consistency of a marked graph. By Theorems 10 and 14, it suffices to check if there is a positive basis and each 3-edge-connected vertex pair has the same sign. Checking the former can be done by constructing any basis and checking if it is positive, since if there is a negative cycle in that basis, then G is inconsistent. Checking the latter, it is sufficient to do it for vertices of degree at least 3 because, obviously, a vertex of degree at most 2 cannot be 3-edge connected to any other vertex.

Algorithm 1.
Input. A 2-connected marked graph G.
Output. The answer to the question: Is G consistent?
Step 1: *Construct a basis of G and check if it is positive.*
Step 2: *Let V_0 be the set of vertices of degree at least 3. Check if any two vertices of V_0 with different signs are 3-edge-connected. Do this as follows:*
Step 2a: *Pick two vertices from $V_+ \cap V_0$ and $V_- \cap V_0$, respectively. Check if they are 3-edge connected. To do so, use the shortest path algorithm to find a path between the two vertices. If there is such a path, modify the graph by eliminating its edges and find another path, and then a third path, if possible. If we can find the three paths, then the two vertices are 3-edge-connected, and, by Theorem 14, the marked graph is not consistent.*
Step 2b: *If we cannot find three paths, then there is an edge cut in the graph consisting of two edges, and this divides the graph into two subgraphs. Repeat Step 2a for each of these subgraphs.*

The complexity of Algorithm 1 is $O(mn)$. To see why, note that we can build a tree basis by finding a spanning tree T and then we can check if it is positive by checking if each tree chord produces a positive cycle. Well-known spanning tree algorithms such as Kruskal's are $O(m)$. There are $O(m)$ tree chords and, for each, since there are only $n-1$ edges in T, checking that the corresponding fundamental cycle is positive requires $O(n)$ steps. So, the overall complexity of Step 1 is $O(mn)$. (If we want to use ordering basis, we can do it as follows. Pick an edge xy. Use an $O(n)$ shortest path algorithm to find a path between x and y. If there is no such path, delete xy and continue. If there is a path, this plus xy forms a cycle of length at most n and we can check whether or not it is positive in $O(n)$ steps. Since we must repeat the procedure for each edge, the total complexity of the procedure is $O(mn)$.) Step 2a requires three uses of an $O(n)$ shortest path algorithm. It is easy to see by induction on n that the number of subgraphs for which Step 2a must be repeated in Step 2b is $O(n)$. Therefore the overall complexity of Algorithm 1 is $O(mn)$, as claimed.

As noted in Theorem 6, the problem of determining if a marked graph is consistent and the problem of determining if a bipartite graph has the property that every cycle has length 0 mod 4 are polynomially equivalent. In the following we give two algorithms to test if a bipartite graph has the property that every cycle has length 0 mod 4. We first give some definitions. Let G be a 2-connected graph. A *tripath* is a path whose inner vertices have degree 2 and whose end vertices have degree at least 3. Let G be a 2-connected graph and suppose that $\{uv, xy\}$ is an edge cut whose edges are not on the same tripath. We contract u and v into one vertex and insert a vertex in edge xy to get a graph G'. The operation is called *sliding* (relative to the first edge).

We first give the following lemmas.

Lemma 3. *If G is a graph and G' is obtained from G by sliding, then every cycle of G has length 0 mod 4 iff every cycle of G' has length 0 mod 4 and, moreover, G is bipartite iff G' is bipartite.*

Proof. It is easy to see there is a one to one correspondence between cycles of G and cycles of G' such that two corresponding cycles have the same length.

Q.E.D.

Lemma 4. *Let G be a graph. If u and v are 3-edge-connected, and so are v and w, then u and w are 3-edge-connected.*

Proof. If u and w are not 3-edge-connected, there is a minimal cut E_0 separating u and w with at most two edges. Suppose v is in the same part with u in $G \backslash E_0$. Then the cut is also a cut separating v and w, a contradiction.

Q.E.D.

We will need one other concept. Suppose G is 2-connected and V_0 is the set of vertices of degree at least 3. Then for a vertex v, there is a vertex u such that u and v are 3-edge connected if and only if $v \in V_0$. Lemma 4 implies that 3-edge connectedness defines an equivalence relation $\sim$ on V_0. We use the notation $r(G)$ for k, the number of equivalence classes.

Algorithm 2.

Input. A 2-connected bipartite graph G, which is not a cycle, with bipartition $V(G) = V' \cup V''$.

Output. The answer to the question: Does every cycle of G have length 0 mod 4?

Step 0. *Set* $k \leftarrow 0$, $V'_k \leftarrow V'$, $V''_k \leftarrow V''$, *and* $G^{(k)} \leftarrow G$.

Step 1: *Let* $V_0^{(k)}$ *be the set of vertices of* $G^{(k)}$ *with degree at least 3. Do the following for graph* $G^{(k)}$. *Check first if one of the vertex classes in the bipartition of* $G^{(k)}$ *has no vertices of* $V_0^{(k)}$. *If so, let* $G* = G^{(k)}$ *and* $V_0^* = V_0^{(k)}$ *and go to Step 3. Pick* $u \in V_0^{(k)} \cap V'_k$, $v \in V_0 \cap V''_k$, *and check if* u *and* v *are 3-edge connected. (Use a procedure analogous to that in Step 2a of Algorithm 1.) If they are 3-edge-connected, then because* u *and* v *are in different partite classes, we get the conclusion that some cycle of* G *does not have length 0 mod 4. Otherwise we find an edge cut* $\{e_1, e_2\}$ *with cardinality 2 separating* u *and* v. *Then edges* e_1 *and* e_2 *are on tripaths* P_1 *and* P_2 *respectively. Say* P_1 *goes from vertex* x *to vertex* y. *By repeatedly sliding using an edge of* P_1 *and edge* e_2, *we eventually merge* x *and* y *and obtain graph* G'. *Note that in* G', *the vertex* y *is no longer present and it is replaced by a vertex of degree 2 in* P_2. *The equivalence classes under* $\sim$ *remain unchanged except that the equivalence classes containing* x *and* y *in* $G^{(k)}$ *are merged to form one equivalence class, omitting* y. *Thus,* $r(G') = r(G^{(k)}) - 1$. *By Lemma 3,* G' *is bipartite with bipartite classes* U'_1 *and* U'_2. *Set* $k \leftarrow k+1$, $G^{(k)} \leftarrow G'$, $V'_k \leftarrow U'_1$, *and* $V''_k \leftarrow U_2$.

Step 2: *Repeat Step 1 (with the new* $G^{(k)}$*) until finally we find some* $G^{(k)}$ *and* $V_0^{(k)}$ *such that in the bipartition of* $G^{(k)}$, *one of the two classes has no vertices of* $V_0^{(k)}$. *We know this will eventually happen since it happens when we get to* $G^{(k)}$ *with* $r(G^{(k)}) = 1$ *(since no pair of 3-edge connected vertices are in different classes in the bipartition in this case). Let* $G* = G^{(k)}$ *and* $V_0^* = V_0^{(k)}$.

Step 3: *By construction,* $G*$ *is bipartite and one of the two classes in the bipartition has no vertices of* V_0^*. *Find a spanning tree* T *of* $G*$ *and check if each fundamental cycle relative to* T *has length 0 mod 4. If there is a fundamental cycle with length not 0 mod 4, output the answer "no". Otherwise, output the answer "yes".*

To see that Algorithm 2 is correct, say that after Steps 1 and 2, V^1 and V^2 are the two classes in the bipartition of $G*$ and one of these, say V^2, has no vertices of degree ≥ 3. Of course, any tripath must go between vertices in V^1 and therefore must have even length. Any common path of two fundamental cycles relative to T is a path on T and joins two vertices in V_0^* and all its middle vertices have degree 2. Thus, the path is a tripath and so has length 0 mod 2. This together with the condition obtained from step 3 that all fundamental cycles have length 0 mod 4 implies any cycles which are a symmetric difference of fundamental cycles have length 0 mod 4. By Theorem 9, all cycles have length 0 mod 4.

The complexity of this algorithm can be calculated as follows. Step 1 can be carried out using a procedure analogous to that of Step 2a of Algorithm 1, so it requires $O(n)$ steps. Step 2 must be carried out at most $O(n)$ times. Step 3 has complexity $O(mn)$, as in Algorithm 1. Thus, the overall complexity of the algorithm is $O(mn)$.

There is a variant on Algorithm 2.

Algorithm 3.

Input. A 2-connected bipartite graph G, which is not a cycle, with bipartition $V(G) = V' \cup V''$.

Output. The answer to the question: Does every cycle of G have length 0 mod 4?

This is the same as Algorithm 2, except that we replace Step 3 by:

Step 3′: *By construction, $G*$ is bipartite and one of the two classes in the bipartition has no vertices of V_0^*. Check if we can partition V_0^* into two sets V_1 and V_2 such that any tripath joining one vertex in V_1 and one in V_2 has length 2 mod 4 and any tripath joining two vertices both in V_1 or both in V_2 has length 0 mod 4. If so, output the answer "yes". If no, output the answer "no".*

Let T be a spanning tree of $G*$. We show that after Steps 0, 1, and 2, Step 3 of Algorithm 2 has a positive answer iff Step 3′ of Algorithm 3 has a positive answer. This will show that Algorithm 3 is correct. Suppose that Step 3 has a positive answer. We divide V_0^* into two parts V_1 and V_2 as follows. Pick a vertex $v \in V_0^*$ and put it in V_1. For any other vertex $u \in V_0^*$, if T_{uv} has length 0 mod 4, we put it in V_1, otherwise V_2. We show that this partition has the property stated in Step 3′. It is clear that if u and w are in V_0^*, then T_{uw} has length 0 mod 4 if u and w are in the same class V_i and 2 mod 4 otherwise. Let P be a tripath from u to w. If P has no tree chords, then $P = T_{uw}$ and so has the desired property. If P has a tree chord, then P plus T_{uw} form a fundamental cycle. We know that P has even length since it joins two vertices of the same class in the bipartition. Since all fundamental cycles relative to T have length 0 mod 4 and T_{uw} has the property required in Step 3′, it follows that P does as well.

Conversely, suppose Step 3′ has a positive answer and V_1, V_2 are the sets in the partition. Since G is not a cycle, $G*$ is also not a cycle. Thus, any fundamental cycle of $G*$ has at least two vertices of V_0^* and we can look at the fundamental cycle as consisting of a sequence of tripaths. Each tripath that has endpoints in the same set V_i has length 0 mod 4 and each tripath that has endpoints in different sets has length 2 mod 4. Since following around the cycle ends us back in the starting set V_i, there are an even number of tripaths that have endpoints in different sets. We conclude that the cycle has length 0 mod 4.

The complexity of Algorithm 3 is $O(mn)$. To see why, note that by the discussion of the complexity of Algorithm 2, it suffices to check that Step 3′ can be completed in $O(mn)$ time. Once we reach Step 3′, one of the classes in the bipartition V^1, V^2 of $G*$, say V^2, has no vertices of degree ≥ 3. Let $H*$ be obtained from $G*$ by replacing each path x, y, z with y in V^2 by an edge xz. Then V_0^* is contained in the vertex set of $H*$ and a tripath P in $G*$ corresponds to a tripath Q in $H*$ of half the length. We can partition vertices of V_0^* in $G*$ to satisfy the condition in step 3′ if and only if we can partition vertices of V_0^* in $H*$ so that tripaths joining vertices in different classes have length 1 mod 2 and tripaths joining vertices in the same class have length 0 mod 2. This is essentially the question of whether or not $H*$ is bipartite, and this can be checked by any of a number of well-known algorithms for bipartiteness in $O(mn)$ time.

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