DIMACS Technical Report 99-13
February 1999

Optimal Orientations of Annular Networks

by

Jean-Claude Bermond [1] Johny Bond * Carole Martin *
Aleksandar Pekeč [2] Fred S. Roberts [3]

[1]I3S CNRS, Université de Nice - Sophia Antipolis, bt. ESSI, BP 145, 930 route des Colles, 06903 Sophia Antipolis, France
[2]Fuqua School of Business, Duke University, Durham, NC 27708, USA. Work partially done while at BRICS (Basic Research in Computer Science, Centre of the Danish National Research Foundation), University of Aarhus, Denmark, and while supported by the ESPRIT Long Term Research Programme of the EU under project number 20244 (ALCOM-IT).
[3]Department of Mathematics and Center for Operations Research Rutgers University, New Brunswick, NJ 08903, USA. Fred Roberts thanks the US Air Force Office of Scientific Research for its support under grants AFOSR 89-0512 and F49620-93-1-0041 and the US National Science Foundation for its support under grant NSF-SBR97-09134, all to Rutgers University.

DIMACS is a partnership of Rutgers University, Princeton University, AT&T Labs-Research, Bell Labs, Bellcore and NEC Research Institute.

DIMACS is an NSF Science and Technology Center, funded under contract STC–91–19999; and also receives support from the New Jersey Commission on Science and Technology.

ABSTRACT

Annular Network $AN(c, s)$ is a graph representing a $c \times s$ grid in polar coordinates. We give lower bounds for the diameter of orientations of $AN(c, s)$ and provide orientations which show that bounds are tight in most cases.

1 Introduction

It is well known [16, 2] that a graph $G = (V, E)$ admits a strongly connected orientation $\vec{G}$ if and only if G is 2-edge-connected (i.e., G is connected and has no bridges). However, in most applications there is a need for an optimal orientation $\vec{G}$ with respect to some measure of optimality.

One such measure can be the diameter (or the radius) of the graph. Recall that the distance $d(x, y)$ between two vertices x,y of a graph G is the length of a shortest path in G with endpoints x and y. The diameter $D = D(G)$ of a graph G is $\max_{x,y} d(x, y)$. $\vec{G}$ stands for an orientation of G and $\vec{d}(x, y) = d_{\vec{G}}(x, y)$ for the length of a shortest path from x to y in $\vec{G}$. $D(\vec{G}) = \max_{x,y} d_{\vec{G}}(x, y)$ is the diameter of the orientation $\vec{G}$. The minimal diameter among all orientations of G is $\vec{D} = \vec{D}(G) = \min_{\vec{G}} D(\vec{G})$. Obviously, $\vec{d}(x, y) \geq d(x, y)$ for any two vertices x and y. Hence, $\vec{D} \geq D$. We shall study $\vec{D}$ for a special class of graphs, the annular networks.

The problem of determining $\vec{D}$ is useful in various applications. One which motivated our study involves improving traffic flow in cities by making streets one-way. This application is described in detail in [17, 18]. Another application which motivated our study is communications in interconnection networks. Gossiping (also called total exchange or all–to–all communication) in interconnection networks is the process in which initially each processor has an item of information that must be distributed to every other processor of the system. Gossiping arises in a large class of parallel computation problems, such as linear system solving, matrix manipulation, and sorting, where both input and output data are required to be distributed across the network [1, 4, 12]. The process is accomplished by means of a sequence of *synchronous calls* between processors. Furthermore we suppose that during each call a processor can communicate with all its neighbors. If both adjacent processors can communicate, this communication model is usually referred to as *Full–Duplex* Δ*–Port* (F_*) [5, 7, 23]. In that case the gossiping time (minimum number of steps needed to complete the protocol) is equal to the diameter D of the graph G representing the interconnection network. If we suppose that simultaneous exchange of a message on the same link is not authorized, we have another popular communication model called *Half–Duplex* Δ*–Port* (H_*). In that case, for a given orientation $\vec{G}$, we can consider the greedy protocol which at each step sends all the information known by a vertex to all of its out-neighbors according to the orientation $\vec{G}$; so we have $D(\vec{G})$ as an upper bound of the gossiping time. Since a lower bound is D, for any network G that admits an orientation $\vec{G}$ for which $D(\vec{G})$ is close to D, the performance of the greedy protocol is provably close to an optimal performance.

Chvátal and Thomassen [3] showed that finding $\vec{G}$ with the smallest possible oriented diameter or radius is NP-hard. Orientations with the smallest possible oriented diameter are known for some classes of graphs. Plesnik [15], Gutin

[6], and Koh and Tan [8, 9] presented such orientations for complete multipartite graphs. McCanna [13] presented such orientations for n-dimensional cube. Roberts & Xu in a series of papers [19, 20, 21, 22] found optimal orientations for rectangular grids with respect to various measures of of optimality. Koh and Tay [10] discussed optimal orientations of products of paths and cycles. König, Krumme and Lazard [11] solved the case of tori (wraparound grids). In this paper we discuss optimal orientations of Annular Networks AN(c,s) which are graphs representing a $c \times s$ grid in polar coordinates. More details and studies of various communications problems on these networks can be found in [14].

An **Annular Network** $AN(c, s)$ can be represented as a 2-dimensional grid consisting of c concentric circles around the center and s straight lines crossing all the circles (i.e., a $c \times s$ grid in the polar coordinate system). More precisely, $AN(c, s)$ has $cs+1$ vertices denoted $(0, 0)$ for the **center** and (i, u) with $1 \leq i \leq c$ and $0 \leq u \leq s-1$ for the other vertices. The center is adjacent to all the vertices $(1, u)$, $0 \leq u \leq s-1$. Otherwise, a vertex (i, u) is adjacent to a vertex (j, v) if and only if they agree in one coordinate and differ by one in the other (in the case of the second coordinate, the difference has to be taken modulo s). For example, Figure 1 shows $AN(3, 8)$.

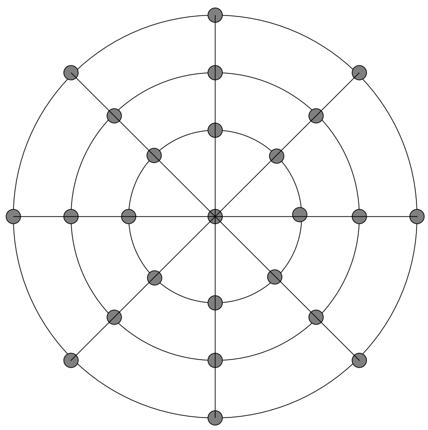

Figure 1: Annular Network $AN(3, 8)$

The **circle** C_i is the subgraph induced by the vertices $\{(i, u) : 1 \leq u \leq s-1\}$ and the **spoke** S_u is the subgraph induced by the vertices $\{(0, 0)\} \cup \{(i, u) : 1 \leq i \leq c\}$. Hence, every C_i, $1 \leq i \leq c$, is a cycle on s vertices and every S_u is a path on $c+1$ vertices. Moreover, C_i and S_u intersect in (i, u), the center $(0, 0)$ is the intersection of any two spokes, and any two circles are disjoint. Hence, every edge of $AN(c, s)$ is in exactly one spoke or circle. Edges belonging to circles are **circle edges**, and edges belonging to spokes are **spoke edges**. Altogether $AN(c, s)$ has

$2cs$ edges (cs circle edges and cs spoke edges).

Before determining the diameter of $AN(c,s)$ we introduce the following notation. If $x = (r,u)$ and $y = (r,v)$ are two vertices on the same circle C_r then $d(x,y)$ is denoted by

$$t(u,v) := d(x,y) = \min\{|u-v|, s-|u-v|\} \leq \lfloor s/2 \rfloor,$$

that is $t(u,v)$ is the distance between the vertices u and v on the cycle of length s. We will also use α to denote clockwise distance between u and v:

$$\alpha = \alpha(u,v) := \begin{cases} v-u & \text{if } v \geq u \\ v-u+s & \text{if } v < u. \end{cases}$$

Lemma 1.1 *Let $x = (i,u)$ and $y = (j,v)$ be two vertices in AN(c,s). Then $d(x,y) = d((i,u),(j,v)) = \min\{i+j, |i-j| + t(u,v)\}$.*

Proof : An x,y-path going through the center uses at least $i+j$ spoke edges and the center path starting at x, continuing on the spoke u to the center and ending at y along the spoke v has length $i+j$. Any path not going through the center must use at least $|i-j|$ spoke edges and $t(u,v)$ circle edges. The path starting at x, going along the spoke u towards the circle j and using $t(u,v)$ edges on circle j ending at y is an example of a non-center path of length $|i-j| + t(u,v)$. □

A pair of vertices (x,y) for which $d(x,y) = D$ is called an **extremal pair** and x and y are called **extremal vertices**.

Theorem 1.2
$D = 2c$ if $c \leq \lfloor s/4 \rfloor$.
$D = c + \lfloor s/4 \rfloor$ if $c \geq \lfloor s/4 \rfloor$.
Furthermore, every vertex $x = (c,u)$ on the last circle (C_c) is an extremal vertex and all extremal pairs x,y are given by y below:

$$\begin{array}{lcl} c \leq \lfloor s/4 \rfloor & : & y = (c,v), \ \ \textit{where } t(u,v) \geq 2c. \\ c \geq \lfloor s/4 \rfloor & : & y = (\lfloor s/4 \rfloor, v), \ \textit{where } t(u,v) = \lfloor s/2 \rfloor. \\ & & \textit{In addition, if } \lfloor s/2 \rfloor \textit{ is odd then also} \\ & & y = (\lfloor s/4 \rfloor + 1, v), \ \ \textit{where } t(u,v) = \lfloor s/2 \rfloor. \end{array}$$

Proof : Assume $i \geq j$. Then

$$d((i,u),(j,v)) = \min\{i+j, i-j+t(u,v)\} = i + \min\{j, t(u,v) - j\},$$

so $i = c$ will maximize this independently of the choice for j,u,v. This shows that every vertex on C_c is extremal since the choice of u is irrelevant as long we are free to choose v. Therefore,

$$D = c + \max_{j,v} \min\{j, t(u,v) - j\}.$$

If $c \leq \lfloor s/4 \rfloor$ then $j \leq \lfloor s/4 \rfloor$ and $\min\{j, t(u,v) - j\} \leq j \leq c$. On the other hand $2c$ can be obtained whenever $j = c$ and $t(u,v) - j \geq c$. Since $j = c$, the latter condition becomes $t(u,v) \geq 2c$.

If $c \geq \lfloor s/4 \rfloor$ then $\min\{j, t(u,v) - j\} \leq \lfloor t(u,v)/2 \rfloor$ and since $\lfloor t(u,v)/2 \rfloor \leq \lfloor \lfloor s/2 \rfloor /2 \rfloor = \lfloor s/4 \rfloor \leq c$, the upper bound can always be obtained and therefore $t(u,v)$ must be $\lfloor s/2 \rfloor$. Then $j = \lfloor s/4 \rfloor$ gives $D = c + \lfloor s/4 \rfloor$ and, in the case of $\lfloor s/2 \rfloor$ odd, $j = \lfloor s/4 \rfloor + 1$ also gives $D = c + \lfloor s/4 \rfloor$. □

$s \setminus c$	$c > k+2$	$c = k+2$	$c = k+1$	$c \leq k$
$s = 4k$	$\vec{D} = D$	$\vec{D} = D+1$	$\vec{D} = D+2$ ($k=1$: $\vec{D}=D+1$)	$\vec{D} = D+2$ ($k=c=1$: $\vec{D}=D+1$)
$s = 4k+1$	$\vec{D} = D+1$	$D+1 \leq \vec{D} \leq D+2$	$D+2 \leq \vec{D} \leq D+3$ ($k=1$: $\vec{D}=D+2$)	$\vec{D} = D+2$
$s = 4k+2$	$\vec{D} = D+1$	$\vec{D} = D+1$	$\vec{D} = D+2$	$\vec{D} = D+2$
$s = 4k+3$	$D+1 \leq \vec{D} \leq D+2$	$D+2 \leq \vec{D} \leq D+3$ ($k=0$: $\vec{D}=D+2$)	$\vec{D} = D+3$ ($k=0$: $\vec{D}=D+2$)	$\vec{D} = D+2$

Table 1: Values of $\vec{D}$ in terms of D for Annular Networks AN(c,s).

The main theme of this paper is finding $\vec{D}$ for AN(c,s). As already mentioned, $\vec{D} \geq D$ for any graph G. The results that we present here are summarized in Table 1. Since $\lfloor s/4 \rfloor$ is a constant which appears naturally in most of our results, **we will use the notation $\mathbf{k} := \lfloor \mathbf{s/4} \rfloor$**.

In the next section we present orientations which attain the upper bounds. Proofs for lower bounds on $\vec{D}$ are given in Section 3. Finally, we summarize our results and discuss the open questions.

2 Optimal Orientations and Upper Bounds

For a vertex $x = (c, u)$ on the last circle (circle c) the edge $[(c-1, u); (c, u)]$, which is the unique spoke edge incident to x, is called the **outerspoke of x** and if the orientation $\vec{G} = (V(G), A(\vec{G}))$ is given we say that

$$os(x) = \quad in \qquad \text{if } [(c,u);(c-1,u)] \text{ is an arc in } \vec{G}$$

$$os(x) = \quad out \qquad \text{if } [(c-1,u);(c,u)] \text{ is an arc in } \vec{G}$$

We say that circle r is **uniformly oriented** in $\vec{G}$ if C_r is a directed cycle and we say that spoke u is **uniformly oriented** in $\vec{G}$ if S_u is a directed path. Note that the orientation of a uniformly oriented spoke is completely determined by the orientation of its outerspoke and we can say that an oriented spoke is going *in* or *out* depending on the orientation of its outerspoke.

In order to prove all the upper bounds from Table 1 we will use only two orientations $\vec{O1} = \vec{O1}(c,s)$ and $\vec{O2} = \vec{O2}(c,s)$. In both of these orientations, the spokes are uniformly oriented and alternate in direction: S_0, S_2,... are oriented *in* (that is ¿from circle c into the center), while S_1, S_3,... are oriented *out*. Hence, the only case when two adjacent spokes have the same orientation is when s (the number of spokes) is odd, and both S_{s-1} and S_0 are oriented *in*.

In the orientation $\vec{O1}$ (Figure 2), all circles except the circle c are uniformly oriented and alternate in direction: C_{c-1}, C_{c-3},... are oriented clockwise, while C_{c-2}, C_{c-4},... are oriented counterclockwise. The edges on circle c alternate in direction: $[(c,1);(c,2)]$, $[(c,3);(c,4)]$,... are oriented clockwise, while $[(c,1);(c,0)]$, $[(c,3);(c,2)]$,... are oriented counterclockwise. Obviously, the only case when there is a directed path of length two on the circle c is when s is odd (path $(c,1),(c,0),(c,s-1)$).

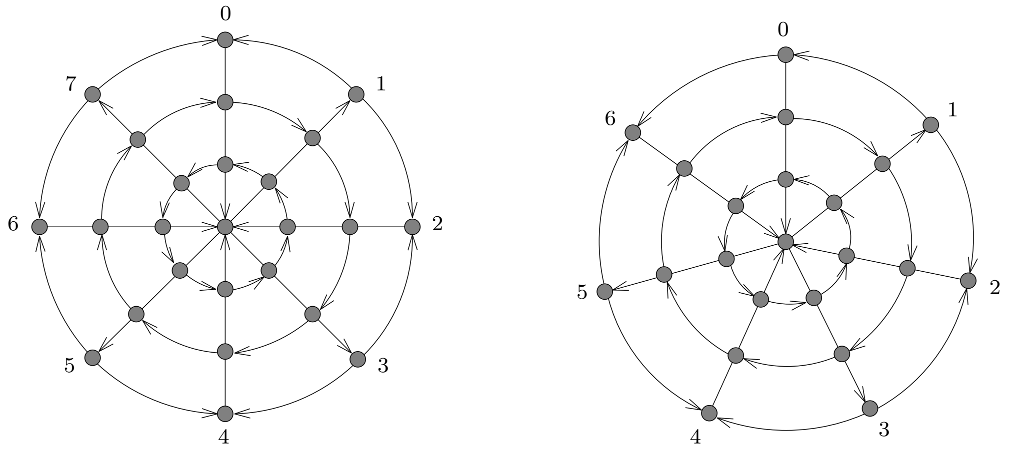

Figure 2: Orientation $\vec{O1}$: AN(3,8) and AN(3,7)

The orientation $\vec{O2}$ (Figure 3) is much simpler: all circles (including circle c) alternate in direction: C_c, C_{c-2},... are oriented clockwise, while C_{c-1}, C_{c-3},... are oriented counterclockwise.

Theorem 2.1 $\vec{D} \leq 2c+2$.

Proof : We will show that $D(\vec{O1}) \leq 2c+2$. Hence, we need to show that $\vec{d}(x,y)_{\vec{O1}} \leq 2c+2$ for any two vertices $x=(i,u)$ and $y=(j,v)$.

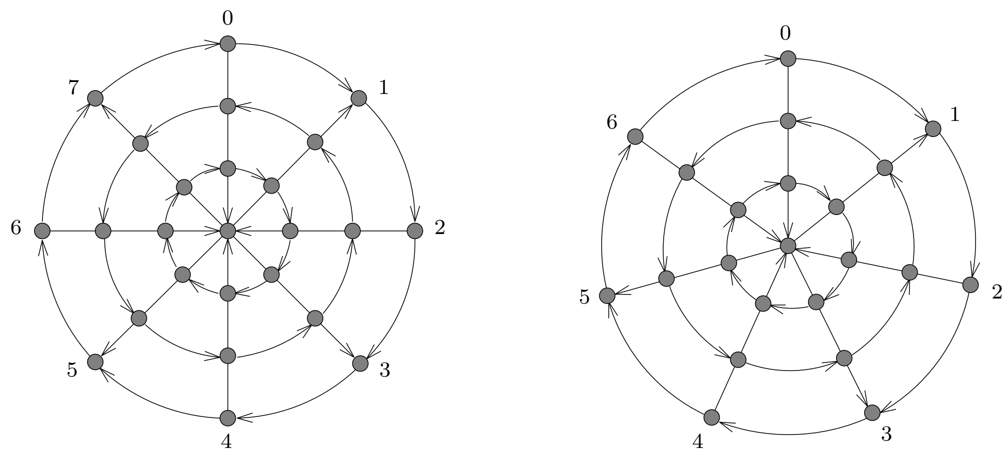

Figure 3: Orientation $\vec{O}2$: AN(3,8) and AN(3,7)

Let $x' = (i, u')$ be a vertex on the circle i such that the spoke u' is *in* and $\vec{d}(x', x)$ is minimal. Note that $\vec{d}(x', x) \leq 1$. Let $y' = (j, v')$ be a vertex on the circle j such that the spoke v' is *out* and $\vec{d}(y, y')$ is minimal. Note that $\vec{d}(y', y) \leq 2$ and $\vec{d}(y', y) \leq 1$ if $j = c$; furthermore $\vec{d}(x', y') = i + j$ since we can use the spoke u' from x' to the center and the spoke v' from the center to y'. Hence, $\vec{d}(x, y) = \vec{d}(x', x) + \vec{d}(x', y') + \vec{d}(y, y') \leq 2c + 2$. □

Corollary 2.2
$\vec{D} \leq D + 2$ if $c \leq k$.
$\vec{D} \leq D + 3$ if s is odd and $c = k + 1$.

Proof : If $c \leq k$ then $D = 2c$, and if $c = k + 1$ then $D = c + k = 2c - 1$ (Theorem 1.2). The result follows from Theorem 2.1. □

Theorem 2.3 *Let s be even and $c \geq k + 1$. Let*

$$A := \begin{cases} \max\{c + k, 2k + 3\} & \text{if } s = 4k \\ \max\{c + k + 1, 2k + 3\} & \text{if } s = 4k + 2 \end{cases}$$

Then $\vec{D} \leq A$.

Proof : It is straightforward (but rather tedious) to check that $D(\vec{O}2) \leq A$ if $c = k + 2$ and $s = 4k + 2$, and that $D(\vec{O}1) \leq A$ in all other cases. The details of the proof are relegated to the Appendix. □

Corollary 2.4
$\vec{D} = D$ if $s = 4k$ and $c > k+2$.
$\vec{D} \le D+1$ if $s = 4k+2$ and $c > k+2$.
$\vec{D} \le D+1$ if s is even and $c = k+2$.
$\vec{D} \le D+2$ if s is even and $c = k+1$.

Proof : Directly from Theorem 1.2 and Theorem 2.3. □

Theorem 2.5 *Let s be odd and $c \ge k+2$. Let*

$$A := \begin{cases} \max\{c+k+1, 2k+4\} & \text{if } s = 4k+1 \\ \max\{c+k+2, 2k+5\} & \text{if } s = 4k+3 \end{cases}$$

Then $\vec{D} \le A$.

Proof : $D(\vec{O}1) \le A$. A detailed proof can be found in the Appendix. □

Corollary 2.6
$\vec{D} \le D+1$ if $s = 4k+1$ and $c > k+2$.
$\vec{D} \le D+2$ if $s = 4k+1$ and $c = k+2$.
$\vec{D} \le D+2$ if $s = 4k+3$ and $c > k+2$.
$\vec{D} \le D+3$ if $s = 4k+3$ and $c = k+2$.

Proof : Directly from Theorem 1.2 and Theorem 2.5. □

It is straightforward to check that $\vec{D}(\vec{O}2)$ for $AN(1,3)$, $AN(1,4)$, $AN(2,4)$, and $AN(2,5)$ gives upper bounds on $\vec{D}$ from Table 1. Also it is not hard to check that $AN(2,3)$ has an orientation $\vec{O}$ such that $D(\vec{O}) = D+2 = 4$ (see Figure 4). These five cases, together with Corollary 2.2, Corollary 2.4, and Corollary 2.6, give all the upper bounds from Table 1.

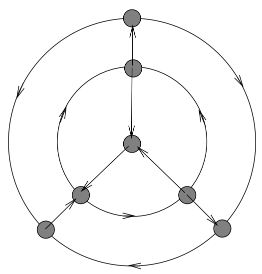

Figure 4: An orientation $\vec{O}$ of $AN(2,3)$ with $4 = D(\vec{O}) = D(AN(2,3)) + 2$

3 Lower Bounds

We will repeatedly use the fact that for any orientation $\vec{G}$ of AN(c,s), there exist other orientations with the same oriented diameter. $E(\vec{G})$ will stand for the set of arcs of an orientation $\vec{G}$.

First of all, for any graph G and its orientation $\vec{G}$, there exists its opposite (or reversed) orientation $\vec{G}^*$ which is obtained from $\vec{G}$ by reversing orientation of every arc. In other words, $[x; y] \in E(\vec{G}^*)$ if and only if $[y; x] \in E(\vec{G})$. Clearly, for any pair of vertices $x, y \in V(G)$, $\vec{d}_{\vec{G}}(x, y) = \vec{d}_{\vec{G}^*}(y, x)$. Hence, $D(\vec{G}) = D(\vec{G}^*)$.

Furthermore, any graph isomorphism ϕ defines an orientation $\phi(\vec{G})$ in a natural way: $[x; y] \in E(\phi(\vec{G}))$ if and only if $[\phi^{-1}(x); \phi^{-1}(y)] \in E(\vec{G})$.

It is not hard to see that the automorphism group of AN(c,s) is isomorphic to the dihedral group D_s. Choosing an automorphism amounts to fixing a cyclic labeling of spokes: we are free to choose which spoke will be labeled by zero and we are free to choose the direction of labeling: clockwise or counterclockwise.

Therefore, if $\vec{G}$ is any orientation of $AN(c, s)$, then we can find some other orientation $\vec{G}'$ with $D(\vec{G}') = D(\vec{G})$ such that one spoke edge is oriented as we want (either $\vec{G}$ or $\vec{G}^*$ satisfies this) and one circle edge is oriented as we want (we are free to choose the direction of labeling). Moreover, the spoke of our choice will be labeled by zero.

Throughout this section we will mainly consider vertices on the circle c and, in order to simplify the notation, we will **denote vertex $(\mathbf{c}, \mathbf{u})$ by $\mathbf{u}$** (e.g., in this notation $os(u)$ stands for $os((c, u))$. Throughout, $\vec{G}$ will always be an orientation of $AN(c, s)$.

Lemma 3.1 *If there exists an extremal pair of vertices x, y such that there is a unique path of length D between them, then $\vec{D} \geq D+1$. In particular, if $s = 4k+2$ and $c \geq k + 2$ then $\vec{D} \geq D + 1$.*

Proof : We cannot obtain simultaneously $\vec{d}(x, y) = \vec{d}(y, x) = D$.

Note that, for $s = 4k + 2$, by Theorem 1.2, $x = (c, 0)$, $y = (k, 2k + 1)$ is an extremal pair in AN(c,s) and the only xy path of length $c + k$ is a center path. □

Lemma 3.2 *If outerspokes alternate in direction, then $\vec{D} \geq \min\{2c+2,\ 2k+3\}$ except for $AN(1, 4)$ and $AN(2, 4)$.*

Proof : First note that s must be even. Note that without loss of generality we may assume that $\mathrm{os}(u) = in$ if and only if u is even.

Case 1: Arcs on circle c do not alternate in direction (Figure 5).

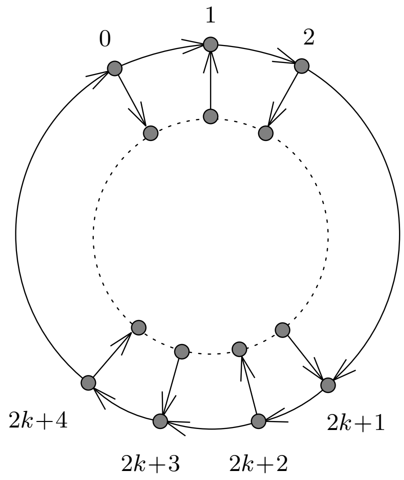

Figure 5: s even, *outerspokes* alternate and $[0;1]$,$[1;2] \in E(\vec{G})$.

Since the arcs on circle c do not alternate, there are two consecutive arcs on circle c having the same orientation. We may assume that $[0;1]$ and $[1;2]$ are such arcs. Therefore, $\vec{d}(1,x) = 1+\vec{d}(2,x)$. (Note that we may assume that $s > 4$ since, for $s = 4$ and $x = (1,0)$, we have $\vec{d}(1,x) = 1 + \vec{d}(2,x) \geq 1 + d(2,x) = 1 + c + 1 \geq 5 = 2k + 3$. The last inequality holds because, by assumption, $c \geq 3$ when $s = 4$.)

Let $x = 2k+2$. Any center path from 2 to $2k+2$ is of length at least $2c+1$ since $2k+2$ is even and $os(2k+2) = in$, so if we use a center path, $\vec{d}(1, 2k+2) \geq 2c+2$. If we use a non-center path and if the circle c between 2 and $2k+2$ is not uniformly oriented clockwise, then $\vec{d}(2, 2k+2) \geq 2k+2$ and then $\vec{d}(1, 2k+2) \geq 2k+3$.

If the circle c is uniformly oriented from 2 to $2k+2$, then $\vec{d}(2k+1, 2) = 2+\vec{d}(2k+2, 1)$. If we use a center path, then $\vec{d}(2k+2, 1) \geq 2c$, so $\vec{d}(2k+1, 2) \geq 2c+2$. If we use a non-center path, then $\vec{d}(2k+2, 1) \geq 2k+1$ and $\vec{d}(2k+1, 2) \geq 2k+3$, except if the circle c is oriented clockwise between $2k+1$ and 0.

Finally, if the circle c is uniformly oriented clockwise, consider $\vec{d}(1, 2k+4) = 2+\vec{d}(2, 2k+3)$. But $\vec{d}(2, 2k+3) \geq 2c$ if we use a center path and $\vec{d}(2, 2k+3) \geq 2k+1$ if we use a non-center path.

Hence, in all the cases, $\vec{D} \geq \min\{2c+2,\ 2k+3\}$.

Case 2: Arcs on the circle c alternate in direction (Figure 6).

By strong connectivity, if u is even we have $[u-1;u]$, $[u+1;u] \in E(\vec{G})$, and if u is odd we have $[u;u-1]$, $[u;u+1] \in E(\vec{G})$.

If $s = 4k+2$, we always have $\vec{d}(1, 2k+2) \geq \min\{2c+2, 2k+3\}$ since any center

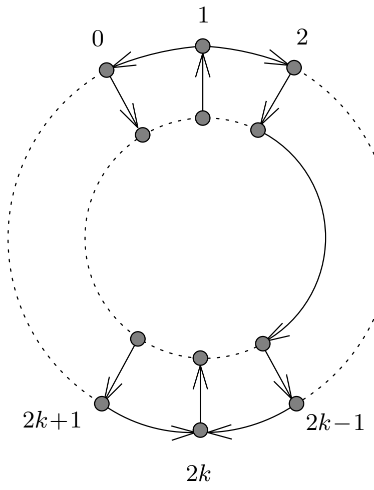

Figure 6: s even, *outerspokes* alternate and arcs on circle c alternate.

path is of length at least $2c+2$ and any non-center path is of length at least $2+2k+1=2k+3$.

For $s=4k$, $k>1$, any center path from 1 to $2k$ is of length at least $2c{+}2$, and any other path should use at least two spoke edges and at least $2k+1$ circle edges if we go counterclockwise on circle $c-1$ or $2k-1$ circle edges if we go clockwise. So, the only way to have a non-center path of length less than $2k+3$ is to have circle $c-1$ oriented clockwise from $(c{-}1,2)$ to $(c{-}1,2k{-}1)$.

Similarly, $\vec{d}(2k{+}1,2) \geq \min\{2c{+}2,2k{+}3\}$ except if the circle $c{-}1$ is oriented counterclockwise from $(c-1,2k)$ to $(c-1,3)$, which contradicts the preceding orientation for all arcs between $(c{-}1,2k{-}1)$ to $(c{-}1,3)$ whenever $k>2$.

If $k=2$, we conclude that $[(c-1,2);(c-1,3)]$ and $[(c-1,4);(c-1,3)]$ are arcs in $\vec{G}$. Applying the same argument to $\vec{d}(3,6)$ and $\vec{d}(7,4)$, $\vec{d}(5,0)$ and $\vec{d}(1,6)$, $\vec{d}(7,2)$ and $\vec{d}(3,0)$, we conclude that arcs on circle $c{-}1$ alternate in direction. So, any non-center path from 1 to 5 is of length at least $2k+4=8$, and any center path is of length at least $2c{+}1$. If $c \geq k+1=3$, then $2c{+}1 \geq 2k{+}3$ and we are done. If $c \leq 2$, then $\min\{2c+2,2k+3\}=2c+2$. If $c=1$, $\vec{d}(1,4) \geq 4 = 2c+2$. If $c=2$, $\vec{d}(1,5) < 2c+2 = 6$ only if we use a center path, which means that spoke 5 is uniformly oriented *out*. In that case, $\vec{d}((c-1,5),1) = 1+\vec{d}(5,y) \geq 1+2c+1 = 2c+2$.

The remaining case is $s=4$. If $c > k+2 = 3$, we have $\vec{D} \geq D = c+k \geq 2k+3$ If $c=k+2=3$, then $\vec{d}(1,3) < 5 = \min\{2c+2,2k+3\}$ only if $[(2,0);(2,3)]$ or $[(2,2);(2,3)]$ is an arc. Without loss of generality we may assume that $[(2,0);(2,3)]$ is an arc (otherwise exchange labels for spokes 0 and

2 throughout). Then, $\vec{d}((2,3),1) < 5$ only if $[(2,3);(2,2)]$ is an arc. But then, $d(2,3) \geq 5$ since every path from 2 to 3 must use at least four spoke edges, or two spoke edges and three circle edges. □

Lemma 3.3 *Let $\vec{G}$ be an orientation of $AN(c,s)$ such that $\vec{G}$ contains a directed path $P = u, u{+}1, u{+}2, \cdots, u{+}m{+}1$ on circle c. If $os(u{+}1) = os(u{+}2) = \cdots = os(u{+}m)$, then $D(\vec{G}) \geq D{+}m$.*

Proof : Without loss of generality we may assume $os(u{+}1) = os(u{+}2) = \cdots = os(u{+}m) = out$. Then $\vec{d}(u+1,y) = m + \vec{d}(u{+}m{+}1,y)$ since any path starting at $u+1$ should go through $u+2, \cdots, u+m+1$. Let z be a vertex at distance D from $u{+}m{+}1$. Then $\vec{D} \geq \vec{d}(u+1,z) \geq m + \vec{d}(u{+}m{+}1,z) \geq m + D$. □

Lemma 3.4 *If outerspokes do not alternate in direction, then $\vec{D} \geq D{+}1$. Furthermore, if $\vec{D} = D{+}1$, then there exists an orientation $\vec{G}$, $\vec{D} = D{+}1 = D(\vec{G})$, such that $out = os(0) = os(1) \neq os(2)$, and such that $[0;1]$, $[0;-1]$, $[1;2] \in E(\vec{G})$ (Figure 7).*

Proof : Since outerspokes do not alternate there exist two adjacent vertices on circle c, u and $u{+}1$, such that $os(u) = os(u{+}1)$. Without loss of generality we may assume that $u = 0$ (otherwise we relabel spokes by changing spoke l into $(l-u) \bmod s$), that $os(0) = os(1) = out$ (otherwise we reverse orientations of all arcs), and that $[0;1] \in E(\vec{G})$ (otherwise we relabel spokes by changing spoke l into $(1{-}l) \bmod s$).

By strong connectivity, $[1;2] \in E(\vec{G})$. $D \geq D{+}1$ follows from Lemma 3.3 (with $P = 0,1,2$).

Let $D(\vec{G}) = D+1$. If $[-1;0] \in E(\vec{G})$ then by Lemma 3.3, $D \geq D{+}2$. Also, if $os(2) = out$, by strong connectivity $[2;3] \in E(\vec{G})$ and by Lemma 3.3, $D \geq D{+}2$. Hence, if $\vec{D} = D{+}1$, then $os(2) = in$ and $[0;-1] \in E(\vec{G})$. □

Corollary 3.5 *Let $s = 4$ and $c \leq 2$. Then $\vec{D} \geq D + 1$.*

Proof : If outerspokes do not alternate in direction then, by Lemma 3.4, $\vec{D} \geq D{+}1$.

Suppose that outerspokes alternate in direction. Then $\vec{d}(0,2) \leq 3$ only if $0,1,2$ or $0,3,2$ is a directed path on circle c. In either case, by Lemma 3.3, $\vec{D} \geq D + 1$. □

Corollary 3.6 *Suppose that $c = k{+}2$ or that $c > k{+}2$ and s odd. Then $\vec{D} \geq D{+}1$.*

Proof : If outerspokes do not alternate then, by Lemma 3.4, $\vec{D} \geq D+1$.
If outerspokes alternate, then s must be even, so $c = k+2$. By Lemma 3.2, $\vec{D} \geq \min\{2c+2, 2k+3\} = 2k+3 = c+k+1 = D+1$. □

In the rest of this section, we will be analyzing cases where $\vec{D} \geq D+2$. In all proofs we will start by assuming that there exists a $\vec{G}$ such that $D(\vec{G}) = D+1$ and will reach a contradiction. If $\vec{D} = D+1$ and outerspokes do not alternate, then there exists a $\vec{G}$ ($D(\vec{G}) = D+1$) satisfying the conditions of Lemma 3.4. We say that $\vec{G}$ contains the **basic configuration** (Figure 7).

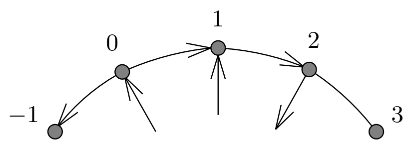

Figure 7: The Basic Configuration: $os(0) = os(1) = out \neq os(2) = in$, and $[0;-1], [0;1], [1;2] \in E(\vec{G})$.

Theorem 3.7 *If $c = k+2$ and $s = 4k+3$ then $\vec{D} \geq D+2$.*

Proof : Outerspokes don't alternate since s is odd. Hence, $\vec{D} \geq D+1$ by Lemma 3.4. Let $\vec{G}$ be any orientation. If $D(\vec{G}) = D+1$ then we may assume that $\vec{G}$ contains the basic configuration (Figure 7).

If $s = 3$, then $\vec{d}(1,0) = 1 + \vec{d}(2,(1,0)) + 1 \leq 2 + d(2,(1,0)) = 4 = D+2$. Hence, we may assume $s \geq 7$.

First suppose $[3;2] \in E(\vec{G})$. Note that $\vec{d}(1, 2k+3) = \vec{d}(2, 2k+3) + 1$. Any center path from 2 to $2k+3$ is of length at least $2c = 2k+4$ and any non-center path is of length at least $2+(2k+1) = 2k+3$. Hence, $\vec{d}(1, 2k+3) \geq 2k+4 = D+2$, a contradiction.

Now suppose that $[2;3] \in E(\vec{G})$. Note that $\vec{d}(2k+2, 2) = \vec{d}(2k+2, 1)+1$. Any center path ¿from $2k+2$ to 1 is of length at least $2c = 2k+4$ and other paths are of length at least $2+(2k+1) = 2k+3$. Hence, $\vec{d}(2k+2, 2) \geq 2k+4 = D+2$, a contradiction again. □

Theorem 3.8 *If $c = k+1$ then $\vec{D} \geq D+2$ except for $AN(2,4)$.*

Proof : Let $\vec{G}$ be any orientation. If outerspokes in $\vec{G}$ alternate, then $\vec{D} \geq \min\{2c+2, 2k+3\} = 2k+3 = c+k+2 = D+2$ by Lemma 3.2. Hence, the theorem is true or there exists a $\vec{G}$, $D(\vec{G}) = D+1$, containing the basic configuration (Figure 7).

If $s = 3$, then $\vec{d}(1, 0) = 3 = D + 2$, so we may assume $s \geq 5$.

First suppose $[3; 2] \in E(\vec{G})$. Note that $\vec{d}(1, 2k{+}2) = \vec{d}(2, 2k{+}2){+}1$. Any center path from 2 to $2k{+}2$ is of length at least $2c = 2k{+}2$ and any non-center path is of length at least $2k + 2$. Hence, $\vec{d}(1, 2k{+}2) \geq 2k{+}3 = D{+}2$, a contradiction.

Now suppose that $[2; 3] \in E(\vec{G})$. Note that $\vec{d}(2k{+}1, 2) = \vec{d}(2k{+}1, 1) + 1$. Any center path ¿from $2k{+}1$ to 1 is of length at least $2c = 2k + 2$ and other paths are of length at least $2k{+}2$. Hence, $\vec{d}(2k{+}1, 2) \geq 2k{+}3 = D{+}2$, a contradiction again. □

Theorem 3.9 *If $c \leq k$ then $\vec{D} \geq D{+}2$ except for $AN(1, 4)$.*

Proof : Let $\vec{G}$ be any orientation. If outerspokes in $\vec{G}$ alternate, then $\vec{D} \geq \min\{2c + 2, 2k + 3\} = 2c + 2 = D + 2$ by Lemma 3.2. Hence, the theorem is true or there exists a $\vec{G}$, $D(\vec{G}) \leq D{+}1$, containing the basic configuration (Figure 7).

Suppose that $[2; 3] \in E(\vec{G})$ (Figure 8 left); then $\vec{d}(y, 2) = \vec{d}(y, 1){+}1$. Let $y = 2k$, $2k{+}1$, or $2k{+}2$. Any non-center path from y to 1 is of length at least $2{+}t(y, 1) \geq 2 + 2k - 1$ because we cannot use exclusively circle c, either clockwise ($[-1; 0] \in E(\vec{G})$) or counterclockwise ($[2; 3] \in E(\vec{G})$). Hence any non-center path from y to 1 is of length at least $2k{+}2 \geq 2c + 2 = D + 2$. Therefore, we must have $os(2k) = os(2k{+}1) = os(2k{+}2) = in$. Now, $\vec{d}(1, 2k{+}2) = \vec{d}(2, 2k{+}2){+}1$, and any center path from 2 to $2k{+}2$ is of length at least $2c{+}1$. Any non-center path is of length $\geq 2{+}2k = D + 2$ except if c is uniformly oriented ¿from 2 to $2k{+}2$. By Lemma 3.4 (with $u = 2k - 1$), we get $D(\vec{G}) \geq D + 2$, a contradiction.

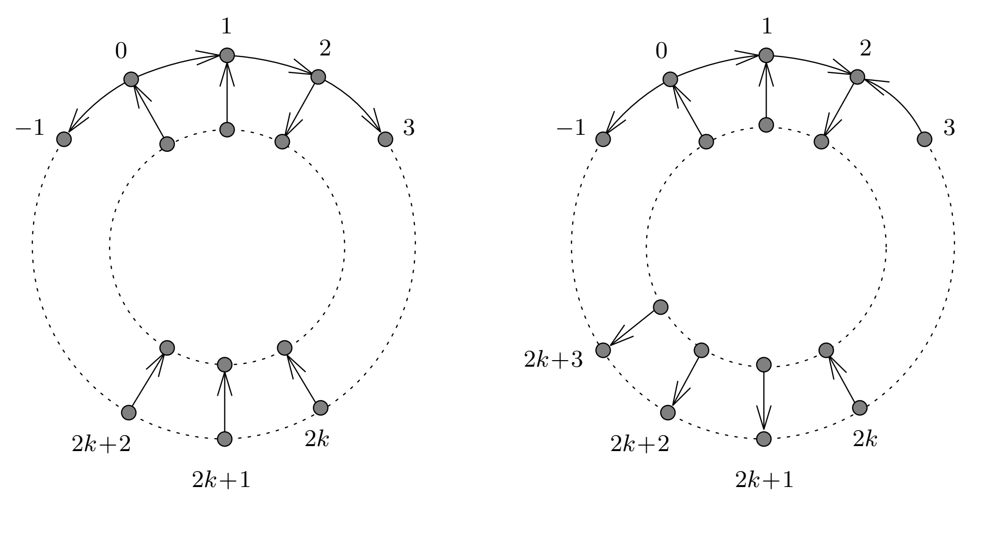

Figure 8: $c \leq k$, $os(0){=}os(1){=}out$ and $os(2){=}in$

Now suppose $[3;2] \in E(\vec{G})$ (Figure 8 right); then $\vec{d}(1,y) = \vec{d}(2,y)+1$. Let $y = 2k+1$, $2k+2$, or $2k+3$. Any non-center path from 2 to y is of length at least $2+t(2,y) = 2k+1$ since we cannot use circle c only. Hence, we must have $os(2k+1) = os(2k+2) = os(2k+3) = out$. Now any center path from $2k+2$ to 2 is of length at least $2c+2$ and any non-center path is of length at least $2k+2$ except if c is oriented counterclockwise from $2k+2$ to 2. By Lemma 3.3, we must have $[2k+2; 2k+3] \in E(\vec{G})$ and $os(2k) = in$. Now any center path from 0 to $2k$ is of length $2c+2$ and any non-center path of length $\geq 2k+2 = 2c+2$, a contradiction again. □

Theorem 3.10 *If $s = 4k+3 \geq 7$ and $c = k+1$, then $\vec{D} \geq D+3$.*

In order to give a proof of this Theorem, we need three lemmas:

Lemma 3.11 *Let $s = 4k+3 \geq 7$ and $c = k+1$. If $D(\vec{G}) \leq D+2$, then the outerspokes corresponding to internal vertices of any directed path P on circle c alternate in direction.*

Proof : Suppose that $\vec{G}$, $D(\vec{G}) \leq D+2$, contains a directed path P on circle c such that the outerspokes corresponding to its internal vertices do not alternate in direction. Without loss of generality we may assume that P is on four vertices (otherwise we consider $P' \subset P$ with two internal vertices having the same direction of outerspokes). Furthermore, as in the beginning of the proof of Lemma 3.4, we may assume that $P = -1, 0, 1, 2$ and that $os(0) = os(1) = out$ (Figure 9).

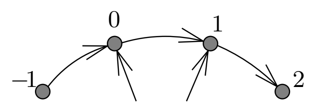

Figure 9: $s = 4k+3$, $c = k+1$. If $D(\vec{G}) \leq D+2$, then the configuration above is impossible.

Then $\vec{d}(0,y) = 2+d(2,y)$. Let $y = 2k+4$. Any center path from 2 to y is of length at least $2c = 2k+2$, and any non-center path is of length at least $\geq 2k+2$. Hence, $\vec{d}(0,y) \geq 2k+4$ $(= D+3)$, a contradiction. □

Lemma 3.12 *Let $s = 4k+3 \geq 7$ and $c = k+1$. Pick $\vec{G}$ such that $D(\vec{G}) \leq D+2$. Let x be a vertex on the circle c such that $[x-1; x], [x; x+1] \in E(\vec{G})$. Then:*
(a): *If $os(x) = out$ and the circle c is not oriented clockwise from x to $x+2k+2$, then the spokes $x+1$, $x+2k+2$, and $x+2k+3$ are uniformly oriented: spoke $x+1$ is in, and spokes $x+2k+2$ and $x+2k+3$ are out. (Figure 10)*

(b): *If $os(x) = in$ and the circle c is not oriented clockwise from $x + 2k + 1$ to x, then the spokes $x - 1$, $x + 2k$, and $x + 2k + 1$ are uniformly oriented: spoke $x - 1$ is out, and spokes $x + 2k$ and $x + 2k + 1$ are in. (Figure 11)*

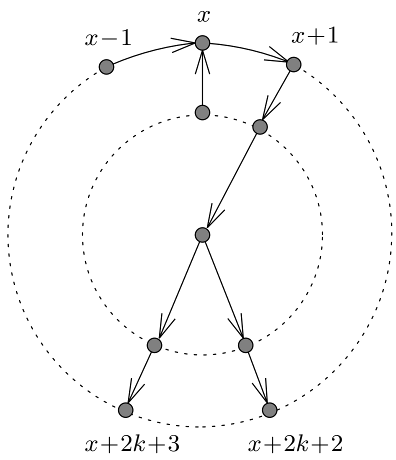

Figure 10: $c = k + 1$ and $s = 4k + 3$. $os(x) = out$ and $[x-1; x], [x; x+1] \in E(\vec{G})$.

Proof : It suffices to prove (a) only since (b) is just one possible restatement of (a) for an orientation equivalent to $\vec{G}$ with respect to symmetries discussed at the beginning of this section.

Note that $\vec{d}(x, y) = 1+d(x+1, y)$. Let $y = x+2k+2$ or $x+2k+3$. Since circle c is not oriented clockwise ¿from $x + 1$ to y, any non-center path from $x + 1$ to y has length at least $2k + 1 + 2$. Therefore, the only way to have $\vec{d}(x, y) \leq 2k + 3$ is to use a center path of length $2c = 2k + 2$ from $x + 1$ to y. Therefore, the spokes $x + 1$, $x + 2k + 2$, and $x + 2k + 3$ must be oriented as described in the statement of the lemma. □

Lemma 3.13 *Let $s = 4k+3 \geq 7$ and $c = k+1$. If the circle c is not uniformly oriented in $\vec{G}$ with $D(\vec{G}) \leq D+2$, then $\vec{G}$ does not contain a directed path on circle c on four vertices.*

Proof : By Lemma 3.11 it suffices to show that $\vec{G}$ does not contain a directed path on circle c, $P = x - 1, x, x + 1, x + 2$ with $os(x) \neq os(x + 1)$. Without loss of generality we may assume that P is oriented clockwise and that $os(x) = out$ and $os(x + 1) = in$.

First we prove that circle c must be oriented clockwise from x to $x + 2k + 2$. Otherwise, we reach a contradiction. Indeed, by Lemma 3.12(a), the spoke $x +$

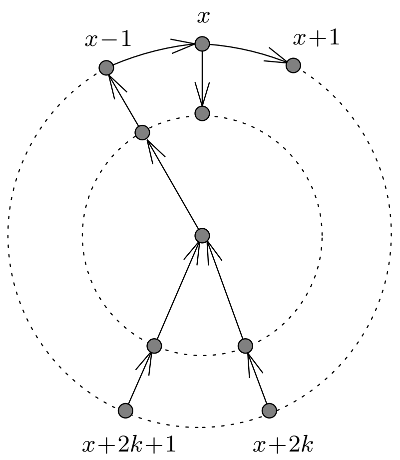

Figure 11: $c = k+1$ and $s = 4k+3$. $os(x) = in$ and $[x-1;x], [x;x+1] \in E(\vec{G})$.

$2k+2$ is *out*, and so, by Lemma 3.12(b) (with $x+1$), circle c must be oriented clockwise ¿from $x+2k+2$ to $x+1$. Then, by Lemma 3.11, $os(x+2k+4) = in$ and then $d(x, x+2k+4) = 2 + d(x+1, x+2k+3) \geq 2k+4$, a contradiction.

By Lemma 3.11, we conclude that outerspokes between x and $x+2k+2$ alternate in direction. Note that the vertices $x+2k-1$, $x+2k$, $x+2k+1$ and $x+2k+2$ form a directed path P' with $os(x+2k) = out$ and $os(x+2k+1) = in$. Hence, we use the same argument (now applied to P' instead of P) to conclude that circle c must be oriented clockwise from vertex $x+2k$ to vertex $x+4k+2 = x-1$. Therefore, circle c is uniformly oriented, a contradiction. $\square$

Now we can give a proof of Theorem 3.10.

Proof : Suppose that $\vec{D} \leq D+2$. Then there exists an orientation $\vec{G}$ such that $os(0) = os(1) = out$ (s is odd; so outerspokes can't alternate) and $[0;1] \in E(\vec{G})$. By strong connectivity, $[1;2] \in E(\vec{G})$ and By Lemma 3.11, $[0;–1] \in E(\vec{G})$. Furthermore, $os(2) = in$ because, $os(2) = out$ would imply $[2;3] \in E(\vec{G})$ by strong connectivity and we would have a configuration (with 0,1,2,3) forbidden by Lemma 3.11. Consequently, $[3;2] \in E(\vec{G})$ by Lemma 3.13. Hence, $\vec{G}$ contains the configuration shown on Figure 12.

Lemma 3.12 (with $x = 1$) implies $os(2k+3) = os(2k+4) = out$. Furthermore, $[2k+3; 2k+2] \in E(\vec{G})$, for otherwise by strong connectivity we would have $[2k+3; 2k+4]$ and $[2k+4; 2k+5] \in E(\vec{G})$, contradicting Lemma 3.11. Similarly, $[2k+4; 2k+5] \in E(\vec{G})$.

Note that for $s = 7$, $[6;0] = [-1;0] \in E(\vec{G})$ which is impossible because $[0;-1] \in E(\vec{G})$. Hence, we may assume $s \geq 11$ ($k \geq 2$).

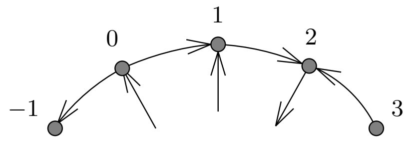

Figure 12: $c = k+1$ and $s = 4k+3$. $os(0) = os(1) = out$, $os(2) = in$, and $[0;-1], [0;1], [1;2], [2;3] \in E(\vec{G})$.

Case 1: $[2k{+}3; 2k{+}4] \in E(\vec{G})$. (Figure 13)

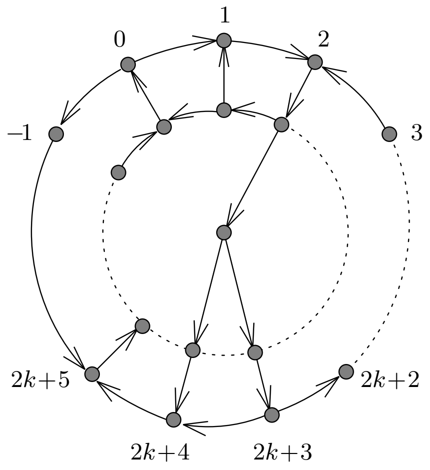

Figure 13: $c = k+1$ and $s = 4k+3$. $os(0) = os(1) = out$, $os(2) = in$ and $[0;-1], [0;1], [1;2], [2;3], [2k+3; 2k+2], [2k+3; 2k+4], [2k+4; 2k+5] \in E(\vec{G})$.

By Lemma 3.11, $os(2k{+}5) = in$. Any center path from 1 to $2k+5$ is of length at least $2c{+}2 = D{+}3$. A non-center path cannot use circle $c{-}2$ (the length would be at least $1{+}2k{+}4$), which implies $[(c{-}1,2);(c{-}1,1)]$, $[(c{-}1,1);(c{-}1,0)] \in E(\vec{G})$.

Now $\vec{d}(2k+4,2) = 1 + d(2k+5,2)$. Any center path and any non-center path from $2k+5$ to 2 using circle $c-2$ is too long. Therefore, any such path should end with $[(c{-}1,0);0]$, $[0;1]$, $[1;2]$. Hence, $[(c{-}1,{-}1);(c{-}1,0)] \in E(\vec{G})$.

But $[(c{-}1,{-}1);(c{-}1,0)] \in E(\vec{G})$ implies that the non-center path from 1 to $2k{+}5$ should use $[(c{-}1,0);0]$ and that circle c is oriented counterclockwise from 0 to $2k{+}5$.

If $k > 2$, we have a directed path on at least four vertices (from -1 to $2k+5$), contradicting Lemma 3.13.

If $k = 2$, then in order to have a path of length $2k+3 = 7$ from 0 to 7 we must have $[(c-1, 9); (c-1, 8)], [(c-1, 8); (c-1, 7)] \in E(\vec{G})$. Similarly, in order to have a path of length $2k+3 = 7$ from 8 to 2 we must have $[(c-1, 9); (c-1; 10)] \in E(\vec{G})$. Now, $\vec{d}(y; (c-1, 9)) = 1 + \vec{d}(y; 9)$. But then any path from 3 to 9 is of length at least seven and so $\vec{d}(3; (c-1, 9)) \geq 8 = D + 3$.

Case 2: $[2k{+}4; 2k{+}3] \in E(\vec{G})$ (Figure 14)

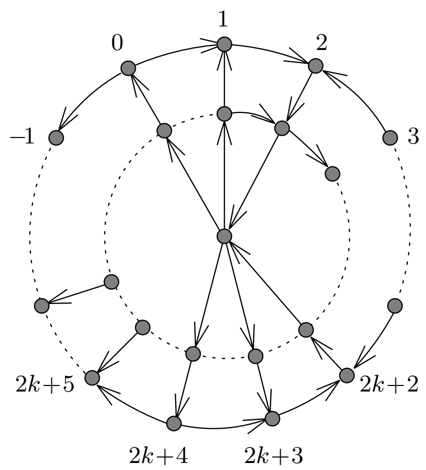

Figure 14: $c = k + 1$ and $s = 4k + 3$. $os(0) = os(1) = out$, $os(2) = in$ and $[0; -1], [0; 1], [1; 2], [2; 3], [2k + 3; 2k + 2], [2k + 4; 2k + 3], [2k + 4; 2k + 5] \in E(\vec{G})$.

By Lemma 3.12 (with $x = 2k + 3$) we conclude that the spokes 0 $(= 2k + 3 - (2k + 3))$ and 1 $(= 2k + 3 - (2k + 2))$ are uniformly oriented *out* and the spoke $(2k + 2)$ is uniformly oriented *in*.

For $\vec{d}(1, 2k{+}2) \leq D{+}2$ we need to have a clockwise non-center path starting by $[1; 2]$, $[2; (c-1, 2)]$, $[(c-1, 2); (c-1, 3)]$ and finishing by $[2k{+}1; 2k{+}2]$. Hence, $[(c-1, 2); (c-1, 3)]$, $[2k{+}1; 2k{+}2] \in E(\vec{G})$.

We must have $[(c-1, 1); (c-1, 2)] \in E(\vec{G})$, for otherwise the only way to reach $(c-1, 2)$ would be through 2 and we would have $\vec{d}(2k+4, (c-1, 2)) \geq \vec{d}(2k{+}4, 2) + 1 \geq 2k + 4 = D + 3$.

Finally, any non-center path from 1 to $2k{+}5$ or $2k{+}6$ has to use arcs $[1; 2]$ and $[2; (c-1, 2)]$, it cannot use circle $c - 2$ (otherwise it would be of length at least $2k{+}4 = D + 3$) so it has to use $[(c{-}1, 2); (c{-}1, 3)]$. But any such path has length at least $2k{+}5$. Therefore, we have to use a center path from 1 to $2k{+}5$ and $2k{+}6$ which implies that the spokes $2k{+}5$ and $2k{+}6$ are *out*. By strong connectivity $[2k{+}5; 2k{+}6]$ and $[2k{+}6; 2k{+}7] \in E(\vec{G})$, contradicting Lemma 3.11 □

We end this section by noting that Theorem 3.9 (case $c \leq k$), Theorem 3.8, Theorem 3.10 (case $c = k + 1$), Lemma 3.1 (case $c \geq k + 2$), Corollary 3.5, and Corollary 3.6 give all the lower bounds stated in Table 1.

4 Concluding Remarks

There are still a few open cases in our analysis of optimal orientations of Annular Networks. Namely, there is a gap of one between lower and upper bounds in the cases $s = 4k + 1$ with $k + 1 \leq c \leq k + 2$ and in the cases $s = 4k + 3$ with $c > k + 1$. Also, it would be interesting to know whether the orientations presented in Section 2 are unique (up to oriented distance-preserving transformations that were presented at the beginning of Section 3). These orientations might not be optimal with respect to some different measures of optimality such as $\min \sum_{x,y} \vec{d}(x, y)$ or $\min \max_{x,y}(\vec{d}(x, y) - d(x, y))$ or $\min_x \max_y \vec{d}(x, y)$ (or their weighted analogues). However, the methods that we have used to obtain lower bounds on $\vec{D}(AN(c, s))$ would most probably be useful in obtaining non-trivial lower bounds in these cases.

A more general problem, or a research topic, would be to develop some general methods or techniques for proving optimality of an orientation of a graph (e.g., minimizing diameter as one of the "simplest" non-trivial objective functions). The extensive case analysis such as in this paper, in the series of papers analyzing the oriented diameter of rectangular grids [19, 20, 21, 22],... is not the only similarity among the methods used to determine optimal values of oriented diameter for various classes of graphs. It seems that many lower bound proofs repeatedly use arguments of a similar flavor. This indicates that there might exist some more general proof technique. However, even though the problem of optimizing oriented diameter of an Annular Network is easy to state, it might be unrealistic to expect simple optimality proofs: the difference between the oriented diameter and the diameter of an Annular Network is at most three and all possible values $\vec{D} = D + i$, $0 \leq i \leq 3$, can be achieved.

Appendix

Proof of Theorem 2.3:
We will show that $D(\vec{O}2) \leq A$ if $c = k + 2$ and $s = 4k + 2$. In all other cases we will show that $D(\vec{O}1) \leq A$. Hence, we need to show that $\vec{d}(x, y) \leq A$ for any two vertices $x = (i, u)$ and $y = (j, v)$. Note that we may assume that $i \geq j$ because $\vec{d}((i, u), (j, v)) = \vec{d}((j, v + 1), (i, u + 1))$ in both $\vec{O}1$ and $\vec{O}2$.
Case $i = j = c$:

First we consider $\vec{O}1$.

We have a path Q of length $\alpha + 2$ using circle $c - 1$ clockwise. We reach the circle c directly from vertex $x = (c, u)$ if $u = in$, and via vertex $(c, u+1)$ if $u = out$.

Similarly, we leave the circle $c-1$ at vertex $(c-1, j)$ if $v = out$ or at $(c-1, j-1)$ if $v = in$ (Figure 15).

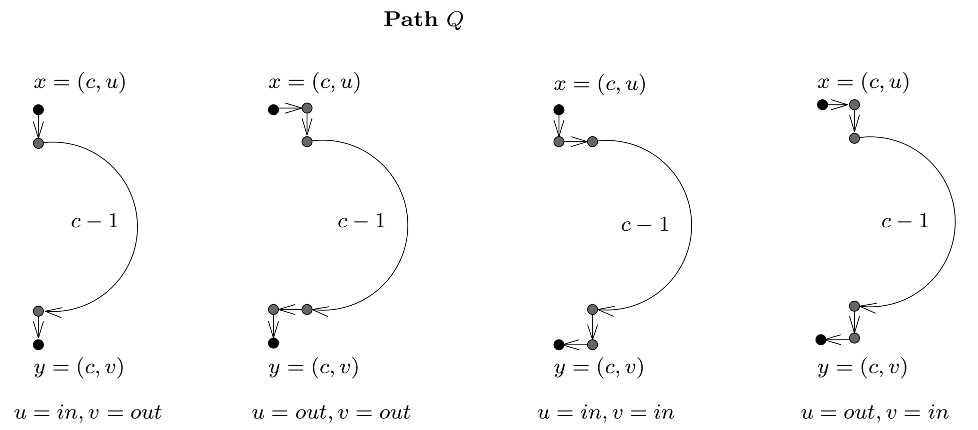

Figure 15: $c \geq k+1$, s even, $i = j = c$, orientation $\vec{O}1$, path Q.

We also have a path R of length $s - \alpha + 4$ using circle $c-2$ counterclockwise (Figure 16).

Therefore, $\vec{d}(x, y) \leq \min\{\alpha + 2, s - \alpha + 4\} \leq 2k + 3 \leq A$ except if $s = 4k + 2$ and $\alpha = 2k + 2$. In this case:
(a) $c \geq k + 3$. Then $\vec{d}(x, y) = 2k + 4 \leq c + k + 1 = A$
(b) $c = k + 1$. Since x and y have the same parity (both are in or both are out) and the center path, using spokes u and $v-1$ if $u = v = in$ (or spokes $u+1$ and v if $u = v = out$), is of length $2c + 1 = 2k + 3 = A$.

We use orientation $\vec{O}2$ when $s = 4k+2$ and $c = k+2$. Here we have a path Q of length α using circle c clockwise and a path R using circle $c-1$ counterclockwise (Figure 17). The length of R is

$$s - \alpha + \begin{cases} 2 & \text{if } u = in \text{ and } v = out \\ 4 & \text{if } u = v = in \text{ or } u = v = out \\ 6 & \text{if } u = out \text{ and } v = in \end{cases} \tag{1}$$

So, $\vec{d}(x, y) \leq \min\{\alpha, s - \alpha + 6\} \leq 2k + 3$ except if $\alpha = 2k + 4$. But if $\alpha = 2k + 4$ then spokes u and v have the same parity and the path R is of length $4k + 2 - (2k + 4) + 4 = 2k + 2$.

Case $i = j \leq c - 1$**:**

Without loss of generality we may assume circle i $(= j)$ is oriented clockwise (if not, replace α by $s - \alpha$ in the proof).

There is a path Q of length α using circle i clockwise and a path R of length at most $s-\alpha+6$ (as in (1)) using circle $i-1$ counterclockwise. The length of R is

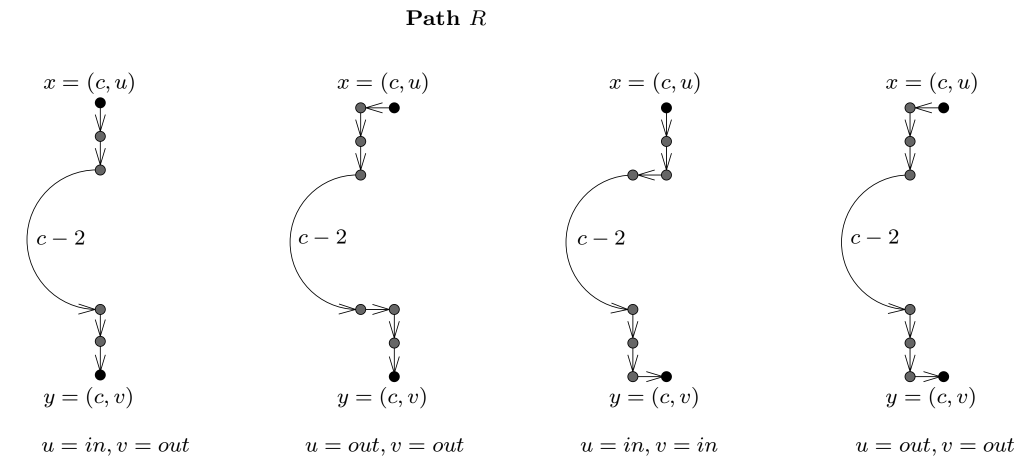

Figure 16: $c \geq k + 1$, s even, $i = j = c$, orientation $\vec{O}1$, path R.

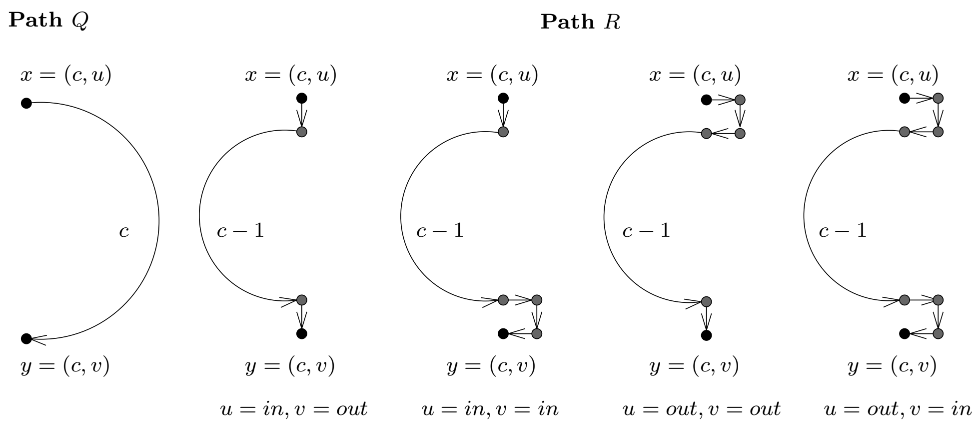

Figure 17: $c \geq k + 1$, s even, $i = j = c$, orientation $\vec{O}2$, paths Q and R.

$s-\alpha+4$ if u and v have the same parity. Hence, $\vec{d}(x,y) \leq \min\{\alpha, s-\alpha+6\} \leq 2k+3$ unless $\alpha = 2k+4$ and $s = 4k+2$, but in that case the path R is of length $2k+2$.

Case $i = c,\ j = c-1$:

We first consider $\vec{O}1$. Note that we may assume $\alpha > 0$ (otherwise, $\vec{d}(x,y) \leq 3 \leq A$). There exists a path Q of length $\alpha+1$ using circle $c-1$ clockwise: we reach the circle $c-1$ directly from vertex $x = (c,u)$ if $u = in$ or via vertex $(c, u{+}1)$ if $u = out$, and we follow circle $c-1$ to vertex (j,v) (Figure 18).

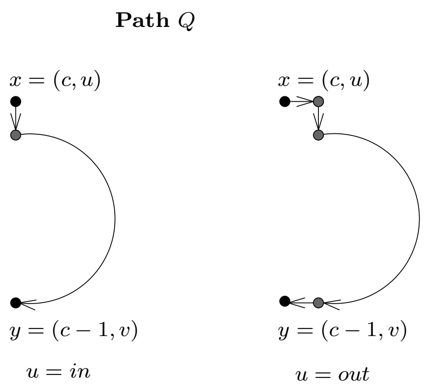

Figure 18: $c \geq k+1$, s even, $i = c, j = c-1$, orientation $\vec{O}1$, path Q.

We also have a path R using circle $c-2$ counterclockwise, of length $s-\alpha+3$ if $v = out$, or of length $s-\alpha+5$ if $v = in$ (Figure 19).

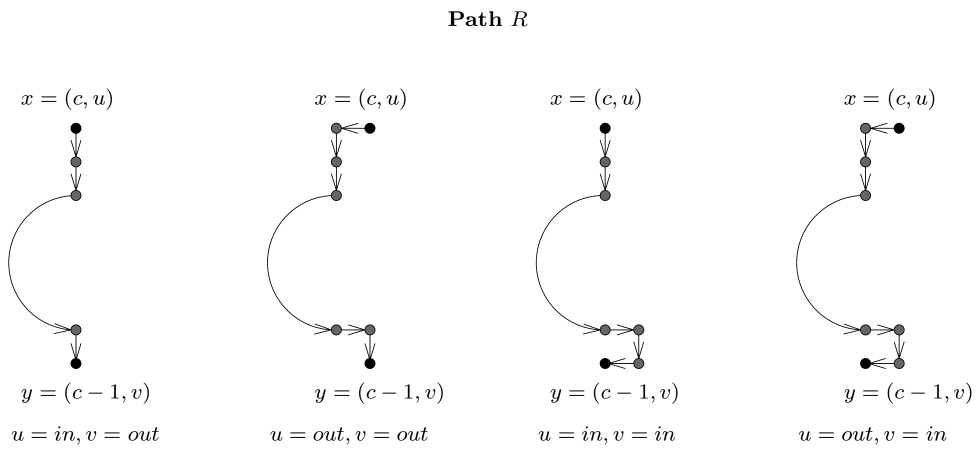

Figure 19: $c \geq k+1$, s even, $i = c, j = c-1$, orientation $\vec{O}1$, path R.

Hence, $\vec{d}(x,y) \leq \min\{\alpha+1, s-\alpha+5\} \leq 2k+3$ except if $\alpha = 2k+3$ and $s = 4k+2$. This is handled as in the case $i = j = c$: if $c \geq k+3$ then $\vec{d}(x,y) \leq 2k+4 \leq c+k+1$, and if $c = k+1$ then the center path is of length $c+c-1+2 = 2c+1 = 2k+3$.

We use $\vec{O}2$ when $s = 4k+2$ and $c = k+2$. We have a path Q using circle c of length $\alpha+1$ if $v = in$ or $\alpha+3$ if $v = out$ and a path R using circle $c-1$ of length $s-\alpha+1$ if $u = in$ or $s-\alpha+3$ if $u = out$ (Figure 20). So

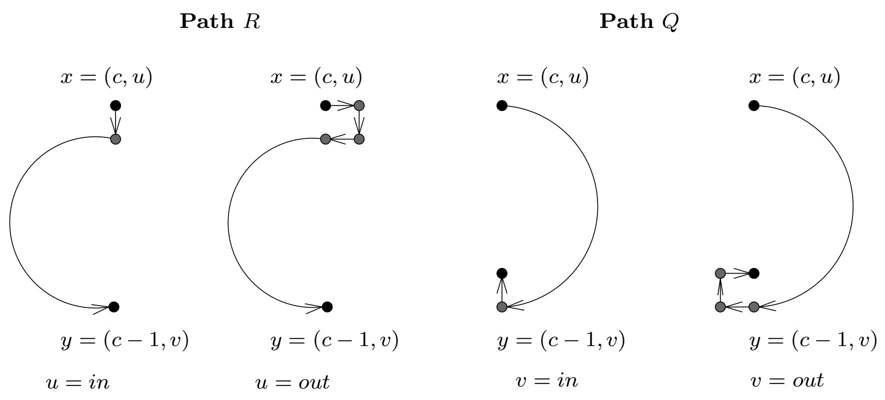

Figure 20: $c \geq k+1$, s even, $i = c, j = c-1$, orientation $\vec{O}2$, paths R and Q.

$\vec{d}(x,y) \leq \min\{\alpha+3, s-\alpha+3\} \leq 2k+3$ unless $\alpha = 2k+1$. But for $\alpha = 2k+1$, u and v have different parity. If $u = in$, then the path R is of length $2k+2$, and if $u = out$, then the path Q is of length $2k+2$.

Case $j \leq c-2$ and $j < i$:

Without loss of generality we may assume that circle j is oriented clockwise (otherwise, replace α with $s-\alpha$ in the proof).

We first assume that $u = in$. We will consider three directed xy-paths (Figure 21):

- Path P of length $i+j$, using spokes u and v if $v = out$, or of length $i+j+1$, using spoke u and spoke $v-1$ if $v = in$.
- Path Q of length $i-j+\alpha$, using spoke u and circle j.
- Path R of length $i-j+s-\alpha$, using spoke u and circle $j+1$ if $v = in$, or of length $i-j+s-\alpha+2$, using spoke u and circle $j-1$ if $v = out$.

Note that these paths exist in both $\vec{O}1$ and $\vec{O}2$.

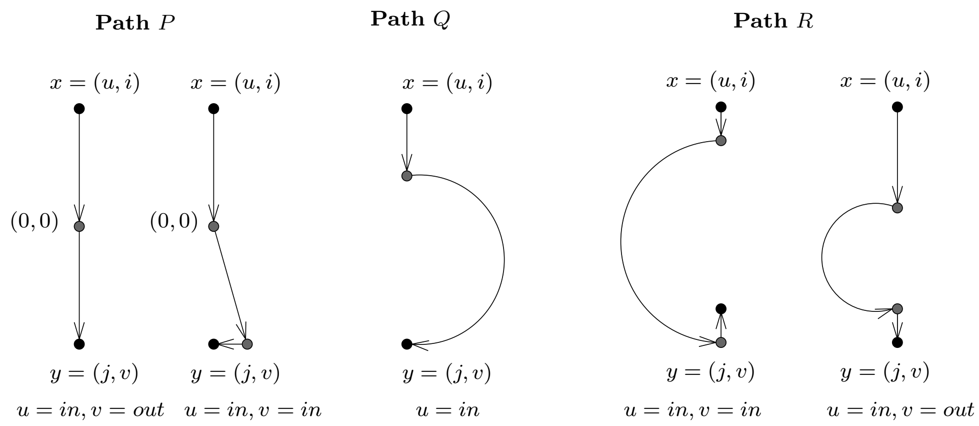

Figure 21: $c \geq k+1$, s even, $j \leq c-2$ and $j < i$, paths P, Q and R.

Hence, if $v = in$,

$$\begin{aligned}
\vec{d}(x,y) &\leq \min\{i+j+1, i-j+\alpha, i-j+s-\alpha\} \\
&= i + \min\{2j+1, \alpha, s-\alpha\} - j \\
&\leq c + \min\{2j+1, \alpha, s-\alpha\} - j && (2) \\
&\leq c + k && (3) \\
&\leq A. && (4)
\end{aligned}$$

Inequality (3) holds either because $j \leq k-1$, or because $j \geq k$ and $\min\{\alpha, s-\alpha\} \leq 2k$; indeed α must be even (and so $\alpha \neq 2k+1$), since both u and v are *in*. Also note that $\vec{d}(x,y) = A$ only if $i = c$ (to make (2) an equality), $s = 4k$ (to make (4) an equality) and $\alpha = 2k$ and $k-1 \leq j \leq k$ (to make (3) an equality). In other words, $\vec{d}(x,y) = A$ implies $x = (c, u)$, and $y = (k, u+2k)$ or $y = (k-1, u+2k)$.

Similarly, if $v = out$,

$$\begin{aligned}
\vec{d}(x,y) &\leq \min\{i+j, i-j+\alpha, i-j+s-\alpha+2\} \\
&= i + \min\{2j, \alpha, s-\alpha+2\} - j \\
&\leq c + \min\{2j, \alpha, s-\alpha+2\} - j && (5) \\
&\leq c + k && (6) \\
&\leq A. && (7)
\end{aligned}$$

Inequality (6) holds either because $j \leq k$, or because $j \geq k+1$ and $\min\{\alpha, s-\alpha+2\} \leq 2k+1$; Furthermore, $\vec{d}(x,y) = A$ only if $i = c$ (to make (5) an equality), $s = 4k$ (to make (7) an equality), and $\alpha = 2k+1$ and $k \leq j \leq k+1$ (to make

(6) an equality and because α must be odd, since $u = in$ and $v = out$, and so $\alpha \neq 2k$ and $\alpha \neq 2k+2$). In other words, $\vec{d}(x,y) = A$ implies $x = (c,u)$, and $y = (k, u+2k+1)$ or $y = (k+1, u+2k+1)$.

Hence, for any $x = (c,u)$ with $u = in$: $\vec{d}(x,y) \leq A$ and the equality holds only if $s = 4k$, and if $y \in \{(k-1, u+2k), (k, u+2k), (k, u+2k+1), (k+1, u+2k+1)\}$.

Finally, we consider the case $u = out$. Let $x^- = (i, u-1)$ and $x^+ = (i, u+1)$. Let $x' \in \{x^-, x^+\}$ such that $[x; x']$ is an arc in $\vec{O}1$ ($\vec{O}2$). Now, $\vec{d}(x,y) = 1 + \vec{d}(x',y) \leq A$ except if $\vec{d}(x',y) = A$. But then $s = 4k$ and $c = i$. In this case we use $\vec{O}1$ and both $[x; x^-]$ and $[x; x^+]$ are arcs in $\vec{O}1$ (since $u = out$), and we have

$$\vec{d}(x,y) \leq 1 + \min\{\vec{d}(x^+,y), \vec{d}(x^-,y)\} \leq A$$

since there is no y such that $\vec{d}(x^+,y) = \vec{d}(x^-,y) = A$. □

Proof of Theorem 2.5:
We will show that $D(\vec{O}1) \leq A$. So, for any two vertices $x = (i,u)$ and $y = (j,v)$, we have to show $\vec{d}(x,y) \leq A$. The proof is very similar to the proof of Theorem 2.3.

Case $i = j = c$ **:**

If $v \neq 0$, we have a path Q of length $\alpha + 2$ using circle $c-1$ clockwise (see Figure 15). If $v = 0$, we modify the end of the path Q, ending it with $\ldots (c-1, s-1), (c-1, 0), (c-1, 1), (c, 1), (c, 0) = y$, obtaining a path of length $\alpha + 4$.

We also have a path R of length $s - \alpha + 4$ using circle $c-2$ counterclockwise (see Figure 16 if $v \neq s-1$; if $v = s-1$, we modify the end of the path R ending it with $\ldots (c-2, 1), (c-1, 1), (c, 1), (c, 0), (c, s-1) = v$).

Therefore, $\vec{d}(x,y) \leq \min\{\alpha+4, s-\alpha+4\} \leq A$.

Case $i = j \leq c-1$**:**

Without loss of generality we may assume circle i $(= j)$ is oriented clockwise (if not, replace α by $s - \alpha$ in the proof).

There is a path Q of length α using circle i clockwise and a path R of length at most $s - \alpha + 6$ (as in (1)) using circle $i-1$ counterclockwise.

Hence, $\vec{d}(x,y) \leq \min\{\alpha, s-\alpha+6\} \leq A$.

Case $i = c$**,** $j = c-1$**:**

Note that we may assume $\alpha > 0$ (otherwise, $\vec{d}(x,y) \leq 3 \leq A$).

There exists a path Q of length $\alpha+1$ using circle $c-1$ clockwise (see Figure 18).

There is also a path R using circle $c-2$ counterclockwise, of length $s-\alpha+3$ if $v = out$, or of length $s-\alpha+5$ if $v = in$ (see Figure 19; if $v = 0$, we modify the end of the path R, ending it with $\ldots (c-2, 1), (c-1, 1), (c, 1), (c, 0), (c-1, 0) = y$, which gives a path of length $s-\alpha+5$).

Hence, $\vec{d}(x,y) \leq \min\{\alpha+1, s-\alpha+5\} \leq A-1$.

Case $i = c-1$**,** $j = c$**:**

As in the previous case, we may assume $\alpha > 0$ (otherwise, $\vec{d}(x, y) \le 5 \le A$).

If $v \ne 0$, there is a path Q of length $\alpha + 1$ using circle $c - 1$ clockwise: Q follows circle $c - 1$ ending with $\ldots (c - 1, v), (c, v) = y$ if $v = out$, or ending with $\ldots (c-1, v-1), (c, v-1), (c, v) = y$ if $v = in$. If $v = 0$, we modify Q to end with $\ldots (c-1, s-1), (c-1, 0), (c-1, 1), (c, 1), (c, 0) = v$, which gives a path of length $\alpha + 3$.

There is also a path R using circle $c-2$ counterclockwise, of length $s - \alpha + 3$ if $u = in$ (starting with $x = (c - 1, u), (c - 2, u), \ldots$), or of length $s - \alpha + 5$ if $u = out$ (starting with $x = (c - 1, u), (c - 1, u + 1), (c - 2, u + 1), \ldots$). The path R ends with $\ldots (c - 2, v), (c - 1, v), (c, v) = y$ if $v = out$, and with $\ldots (c - 2, v + 1), (c - 1, v + 1), (c, v + 1, (c, v) = y$ if $v = in$ (if $v = s - 1$, R ends with $\ldots (c-2, 1), (c-1, 1), (c, 1), (c, 0), (c, s-1))$.

Hence, $\vec{d}(x, y) \le \min\{\alpha + 3, s - \alpha + 5\} \le A$.

Case $j \le c - 2$, $j < i$:

Without loss of generality we may assume that circle j is oriented clockwise (otherwise, replace α by $s - \alpha$ in the proof).

We first assume that $u = in$. Exactly as in the corresponding case from the proof of Theorem 2.3, we conclude (from (3) if $v = in$, and from (6) if $v = out$) that $\vec{d}(x, y) \le c + k$. Since $c + k \le A - 1$, we have $\vec{d}(x, y) \le A - 1$.

If $u = out$, then $[x; x']$ is an arc in $\vec{O}1$ for either $x' = (i, u + 1)$ or for $x' = (i, u - 1)$. Hence, $\vec{d}(x, y) \le 1 + \vec{d}(x', y) \le 1 + (A - 1) = A$ since the spoke of x' is in.

Case $i \le c - 2$, $i < j$:

Without loss of generality we may assume that circle i is oriented clockwise (otherwise, replace α with $s - \alpha$ in the proof).

We first assume that $v = out$. We will consider three directed xy-paths:

- Path P of length $i + j$ using spokes u and v if $u = in$, or of length $i + j + 1$ using spoke $u + 1$ and spoke v if $u = out$.
- Path Q of length $\alpha + j - i$ using circle i and spoke v.
- Path R of length $s - \alpha + j - i$ using circle $i + 1$ and spoke v if $u = out$, or of length $s - \alpha + 2 + j - i$ using circle $i - 1$ and spoke v if $u = in$.

Hence, if $u = out$,

$$\begin{aligned} \vec{d}(x, y) &\le \min\{i + j + 1, \alpha + j - i, s - \alpha + j - i\} \\ &= j + \min\{2i + 1, \alpha, s - \alpha\} - i \\ &\le c + \min\{2i + 1, \lfloor s/2 \rfloor\} - i \\ &\le A - 1. \end{aligned}$$

The last inequality holds since $\min\{2i+1, \lfloor s/2 \rfloor\} - i$ is at most k when $s = 4k+1$, and it is at most $k + 1$ when $s = 4k + 3$.

Similarly, if $u = in$,

$$\begin{aligned}\vec{d}(x,y) &\leq \min\{i+j, \alpha+j-i, s-\alpha+2+j-i\} \\ &= j+\min\{2i, \alpha, s-\alpha+2\}-i \\ &\leq c+\min\{2i, \lceil s/2\rceil\}-i \\ &\leq A-1.\end{aligned}$$

The last inequality holds since $\min\{2i, \lceil s/2\rceil\} - i$ is at most k when $s = 4k+1$, and it is at most $k+1$ when $s = 4k+3$.

Therefore, if $v = out$, $\vec{d}(x,y) \leq A-1$.

If $v = in$, then $[y'; y]$ is an arc in $\vec{O}1$ for either $y' = (j, v{+}1)$ or for $y' = (j, v{-}1)$. Hence, $\vec{d}(x,y) \leq \vec{d}(x,y') + 1 \leq (A-1)+1 = A$ since the spoke of y' is *out*. □

References

[1] D.P. Bertsekas, and J.N. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods*, Prentice–Hall, Englewood Cliffs, NJ, 1989.

[2] F. Boesch and R. Tindell, “Robbins’ theorem for mixed multigraphs”, *Am. Math. Monthly*, **87** (1980), 716–719.

[3] V. Chvátal and C. Thomassen, Distances in orientations of graphs. *J. of Combinatorial Theory (B)*, **24** (1978), 61–75.

[4] G. Fox, M. Johnsson, G. Lyzenga, S. Otto, J. Salmon and D. Walker, *Solving Problems on Concurrent Processors, Volume I*, Prentice Hall, 1988.

[5] P. Fraignaud and E. Lazard, “Methods and Problems of Communication in Usual Networks”, *Discrete Applied Mathematics*, **53** [1994), 79–134.

[6] Z.G. Gutin, “Minimizing and maximizing the diameter in orientations of graphs”, *Graphs and Combinatorics*, **10** (1994), 225–230.

[7] S.M. Hedetniemi, S.T. Hedetniemi and A. Liestman, “A Survey of Gossiping and Broadcasting in Communication Networks”, *Networks*, **18** (1988), 129–134.

[8] K.M. Koh and B.P. Tan, “The diameter of an orientation of a complete multipartite graph”, *Discrete Mathematics*, **149** (1996), 131-139. See also addendum *Discrete Mathematics*, **173** (1997), 297-298.

[9] K.M. Koh and B.P. Tan, “The minimum diameter of orientations of complete multipartite graphs”, *Graphs and Combinatorics* **12** (1996), 333-339.

[10] K.M. Koh and E.G. Tay "Optimal orientations of products of paths and cycles", *Discrete Applied Mathematics*, bf 78 (1997), no. 1-3, 163–174.

[11] J-C. König, D.W. Krumme and E. Lazard, "Diameter-preserving orientation of the torus", *Networks*, **32** (1998), 1–11.

[12] D.W. Krumme, K.N. Venkataraman and G. Cybenko, "Gossiping in Minimal Time", *SIAM J. on Computing*, **21** (1992), 111–139.

[13] J.E. McCanna, "Orientations of the n-cube with minimum diameter", *Discrete Mathematics* **68** (1988), 309–310.

[14] C. Martin, "La Cité annulaire", PhD thesis, Université de Nice–Sophia Antipolis (1995).

[15] J. Plesnik, "Remarks on diameters of orientations of graphs", *Math. Univ. Comeniance* **46/47** (1986), 2225-236.

[16] H.E. Robbins, "A theorem on graphs, with an application to a problem of traffic control", *Am. Math. Monthly* **46** (1939), 281–283.

[17] F.S. Roberts, *Discrete Mathematical Models, with Applications to Social, Biological, and Environmental Problems*, Prentice–Hall, Englewood Cliffs, NJ, 1976.

[18] F.S. Roberts, *Graph Theory and its Applications to Problems of Society*, NSF–CMBS Monograph No. 29, SIAM, Philadelphia, PA, 1978.

[19] F.S. Roberts and Y. Xu, "On the optimal strongly connected orientations of city street graphs I: Large grids", *SIAM J. Discrete Mathematics* **1** (1988), 199–222.

[20] F.S. Roberts and Y. Xu, "On the optimal strongly connected orientations of city street graphs II: Two east-west avenues or north-south streets", *Networks* **19** (1989), 221–233.

[21] F.S. Roberts and Y. Xu, "On the optimal strongly connected orientations of city street graphs III: Three east-west avenues or north-south streets", *Networks* **22** (1992), 109–143.

[22] F.S. Roberts and Y. Xu, "On the optimal strongly connected orientations of city street graphs IV: Four east-west avenues or north-south streets", *Discrete Applied Mathematics* **49** (1994), 331–356.

[23] J. de Rumeur, *Communications dans les rseaux de processeurs*, MASSON, Paris (1994).