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Stable Effectivity Functions and Perfect Graphs[1,2]

by

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ABSTRACT

We consider the problem of characterizing the stability of effectivity functions (EFF), via a combinatorial correspondance between game theoretic and well-known combinatorial concepts. To every EFF we assign a pair of hypergraphs, representing clique covers of two associated graphs, and obtain some necessary and some sufficient conditions for the stability of EFFs in terms of graph-properties. These conditions imply e.g. that to check the stability of an EFF is an NP-complete problem. We also translate some well known conjectures of graph theory into game theoretic language and vice versa.

Keywords: core, kernel, stable effectivity function, perfect graph, kernel-solvable graph

1 Introduction.

Let us consider multiplayer games, in which I and A denote finite sets of *players* and *outcomes*, respectively. Subsets of players are called *coalitions*, while subsets of outcomes are called *blocks*. An *effectivity function* (or EFF in short) is a mapping $\mathcal{E} : 2^I \times 2^A \mapsto \{0,1\}$ representing the "power" of the players, in very general terms. Namely, $\mathcal{E}(K,B) = 1$ for $K \subseteq I$ and $B \subseteq A$, i.e. K is said to be *effective* for B, exactly when the coalition K is able to guarantee that an outcome from the block B will be realized, see e.g. Moulin (1983), Peleg (1984). (Obviously, the players of a coalition K may choose to cooperate in many different ways, hence the same coalition may be effective for many different blocks of the outcomes.)

We shall consider *monotone* effectivity functions, i.e. for which $\mathcal{E}(K',B') \geq \mathcal{E}(K,B)$ holds for all $K' \supseteq K$ and $B' \supseteq B$. It is quite natural to assume both that a larger set of players have a greater influence, and that it is easier to guarantee the final outcome to fall into a larger set. Hence monotonicity is a natural assumption for effectivity functions corresponding to games.

A simple example could be a voting game, i.e. in which players are voters, and outcomes are the candidates. In this case the effectivity function describes the power of certain groups of voters (coalitions) to eliminate some of the candidates from the final race.

A more general example is the family of games described by game forms $< I, A, X, g >$, where $X = \times_{i \in I} X_i$, and the (finite) set X_i represents the possible strategies of the player $i \in I$, and where the mapping $g : X \mapsto A$ defines the game form, i.e. determines the outcome for each situation in which all the players have chosen a strategy. A coalition $K \subseteq I$ is effective for a block $B \subseteq A$ whenever the players $i \in K$ can choose a strategy vector $x_K \in \times_{i \in K} X_i$ such that the restriction $g(x_K, *)$ takes values only from B, i.e. if the outcome of the game falls into the set B no matter what strategy the players outside of K follow.

Not all effectivity functions correspond, of course, to game forms. Besides the monotonicity, there are other (fairly natural) properties satisfied by effectivity functions of game forms. One of them, the so called *superadditivity* states that if two disjoint coalitions K_1 and K_2 are respectively effective for the blocks B_1 and B_2, then, since the players of both coalitions can independently practice their power, the union coalition $K_1 \cup K_2$ must also be effective for the intersection $B_1 \cap B_2$. Moulin and Peleg (1982) proved that monotonicity and superadditivity together with some boundary conditions characterize the effectivity functions corresponding to game forms.

The individual preferences of the players over the different outcomes is represented by a *utility function*, i.e. by a mapping of the form $u : I \times A \mapsto \mathbb{R}$, where the real value $u(i,a)$ for a player $i \in I$ and outcome $a \in A$ is interpreted as the profit of player i in the case of outcome a is realized. For a coalition $K \subseteq I$ and outcome $a \in A$ let $PR(K,a,u) = \{a' \in A | u(i,a') > u(i,a) \text{ for all players } i \in I\}$ denote the set of outcomes preferred over a uniformly by all players of K. We shall say that a coalition K can *reject* an outcome a if $\mathcal{E}(K, PR(K,a,u)) = 1$, i.e. if the players in K have the power guaranteeing that the outcome of the game will be preferred by all of them over a.

For a given effectivity function $\mathcal{E}$ and utility u, the *core* $\mathcal{C}(\mathcal{E}, u)$ is defined as the set of outcomes not rejected by any of the coalitions. One of the central problems of game theory is to find conditions guaranteeing that the core $\mathcal{C}(\mathcal{E}, u)$ is not empty. A somewhat surprising fact is that the non-emptiness of $\mathcal{C}(\mathcal{E}, u)$ can be guaranteed in certain cases, even if the utility function is not specified. In other words, one can find conditions based only on the structure of the effectivity function, which imply that $\mathcal{C}(\mathcal{E}, u) \neq \emptyset$ for all utility functions u. Effectivity functions for which this happens are called *stable*.

In this paper we study the stability of effectivity functions, present some necessary and some sufficient conditions for stability, and show, among other results that testing the stability of a given effectivity function is an NP-complete problem.

2 Results.

The prime tools in our analysis are combinatorial. We show that an effectivity function $\mathcal{E}$ can equivalently be represented by a pair of hypergraphs, forming clique covers of two graphs on the same set of vertices. We shall start with the interesting special case when the two graphs are complementary to each other. (We follow standard graph theoretical notations, some of them, for completeness, included in the Appendix.)

Let us consider a graph $G =< V, E >$, and let us assign a player to every maximal clique and an outcome to every maximal stable set of G. Let us denote by I the set of all the players (maximal cliques) and by A the set of all the outcomes (maximal stable sets). For every vertex $v \in V$ let us denote by K_v the family of maximal cliques containing vertex v, and similarly, let B_v denote the family of maximal stable sets containing vertex v. Finally, let us associate to the graph G an effectivity function $\mathcal{E}_G$ by defining $\mathcal{E}_G(K, B) = 1$ for a coalition $K \subseteq I$ and block $B \subseteq A$ if and only if $K_v \subseteq K$ and $B_v \subseteq B$ for some vertex $v \in V$ (and defining $\mathcal{E}_G(K, B) = 0$ otherwise.)

This correspondence was introduced and several results were shown by Boros and Gurvich (1994). In order to state the new results, we shall recall first some of the properties shown earlier. (For the precise definitions see the Appendix and/or Sections 3-4.)

Proposition 1 *A graph G is perfect if and only if the corresponding EFF $\mathcal{E}_G$ is balanced.*

This follows from a characterization of perfect graphs given by Lovasz (1972, Theorem 2.)

Proposition 2 *A graph G is kernel-solvable if and only if the corresponding EFF $\mathcal{E}_G$ is stable.*

This property is in fact a direct consequence of the criterion of stability stated by Keiding (1985), (see Section 3.)

Scarf (1967) proved that balanced NTU-games have non-empty cores, and later Danilov and Sotskov (1987, 1991, 1992) reformulated this result in terms of EFFs.

Proposition 3 *Balanced EFFs are stable.*

As a consequence of all above one can arrive to the following theorem.

Theorem 1 *Perfect graphs are kernel-solvable.*

This statement was conjectured by Berge and Duchet (1983). The converse direction, namely that

Conjecture 1 *Kernel-solvable graphs are perfect.*

was also conjectured in the same paper, and it is still an open problem. Berge and Duchet (1983) observed that Conjecture 1 would result from the Strong Perfect Graph Conjecture. Boros and Gurvich (1994) proved that Conjecture 1 is equivalent to either of the following conjectures.

Conjecture 2 *If a graph G is kernel-solvable then its complementary graph G^c is kernel-solvable, too.*

Conjecture 3 *If a graph G is kernel-solvable and a vertex of G is substituted by an edge (or by a clique) then the resulting graph G' is kernel-solvable, too.*

Thus, assuming that the above conjectures hold true, for EFFs corresponding to graphs a reasonably simple criterion of stability exists, stated in Proposition 2. However, only some very special EFFs are generated by graphs in the above way.

Let us consider in the sequel the general case, and let us represent an arbitrary EFF $\mathcal{E}$ by specifying explicitly the list of coalition - block pairs for which $\mathcal{E}$ takes value 1. More precisely, let us consider $\mathcal{E} =< K_j, B_j; J; I, A >$, where I denotes the set of players, and A stands for the set of outcomes, as above, and J denotes the set of indices of the pairs $K_j \subseteq I$, $B_j \subseteq A$ for $j \in J$ for which $\mathcal{E}(K_j, B_j) = 1$, i.e. for which coalition K_j is effective for block B_j. We can interpret the EFF $\mathcal{E}$ as a pair of hypergraphs $\mathcal{K}$ and $\mathcal{B}$ on disjoint sets of vertices I and A, respectively, the edges of which have a common set of indices J. Let us consider the dual (transposed) hypergraphs $\mathcal{C}$, and $\mathcal{S}$ on the common vertex set J defined by

$$\mathcal{C} = \{C_i \subseteq J | i \in I\}, \qquad \mathcal{S} = \{S_a \subseteq J | a \in A\},$$

where $C_i = \{j \in J | K_j \ni i\}$, and $S_a = \{j \in J | B_j \ni a\}$.

An EFF $\mathcal{E}$ can equally be specified by the hypergraphs $\mathcal{C}$, and $\mathcal{S}$, as well. In the sequel, we shall use both notations $\mathcal{E} =< C_i, S_a; I, A; J >$ and $\mathcal{E} =< K_j, B_j; J; I, A >$.

Given an EFF $\mathcal{E}$, Peleg (1984) introduced the dual EFF $\mathcal{E}^d$ by exchanging the roles of players and outcomes, i.e. $\mathcal{E}^d =< S_a, C_i, A, I, J >=< B_j, K_j, J, A, I >$ using the notations above.

Let us observe, e.g. that if an EFF $\mathcal{E}_G$ is generated by a graph $G =< V, E >$, then $J = V$ and the hyperedges of $\mathcal{C}$ and $\mathcal{S}$ are, respectively, the maximal cliques and maximal stable sets of G. Hence we have the relation

$$\mathcal{E}_G^d = \mathcal{E}_{G^c},$$

where G^c denotes the complementary graph to G.

We can now state some necessary and some sufficient conditions for the stability of an arbitrary EFF $\mathcal{E}$ in terms of the corresponding hypergraphs $\mathcal{C}$, and $\mathcal{S}$.

Theorem 2 *If an EFF $\mathcal{E} =< C_i, S_a; I, A; J >$ or its dual $\mathcal{E}^d$ is stable then,*

(i) *for every subset $J' \subseteq J$ such that $|C_i \cap J'| \leq 1$ for all $i \in I$ there exists an $a \in A$ such that $J' \subseteq S_a$, and*

(ii) *for every $J'' \subseteq J$ such that $|S_a \cap J''| \leq 1$ for all $a \in A$ there exists an $i \in I$ such that $J'' \subseteq C_i$.*

A simple consequence of this statement is the following corollary.

Corollary 1 *If an EFF $\mathcal{E} =< C_i, S_a; I, A; J >=< K_j, B_j; J; I, A >$ is stable then every pair $j, j' \in J$, $j \neq j'$ must belong either to C_i for some $i \in I$ or to S_a for some $a \in A$. In other words, if $\mathcal{E}(K_1, B_1) = \mathcal{E}(K_2, B_2) = 1$ and $K_1 \cap K_2 = B_1 \cap B_2 = \emptyset$ then the EFF $\mathcal{E}$ cannot be stable.*

Let us remark that conditions (i) and (ii) of Theorem 2 can be reformulated in dual terms, namely if for some $J' \subseteq J$ we have

(i^d) $\mathcal{E}(K_j, B_j) = 1$ for all $j \in J'$ and $K_j \cap K_{j'} = \emptyset$ for all $j \neq j' \in J'$, then $\cap_{i \in J'} B_j \neq \emptyset$ must hold; and similarly

(ii^d) $\mathcal{E}(K_j, B_j) = 1$ for all $j \in J'$ and $B_j \cap B_{j'} = \emptyset$ for all $j \neq j' \in J'$ must imply that $\cap_{i \in J'} K_j \neq \emptyset$.

Remark 1 *Gurvich and Vasin (1978) proved that condition (i^d) holds iff the EFF $\mathcal{E}$ is* playing-minor, *(i.e. iff $\mathcal{E} \leq \mathcal{E}_g$ holds for some EFF $\mathcal{E}_g$ corresponding to a normal game form g.) Later, Boros and Gurvich (1994) gave a shorter proof based on a theorem by Moulin and Peleg (1983) characterizing EFFs generated by game forms (as monotone and superadditive). Respectively, we can see that condition (ii^d) holds iff the dual EFF $\mathcal{E}^d$ is playing-minor. Thus playing-minority of both EFFs $\mathcal{E}$ and $\mathcal{E}^d$ are necessary for the stability of either one of $\mathcal{E}$ or $\mathcal{E}^d$.*

The main contribution of this paper is the following "perfect split criterion".

Theorem 3 *Both EFFs $\mathcal{E} =< C_i, S_a; I, A; J >$ and its dual $\mathcal{E}^d$ are stable if there exists a graph $G =< J, E >$ such that*

(i) *G is perfect;*

(ii) *every (maximal) clique of G is a subset of some C_i, $i \in I$;*

(ii') *every (maximal) stable set of G is a subset of some S_a, $a \in A$.*

From Theorems 2 and 3 we will be able to derive that

Theorem 4 *To check the stability of an EFF, given as $\mathcal{E} =< C_i, S_a;\ I, A;\ J >$ is an NP-complete problem.*

Using Theorems 2 and 3, we can extend the list of equivalent conjectures 1-3 on kernel-solvability and translate them into game theoretic language.

Conjecture 4 *If an EFF $\mathcal{E} =< K_j, B_j; J; I, A >$ is stable while its dual $\mathcal{E}^d$ not, then there exist coalitions K_1, K_2 and blocks B_1, and B_2 such that $\mathcal{E}(K_1, B_1) = \mathcal{E}(K_2, B_2) = 1$, $K_1 \cap K_2 \neq \emptyset$, and $B_1 \cap B_2 \neq \emptyset$ hold.*

In dual terms we can state this conjecture equivalently as

Conjecture 5 *If an EFF $\mathcal{E} =< C_i, S_a; I, A; J >$ is stable, and its dual $\mathcal{E}^d$ not, then there must exist a player $i \in I$ and an outcome $a \in A$ such that $|C_i \cap S_a| \geq 2$.*

In particular, we conjecture that such an EFF cannot be generated by a graph. The Strong Perfect Graph Conjecture (SPGC) can also be reformulated in terms of EFFs.

Conjecture 6 *If an EFF $\mathcal{E}_G$ corresponding to a graph G is not stable then the graph G has an odd hole or an odd antihole.*

In Section 5 we will prove that Conjectures 4 and 5 are equivalent to Conjecture 1 and that Conjecture 6 is equivalent to SPGC.

3 Keiding's theorem and its dual.

The following simple criterion of stability was obtained by Keiding (1985). Given an EFF $\mathcal{E} =< K_j, B_j;\ J;\ I,\ A >$ and a utility function $u : I \times A \mapsto \mathbb{R}$, let us denote by R_j for $j \in J$ a subset of the outcomes (not necessarily all of them) which are strictly worse than any one from B_j for all the players of the coalition K_j. In other words,

$$R_j \subseteq \{a \in A \mid u(i,a) < u(i,b) \text{ for all } b \in B_j \text{ and } i \in K_j\}. \tag{R}$$

By this definition, $B_j \cap R_j = \emptyset$ for every $j \in J$, and hence the coalition K_j being effective for B_j, can reject all the outcomes in R_j. Consequently, these outcomes cannot belong to the core $C(\mathcal{E}, u)$.

Let us consider now an extended list $T = \{(K_j, B_j, R_j) | j \in J\}$ such that K_j is effective for B_j and rejects R_j for every $j \in J$. The core $C(\mathcal{E}, u)$ is empty if all the outcomes are rejected, i.e. $\cup_{j \in J} R_j = A$. In this case we shall call the list $T = \{(K_j, B_j, R_j) | j \in J\}$ a *rejecting table* for the pair $(\mathcal{E}, u)$. Let us denote the family of such rejecting tables by $RT(\mathcal{E}, u)$.

Let us consider next the case when we are given only an EFF $\mathcal{E}$ but no utility u is specified. We shall call a list of the form $T = \{(K_j, B_j, R_j) | j \in J\}$ a *rejecting table* of the EFF $\mathcal{E}$, if $\mathcal{E}(K_j, B_j) = 1$ for every $j \in J$ and $\cup_{j \in J} R_j = A$. Let us denote the family of such lists by $RT(\mathcal{E})$. Let us call a rejecting table $T \in RT(\mathcal{E})$ *utilizable* if $T \in RT(\mathcal{E}, u)$ for some utility function u. By definition of stability, an EFF $\mathcal{E}$ is not stable iff it has a utilizable rejecting table $T \in RT(\mathcal{E})$.

To have a combinatorial characterization of utilizability of rejecting tables, we shall need the following definitions. Let us call a subset $J_C = \{j_1, ..., j_r\} \subseteq J$, $r \geq 2$ of the rows of a rejecting table $T \in RT(\mathcal{E})$ a *cycle*, if $B_{j_1} \cap R_{j_2} \neq \emptyset$, $B_{j_2} \cap R_{j_3} \neq \emptyset$, ..., and $B_{j_r} \cap R_{j_1} \neq \emptyset$. Such a cycle J_C will be called a *common-player cycle* if all the corresponding coalitions have a player in common, i.e. if $\cap_{j \in J_C} K_j \neq \emptyset$.

Theorem 5 (Keiding (1985).) *An EFF $\mathcal{E}$ is stable if and only if every rejecting table $T \in RT(\mathcal{E})$ contains a common-player cycle.*

The proof results immediately from the following lemma.

Lemma 1 *A rejecting table $T \in RT(\mathcal{E})$ is utilizable if and only if it contains no common-player cycle.*

Proof. If there exists a common-player cycle J_C in $T \in RT(\mathcal{E})$, then T can not be utilizable because the preference of any common player $i \in \cap_{j \in J_C} K_j$, according to (R), would be cyclic over the outcomes $a_1 \in B_{j_1} \cap R_{j_2}$, $a_2 \in B_{j_2} \cap R_{j_3}$, ..., $a_r \in B_{j_r} \cap R_{j_1}$.

Conversely, if there are no common-player cycles in $T \in RT(\mathcal{E})$, then the inequalities in (R) induce an acyclic partial preference over the outcomes for every player $i \in I$. Due to the acyclicity, we can choose a utility $u(i, *)$ for every player $i \in I$, which realizes the same preferences. For such a utility function u we shall have $T \in RT(\mathcal{E}, u)$, completing the proof of the lemma. $\square$

Keiding's theorem can be reformulated in terms of the hypergraphs $\mathcal{C}$ and $\mathcal{S}$ corresponding to the effectivity function $\mathcal{E}$.

Let us begin with the special case of EFFs generated by graphs. Given a graph $G =< V, E >$, let us direct some of its edges. (We assume that some edges may remain non-directed but no edge can be bidirected.) The obtained partially directed graph D is called a *suborientation* of G. We say that a vertex $v \in V$ of the suborientation D *rejects* a subset $V' \subseteq V$ if $v \notin V'$ and every edge (v, v') for $v' \in V'$ is directed from v' to v in D. A suborientation D of G is called *rejecting* if every (maximal) stable set of G is rejected in D by a vertex. A directed cycle of D whose vertices form a clique in G is called a *clique-cycle* of D. A rejecting suborientation without clique cycles is called a *clique-acyclic rejecting suborientation* (or in short a CARS).

Proposition 4 (Boros and Gurvich (1994)) *A graph G has no CARS iff the corresponding EFF $\mathcal{E}_G$ is stable.*

Proof. In fact this proposition is "dual" to Keiding's theorem. Let us suppose that the EFF $\mathcal{E}_G$ is not stable and consider a rejecting table $T \in RT(\mathcal{E}_G)$ which has no common-player cycles. Every outcome $a \in A$ is rejected by some coalition according to T, hence there exists a corresponding row (K, B, R) in T for which $\mathcal{E}_G(K, B) = 1$, and $a \in R$. This implies, according to the definition of $\mathcal{E}_G$, the existence of a vertex $v \in V$ for which $K_v \subseteq K$ and $B_v \subseteq B$. Then $v \notin S_a$, where S_a is the maximal stable set corresponding to the outcome a, since all such maximal stable sets are included in B_v by the definition of $\mathcal{E}_G$, and $B \cap R = \emptyset$ by the definition of a rejecting table. Let us orient all the edges of G between the stable set S_a and vertex v towards vertex v. By repeating the same for all the outcomes, we obtain an orientation D of (some of) the edges of G, and we claim that it is a CARS. Indeed, D is rejecting because $T \in RT(\mathcal{E}_G)$ is a rejecting table and D is clique-acyclic (and in particular, no edge of G is bidirected) because $T \in RT(\mathcal{E}_G)$ has no common-player cycles.

For the reverse direction, we can construct a rejecting table of the form $T = \{(K_v, B_v, R_v) | v \in V\} \in RT(\mathcal{E}_G)$ from any CARS D of G, by defining R_v as the set of maximal stable sets rejected in D at vertex v. It is easy to verify that T is indeed a rejecting table for $\mathcal{E}_G$, since D is a rejecting suborientation, and that T has no common player cycles, since such a cycle in T would correspond to a clique-cycle in D.

Thus, the above constructions provide a correspondence between CARS of a graph G and rejecting tables without common-player cycles of the corresponding EFF $\mathcal{E}_G$, in both directions. $\square$

There is a simple relation between CARS and kernel solvability of graphs. Given a graph $G =< V, E >$, let us direct *all* of its edges, (allowing some edges to be bidirected.) The obtained "overdirected" graph D^+ is called a *superorientation* of G. Given a superorientation

D^+, a subset $S \subseteq V$ is called a *kernel* if S is a stable set of G and every vertex outside of S has a successor in S, according to D^+. In other words, if no edge of G has both endpoints in S and for every vertex $v \notin S$ there exists a vertex $v' \in S$ such that the edge (v,v') in D^+ is directed, from v towards v' (or bidirected.) Obviously, only a maximal stable set of G can be a kernel. If the graph G is complete (i.e. G is a clique) then only a single vertex can be a kernel, and a vertex v is a kernel iff v is a sink in D^+, i.e. iff all the edges incident to v are directed towards v (or are bidirected.) Finally, a graph G is called *kernel-solvable* if a superorientation D^+ of G has a kernel whenever every clique of G has a kernel.

There is a simple one-to-one correspondence between sub- and superorientations of a graph G, via interchanging unoriented edges with bidirected ones, and vice versa. It is easy to observe that a suborientation is a CARS iff the corresponding superorientation has no kernel. Thus we obtain the following claim.

Proposition 5 *A graph G has no CARS iff G is kernel-solvable.*

Theorem 6 (Boros and Gurvich (1994)) *A perfect graph has no CARS.*

By Proposition 5, this theorem implies Theorem 1. According to Conjecture 1, the converse, i.e. that graphs with no CARS are perfect, is conjectured to hold too.

Let us now consider a general EFF $\mathcal{E} =< C_i, S_a; I, A; J >$, and let us assign to the hypergraph $\mathcal{C} = \{C_i | i \in I\}$ a graph $G(\mathcal{C}) =< J, E(\mathcal{C}) >$ on the vertex set J, whose edges are those pairs $j, j' \in J$, $j \neq j'$ for which $\{j, j'\} \subseteq C_i$ for some $i \in I$. Given a suborientation D of $G(\mathcal{C})$, we shall call it $\mathcal{C}$*-acyclic* if there exists no directed cycle whose vertices all belong to a clique C_i for some $i \in I$, and we shall say that D is $\mathcal{S}$*-rejecting* if the subsets $S_a \subseteq J$ are rejected in D for all $a \in A$. We will use the same abbreviation, CARS, for a $\mathcal{C}$-acyclic and $\mathcal{S}$-rejecting suborientation of the graph $G(\mathcal{C})$.

Proposition 6 *An EFF $\mathcal{E} =< C_i, S_a; I, A; J >$ is stable iff the graph $G(\mathcal{C})$ has no CARS.*

The proof goes along exactly the same lines as for EFFs generated by graphs, in Proposition 4, and we omit it here.

Kernel-solvability also generalizes for an arbitrary effectivity function $\mathcal{E} =< C_i, S_a; I, A; J >$ as follows.

Proposition 7 *A graph $G(\mathcal{C})$ has no CARS iff for any linear order of the vertices in the hyperedges C_i, $i \in I$, there exists always an $a \in A$ such that the hyperedge S_a is dominating, i.e. for every vertex $j \in J \setminus S_a$ there is an $i \in I$ such that $j \in C_i$, $C_i \cap S_a \neq \emptyset$ and there exists a vertex $j' \in C_i \cap S_a$ which is greater than j in the given order of C_i.*

The proof goes along the same lines as for Proposition 5 above, and we omit it here.

Let us consider an interpretation of the above. Let J denote a set of students organized into several competing (and possibly overlaping) teams, denoted by $S_a \subseteq J$, $a \in A$. Furthermore, let I denote the set of tests, and $C_i \subseteq J$ for $i \in I$, denote the set of students taking the i-th test. Then the EFF $\mathcal{E} =< C_i, S_a; I, A; J >$ is stable iff for every result of these tests there exists a “winning” team, i.e. a team S_a such that for every participant $j \notin S_a$ there exists a member of the team $j' \in S_a$ who is better than j in some test(s) $i \in I$.

4 Monotonicity.

The family of EFFs $\mathcal{E} : 2^I \times 2^A \to \{0, 1\}$ admit a natural partial order, defined by $\mathcal{E} \leq \mathcal{E}'$ iff $\mathcal{E}(K, B) \leq \mathcal{E}'(K, B)$ for all $K \subseteq I$, and $B \subseteq A$, or in other words, iff $\mathcal{E}(K, B) = 1$ implies $\mathcal{E}'(K, B) = 1$. Thus, for two effectivity functions $\mathcal{E} =< K_j, B_j; J; I, A >$ and $\mathcal{E}' =< K'_j, B'_j; J'; I', A' >$, the relation $\mathcal{E} \leq \mathcal{E}'$ holds iff for every $j \in J$ there exists $j' \in J'$ such that $K'_{j'} \subseteq K_j$ and $B'_{j'} \subseteq B_j$. In particular, an EFF $\mathcal{E} =< K_j, B_j; J; I, A >$ will remain the same if we remove all the pairs (K_j, B_j) which are not inclusion-minimal.

Analogously, for EEFs $\mathcal{E} =< C_i, S_a; I, A; J >$ and $\mathcal{E}' =< C'_i, S'_a; I', A'; J' >$ we have $\mathcal{E} \leq \mathcal{E}'$ iff for every $i \in I$ there exists $i' \in I'$ such that $C'_{i'} \subseteq C_i$. and for every $a \in A$ there exists $a' \in A'$ such that $S'_{a'} \subseteq S_a$. In particular, the EFF $\mathcal{E} =< C_i, S_a; I, A; J >$, will remain the same if we remove all the players C_i and outcomes S_a which are not inclusion-minimal.

It is both obvious and well known that stability is antimonotone.

Lemma 2 *If EFF $\mathcal{E}$ is stable and $\mathcal{E}' \leq \mathcal{E}$ then EFF $\mathcal{E}'$ is stable, too.*

Let us apply this to EFFs generated by graphs. Given a graph G and an induced subgraph G' in G, the above definitions imply that $\mathcal{E}_{G'} \leq \mathcal{E}_G$. Thus, according to Lemma 2, $\mathcal{E}_{G'}$ is stable whenever $\mathcal{E}_G$ is stable. In other words, if G' has a CARS then G also has one, i.e. G' is kernel-solvable whenever G is kernel-solvable.

Analogously, we can define “induced subEFFs” of an arbitrary EFF. Given an EFF $\mathcal{E} =< C_i, S_a; I, A; J >$ and a subset $J' \subseteq J$, the subEFF $\mathcal{E}_{J'} =< C'_i, S'_a; I', A'; J' >$ induced by J' is defined by setting $I' = \{i \in I | C_i \cap J' \neq \emptyset\}$, and $A' = \{a \in A | S_a \cap J' \neq \emptyset\}$, and defining $C'_i = C_i \cap J'$ for all $i \in I'$, and $S'_a = S_a \cap J'$ for all $a \in A'$. Obviously, $\mathcal{E}_{J'} \leq \mathcal{E}$ for any $J' \subseteq J$, thus the following is implied by Lemma 2.

Lemma 3 *An induced subEFF $\mathcal{E}_{J'}$ is stable whenever the EFF $\mathcal{E}$ is stable.*

As an application, let us consider an EFF $\mathcal{E}_0 =< C_i, S_a; I, A; J >$ for which $I = \{i_1, i_2\}$, $A = \{a_1, a_2\}$, and $J = \{1, 2\}$, and where $C_1 = S_1 = \{1\}$, and $C_2 = S_2 = \{2\}$. It is easy to verify that $T_0 = \{(\{i_1\}, \{a_1\}, \{a_2\}),\ (\{i_2\}, \{a_2\},\ \{a_1\})\}$ is a rejecting table for $\mathcal{E}_0$, with no common player cycles, and hence $\mathcal{E}_0$ is not stable, by Theorem 5. Together with Lemma 3 this implies Corollary 1 of Theorem 2.

5 Proofs of main theorems and their applications.

Proof of Theorem 2. Given an EFF $\mathcal{E} =< C_i, S_a; I, A; J >$, let us suppose, on the contrary to (i) that there exists a subset $J' \subseteq J$ such that $|C_i \cap J'| \leq 1$ for every $i \in I$ but there exists no $a \in A$ for which $J' \subseteq S_a$. We shall prove that under these conditions $\mathcal{E}$ is not stable. We can assume that J' is an inclusion-minimal subset with these properties, i.e. that for every $J'' \subseteq J'$ there exists an $a \in A$ such that $J'' \subseteq S_a$. This implies immediately that $|J'| \neq 1$, since otherwise $J' \subseteq S_a$ would hold for some $a \in A$. Let us consider the subEFF $\mathcal{E}_{J'} =< C'_i, S'_a; I', A'; J' >$ induced by the set J'. By its definition and by the minimality of the set J', the equation $\mathcal{E}_{J'}(\{j\}, J' - j) = 1$ holds for all $j \in J'$. Since such an EFF cannot be stable whenever $|J'| > 1$, we obtain that $\mathcal{E}$ is not stable, by Lemma 3. otherwise subEFF $\mathcal{E}_{J'}$ would be stable, too.

The second part (ii) of the theorem is dual to the first part. □

Let us note that in fact we have shown a bit more, than what was claimed. The conditions of Theorem 2 are necessary not only for the stability of $\mathcal{E}$ but also for the stability of its dual $\mathcal{E}^d$.

Proof of Theorem 3. Let us note first that conditions (ii) and (ii') are equivalent with the relation $\mathcal{E} \leq \mathcal{E}_G$. Since G is assumed to be perfect by (i), $\mathcal{E}_G$ is stable by Proposition 4 and Theorem 6 (see Boros and Gurvich (1994)), and thus $\mathcal{E}$ is stable again by Lemma 3. □

Let us remark that again we have shown a somewhat stronger result. The conditions of Theorem 3 are sufficient not only for the stability of $\mathcal{E}$ but also for the stability of its dual $\mathcal{E}^d$. Indeed, let us consider the complementary graph G^c instead of G. Cliques of G^c are stable sets of G and vice versa, and by the definition, we have $\mathcal{E}^d = \mathcal{E}_{G^c}$. It is known that a graph G is perfect iff its complement G^c is perfect (see Lovasz (1972)), thus if conditions (i), (ii), and (ii') of Theorem 3 hold for $\mathcal{E}$ and G then they hold for $\mathcal{E}^d$ and G^c, as well.

Let us note that the conditions of Theorem 2 are necessary but not sufficient for stability and the conditions of Theorem 3 are sufficient but not necessary. This is so, e.g. because otherwise the stability of $\mathcal{E}$ would be equivalent to the stability of $\mathcal{E}^d$, which is not the case, as we shall see in the next section. The following example will clarify these limits of Theorems 2 and 3.

Example 1 *For every positive integer k let us define an EFF $\mathcal{E}_k$ by setting $J = \{1, ..., k\}$, and $I = A = \{(i, j)|1 \leq i < j \leq k\}$, and by defining $C_{(i,j)} = S_{(i,j)} = \{i, j\}$ for all $(i, j) \in I (= A)$. Direct computations show that $\mathcal{E}_k$ is stable if $k \leq 6$, but it is not stable for $k = 7$. A corresponding CARS for $\mathcal{E}_7$ is given by the following three directed cycles $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 1$, $1 \to 3 \to 5 \to 7 \to 2 \to 4 \to 6 \to 1$, and $1 \to 5 \to 2 \to 6 \to 3 \to 7 \to 4 \to 1$. Since $\mathcal{E}_7$ is an induced subEFF of $\mathcal{E}_k$ for any $k > 7$, it follows that $\mathcal{E}_k$ is not stable for all $k \geq 7$.*

Let us note that the conditions of Theorem 2 hold automatically for every k, but $\mathcal{E}_k$ is not stable if $k \geq 7$. On their turn, the conditions of Theorem 3 hold only for $k \leq 4$, while $\mathcal{E}_5$ and $\mathcal{E}_6$ are still stable.

E.g. if $k = 5$ the only graph satisfying (ii) and (ii') of Theorem 3 is the hole C_5 but it is not perfect, while if $k = 6$ then either G or G^c must contain a triangle, consequently either (ii) or (ii') must be violated.

Thus for $\mathcal{E}_5$ and $\mathcal{E}_6$ no "perfect split" exist, but still these EFFs are stable.

Let us consider now an important class of EFFs for which Theorems 2 and 3 are applicable.

Given an EFF $\mathcal{E} =< C_i, S_a; I, A; J >=< K_j, B_j; J; I, A >$, let us assume that

$$|C_i \cap S_a| \leq 1 \text{ for all } i \in I \text{ and } a \in A, \tag{P}$$

or in dual terms,

$$K_j \cap K_{j'} = \emptyset \text{ or } B_j \cap B_{j'} = \emptyset \text{ (or both) for every } j \neq j' \in J. \tag{P^d}$$

Let us suppose first that there exists a pair $j \neq j' \in J$ such that both sets $K_j \cap K_{j'}$ and $B_j \cap B_{j'}$ are empty, or in dual terms, that there exists a pair $j, j' \in J$, $j \neq j'$ which does not belong to the same sets C_i, $i \in I$ or S_a, $a \in A$. Then, according to Corollary 1 of Theorem 2, neither $\mathcal{E}$ nor $\mathcal{E}^d$ is stable.

So let us now assume that for every pair $j \neq j' \in J$ exactly one of the following two options holds

(i) $K_j \cap K_{j'} \neq \emptyset$ or

(ii) $B_j \cap B_{j'} \neq \emptyset$,

or in dual terms,

(i^d) $\{j, j'\} \subseteq C_i$ for some $i \in I$ or

(ii^d) $\{j, j'\} \subseteq S_a$ for some $a \in A$.

Let us denote by $G =< J, E >$ and $G^c =< J, E^c >$ the two complementary graphs having the same set of vertices J, and the edges of which are defined by (i) and (ii), respectively. Theorem 2 claims that neither $\mathcal{E}$ nor its dual $\mathcal{E}^d$ can be stable unless $\mathcal{E} = \mathcal{E}_G$ and $\mathcal{E}^d = \mathcal{E}_{G^c}$. So let us now assume that both these equalities hold. Furthermore, Theorem 3 claims that both $\mathcal{E}$ and $\mathcal{E}^d$ are stable if the graph G is perfect.

Finally, let us suppose that the graph G is not perfect. This is the only open case. According to Conjecture 1, in this case the EFF $\mathcal{E} = \mathcal{E}_G$ is not stable. According to the results of Lovasz (1972), the complementary graph G^c is not perfect either, in this case, thus according to Conjecture 1, the dual EFF $\mathcal{E}^d = \mathcal{E}_{G^c}$ is not stable either.

Proposition 8 *Conjectures 1 and 4 are equivalent.*

Proof. The above arguments prove that if assumption (P) and Conjecture 1 hold then for the dual pair of EFFs $\mathcal{E}$ and $\mathcal{E}^d$, either both or neither of them are stable, i.e. Conjecture 1 implies Conjecture 4.

Let us suppose that Conjecture 1 does not hold then the equivalent Conjecture 2 fails too, i.e. there exists a complementary pair of graphs G and G^c such that G is kernel-solvable while G^c is not. Then, according to Propositions 1–3, the EFF $\mathcal{E}_G$ is stable while its dual $\mathcal{E}^d = \mathcal{E}_{G^c}$ is not. However, assumption (P) holds for EFFs generated by graphs, hence Conjecture 4 fails. $\square$

Proposition 9 *SPGC and Conjecture 6 are equivalent.*

Proof. **SPGC $\Longrightarrow$ Conjecture 6**. Let us suppose that the graph G has neither odd holes nor odd antiholes. SPGC claims that G is perfect. According to Theorem 1, G is then kernel-solvable, and according to Propositions 1–3, the EFF $\mathcal{E}_G$ is stable.
Conjecture 1 and Conjecture 6 $\Longrightarrow$ SPGC. Let us suppose that the graph G contains neither odd holes nor odd antiholes. According to Conjecture 6, the EFF $\mathcal{E}_G$ is stable. According to Propositions 1–3, the graph G is then kernel-solvable, implying finally that, according to Conjecture 1, the graph G is perfect.
Conjecture 6 $\Longrightarrow$ Conjecture 1. Since Conjecture 1 is equivalent to Conjecture 2 (see Boros and Gurvich (1994)), we shall instead show that Conjecture 6 implies Conjecture 2.

Odd holes and odd antiholes are complementary. Thus the graph G contains one of these iff the complementary graph G^c contains the complementary one. Let us suppose that both graphs G and G^c contain an odd hole or an odd antihole, implying that they both are not kernel-solvable, (see e.g. Berge and Duchet (1983) or Boros and Gurvich (1994)). On the other hand, if both graphs contain neither odd holes nor odd antiholes then both graphs are kernel-solvable, according to Conjecture 5. Thus, in all cases the graph G is kernel solvable iff its complement G^c is also kernel solvable, which is exactly Conjecture 2. $\square$

Now we will derive Theorem 4 from Theorems 2, 3, and from Lemma 4 below.

Given an integer $b \in \mathbb{N}$, let us define a graph $G_n =< V_n, E_n >$ consisting of $2n$ vertices $V_n = \{1, 2, ..., 2n\}$, and n pairwise non-adjacent edges $E_n = \{(i, n+i) | 1 \leq i \leq n\}$.

Lemma 4 *Given a graph $G_n =< V_n, E_n >$, and given k subsets $\mathcal{M} = \{M_1, \ ..., \ M_k\}$ of the vertex-set V_n, it is an NP-complete problem to check the validity of the following statement:*

(Q) *Every (maximal) stable set of G_n is contained in some subset $M_i \in \mathcal{M}$.*

Proof. We shall prove the lemma by reducing the well-known NP-complete problem of tautology for DNFs (see e.g. Garey and Johnson (1979)) to (Q).

Obviously, the problem of testing the validity of (Q) belongs to NP.

Let us consider an arbitrary DNF on n variables consisting of k terms

$$D(X_1, ..., X_n) = \bigvee_{i=1}^{k} \bigwedge_{j \in P_i} X_j \bigwedge_{j \in N_i} \overline{X}_j$$

where X_j, $j = 1, ..., n$ denote the Boolean variables, and $\overline{X}$ denotes the negation of X. The tautology problem for D consists in deciding if D evaluates to true for every true-false assignments to the variables X_1, ..., X_n.

Let us now associate to such a DNF a family $\mathcal{M} = \mathcal{M}_D$ of k subsets of V_n, as in the lemma. Let us define

$$M_i = \{j | 1 \leq j \leq n, \; j \notin P_i\} \cup \{j + n | 1 \leq j \leq n, \; j \notin N_i\},$$

for $i = 1, ..., k$. It is then straightforward to verify that D is a tautology (i.e. it evaluates always to true) iff (Q) holds true for the family $\mathcal{M}_D$.

Thus, the general problem of tautology for DNFs is polynomially reduced to testing the validity of (Q). □

Proof of Theorem 4. We shall prove the theorem by reducing the problem of testing the validity of (Q) to the stability testing of effectivity functions.

Obviously, testing the stability of an effectivity function belongs to NP by Theorem 5 (Keiding (1985).)

Let us consider the graph $G_n =< V_n, E_n >$ and a family $\mathcal{M} = \{M_1, ..., M_k\}$ of k subsets of V_n, and let us associate an EFF $\mathcal{E} = \mathcal{E}(G_n, \mathcal{M}) =< C_i, S_a; I, A; J >$ to this pair, as follows. The EFF $\mathcal{E}(G_n, \mathcal{M})$ consists of n players $I = \{1, ..., n\}$ and k outcomes $A = \{1, ..., k\}$, and we have $J = V_n$, $C_i = \{i, i + n\}$ for $i = 1, ..., n$, and $S_a = M_a$, for $a = 1, ..., k$ (i.e. $\mathcal{C} = E_n$ and $\mathcal{S} = \mathcal{M}$.)

For this EFF $\mathcal{E}(G_n, \mathcal{M})$ we have that if (Q) holds for $\mathcal{M}$, then it is stable by Theorem 3, since G_n is a perfect graph, and if (Q) does not hold for $\mathcal{M}$, then it cannot be stable by Theorem 2.

Thus we have reduced (Q) to the problem of testing the stability of EEFs, and thus the theorem follows by Lemma 4. □

Remark 2 *Let us observe that condition (Q) is equivalent to condition (i) of Theorem 2 for the EFF $\mathcal{E}(G_n, \mathcal{M})$, and it is equivalent with the fact that $\mathcal{E}(G_n, \mathcal{M})$ is playing-minor, see Remark 1.*

Thus in fact, it is an NP-complete problem to check condition (i) (i.e. playing-minority) even in a special case when it is sufficient (and not only necessary) for stability.

6 Stability is not selfdual.

As we mentioned earlier, all the conditions of Theorems 2 and 3 hold (or do not hold) for the dual EFFs $\mathcal{E}$ and $\mathcal{E}^d$ simultaneously. However in general, it may happen that an EFF $\mathcal{E}$ is stable while its dual EFF $\mathcal{E}^d$ is not.

Example 2 (Peleg (1984); Example 6.3.16 and Remark 6.3.17) *Let us consider the EFF* $\mathcal{E} =< K_j,\ B_j;\ J;\ I, A >$ *with* $I = A = J = \{1, 2, 3, 4\}$ *defined by the effective pairs in the table below. It is easy to see that by defining* $R_1 = \{2\}$*,* $R_2 = \{3\}$*,* $R_3 = \{1\}$*,*

j	K_j	B_j
1	2 3	1 4
2	1 3	2 4
3	1 2	3 4
4	4	1 2 3

and $R_4 = \{4\}$ *we obtain a rejecting table* $T = \{(K_j, B_j, R_j)|j = 1, ..., 4\}$*, in which all the outcomes are rejected, and hence the EFF* $\mathcal{E}$ *is not stable. It is not difficult to check, however that its dual* $\mathcal{E}^d$ *is stable.*

In a sense this example is minimal. If $\mathcal{E}$ *is stable and* $\mathcal{E}^d$ *is not then both the number of players and the number of outcomes cannot be less than 4.*

For EFFs generated by graphs such examples probably do not exist. Let us consider an arbitrary pair of complementary graphs G and G^c and the corresponding dual EFFs $\mathcal{E} = \mathcal{E}_G$ and $\mathcal{E}^d = \mathcal{E}_{G^c}$. If graphs G and G^c are perfect then both EFFs are stable, according to Theorems 1 and 4. If these graphs are not perfect then probably both EFFs are not stable (see Conjectures 1 and 4), although this is still an open problem.

Given a pair of graphs $G_1 =< J, E_1 >$ and $G_2 =< J, E_2 >$ with the same set J of vertices, let us associate an EFF $\mathcal{E}(G_1, G_2) =< C_i, S_a; I, A; J >$ to this pair of graphs by defining the hypergraphs $\mathcal{C}$ and $\mathcal{S}$ as the sets of all the maximal cliques of G_1 and G_2, respectively. In other words, $\mathcal{E}(G_1, G_2) =< K_j, B_j; J : I, A >$, where I is the set of maximal cliques of G_1, A is the set of maximal cliques of G_2, $K_j = \{i \in I|C_i \ni j\}$ and $B_j = \{a \in A|S_a \ni j\}$ for $j \in J$, and $\mathcal{E}(K, B) = 1$ iff $K_j \subseteq K$ and $B_j \subseteq B$ for some $j \in J$.

By this definition, the EFFs $\mathcal{E}(G_1, G_2)$ and $\mathcal{E}(G_2, G_1)$ are dual. Furthermore, if the edge sets of the graphs G_1 and G_2 cover all pairs (j, j'), $j, j' \in J$, $j \neq j'$, then the conditions (i) and (ii) of Theorem 2 are satisfied. Let us call an effectivity function *bigraphic*, if it can be represented as such an associated EFF $\mathcal{E}(G_1, G_2)$.

As we shall see, stability is not selfdual even for these bigraphic EFFs.

Example 3 *Let* $G_1 = C_9^+$ *be the graph shown in Figure 1 and let* G_2 *be the 9-antihole* C_9^c*. The graphs* G_1 *and* G_2 *have the common set of vertices* $J = \{1, ..., 9\}$*, their edges together contain all the pairs of* J*, and they share three common edges* $(2, 6)$*,* $(3, 8)$ *and* $(5, 9)$*. It was* *verified by computer that the EFF* $\mathcal{E}(G_1, G_2)$ *is stable. Indeed, a case analyses shows that in any CARS of* C_9^+ *the 9-cycle must be cyclically directed, without any loss of generality, we can assume that it is directed clockwise, i.e. all the edges* $(j, j+1)$ *are directed from* j *to* $j+1$*, see Figure 1. Then, in order to reject all nine 4-cliques of* C_9^c *one must direct the three (shared) central edges of as* $[6,2)$, $[3,8)$, *and* $[9,5)$. *This still leaves us, with no more edges to direct, with the clique* $\{258\}$ *un-rejected. Thus indeed,* $\mathcal{E}(G_1, G_2)$ *is stable.*

At the same time, the dual EFF $\mathcal{E}(G_2, G_1)$ *is not stable since there exists a CARS of* C_9^c *given by the directed edges* $9 \to 7$, $8 \to 6$, $7 \to 5$, $6 \to 4$, $5 \to 3$, $4 \to 2$, $3 \to 1$, $2 \to 9$, $1 \to 8$, $2 \to 7$, $8 \to 4$, $5 \to 1$, *while keeping all the other edges of* C_9^c *un-directed.*

Let us note that in this example the set of common edges of graphs G_1 *and* G_2 *is critical, in the following sense. Let* $G_2' = C_9^c$ *be the 9-antihole again, while let* G_1' *contain the complementary 9-cycle and maybe some other edges. Then the EFF* $\mathcal{E}(G_1', G_2')$ *is not stable whenever at least one of the three edges* $(2,6)$, $(3,8)$ *and* $(5,9)$ *is missing. The corresponding critical CARS' are given in Figure 2.*

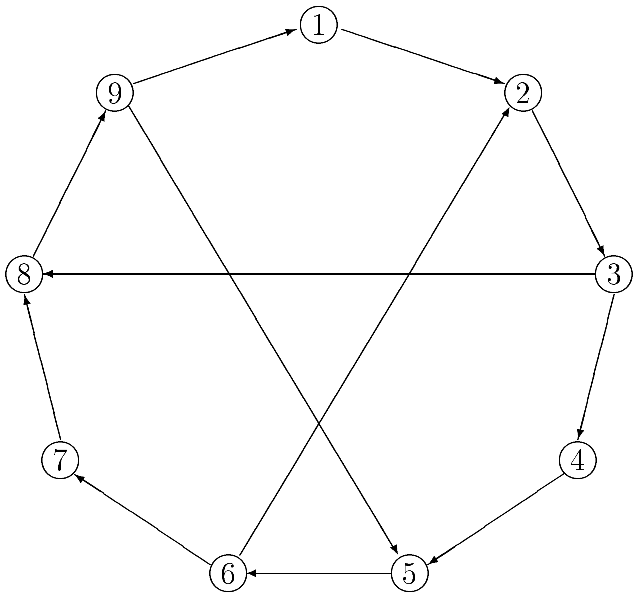

Figure 1: An "almost" CARS: all the stable sets, except $\{2,5,8\}$, are rejected.

The stability of an EFF $\mathcal{E}_G$ generated by a graph G is equivalent to the perfectness of this graph if Conjecture 1 is true, implying that in this case the stability of $\mathcal{E}_G$ could be expressed in terms of the clique and chromatic numbers of the graph G. The previous example shows that the stability of a bigraphic EFF $\mathcal{E}(G_1, G_2)$ cannot be expressed in such terms. Indeed, let us consider the following two pairs of graphs (G_1, G_2): let $G_1 = C_9^c$ the 9-antihole and let $G_2 = C_9$ the 9-hole first, and let $G_1' = G_1$, and $G_2' = C_9^+$ the graph in Figure 1, second. The graphs C_9^+ and C_9 have the same clique and chromatic numbers (2

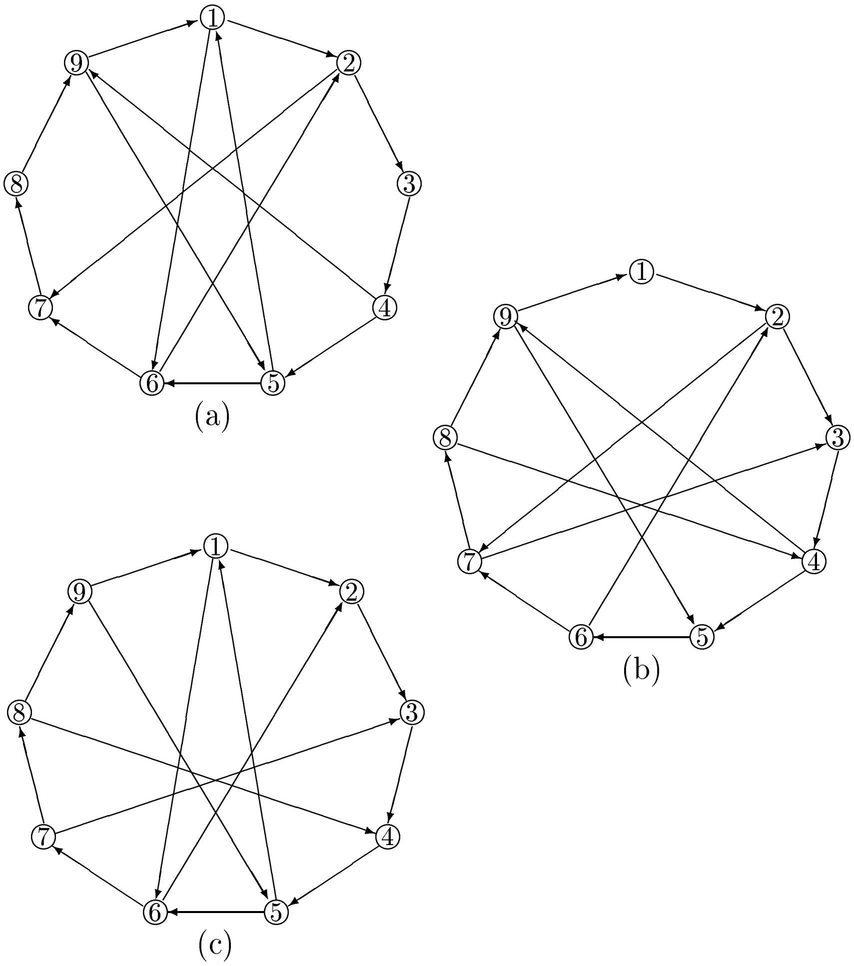

Figure 2: CARS, all the stable sets are rejected.

and 3, resp.), moreover their complements the 9-antihole C_9^c and C_9^{+c} also have the same clique and chromatic numbers (4 and 5, resp.). However, the EFF $\mathcal{E}(G_1', G_2')$ generated by the second pair is stable, while the EFF $\mathcal{E}(G_1, G_2)$ generated by the first pair is not.

Let us call an EFF $\mathcal{E}$ *convex* if $\mathcal{E}(K_1, B_1) = 1$ and $\mathcal{E}(K_2, B_2) = 1$ imply that either $\mathcal{E}(K_1 \cup K_2, B_1 \cap B_2) = 1$ or $\mathcal{E}(K_1 \cap K_2, B_1 \cup B_2) = 1$ (or maybe both) for every $K_1, K_2 \subseteq I$, and $B_1, B_2 \subseteq A$.

By this definition, the family of convex EFFs is selfdual, i.e. $\mathcal{E}$ is convex iff $\mathcal{E}^d$ is convex.

Theorem 7 (Peleg (1984)) *Convex EFFs are stable.*

One could conjecture that all the maximal stable EFFs are convex, or in other words, that for every stable EFF $\mathcal{E}$ there exists a convex EFF $\mathcal{E}'$ such that $\mathcal{E}' \geq \mathcal{E}$. This conjecture would provide a nice characterization of stable EFFs, if it were true. In fact, every EFF $\mathcal{E}$, such that it is stable and its dual $\mathcal{E}^d$ is not stable, provides a counterexample. To see this, let us suppose that $\mathcal{E}$ is a stable EFF for which $\mathcal{E}^d$ is not stable, and that $\mathcal{E} \leq \mathcal{E}'$ for some convex EFF $\mathcal{E}'$. Since convexity is selfdual, $\mathcal{E}'^d$ is also convex, and hence it is also stable by Theorem 7. Because the inequality $\mathcal{E}'^d \geq \mathcal{E}^d$ holds, the stability of $\mathcal{E}^d$ is implied by Lemma 2, a contradiction with our assumptions.

Beside the duality (d) there is another important type of "duality" for EFFs: Given an EFF $\mathcal{E} =< K_j, B_j; J; I, A >$ let us define EFF $\mathcal{E}^*$ by $\mathcal{E}^*(K, B) = 1 - \mathcal{E}(I - K, A - B)$ for all $K \subseteq I$, and $B \subseteq A$. Obviously, both (d) and $(*)$ are involutive, i.e. $\mathcal{E}^{dd} = \mathcal{E}^{**} = \mathcal{E}$. Let us note that (d) and $(*)$ are also commutative, i.e. $\mathcal{E}^{*d} = \mathcal{E}^{d*}$, that (d) is monotone, i.e. $\mathcal{E} \leq \mathcal{E}'$ iff $\mathcal{E}^d \leq \mathcal{E}'^d$, and finally that $(*)$ is antimonotone, i.e. $\mathcal{E} \leq \mathcal{E}'$ iff $\mathcal{E}^* \geq \mathcal{E}'^*$.

Let us call an EFF $\mathcal{E}$ **-selfdual* if $\mathcal{E}(K, B) + \mathcal{E}(I - K, A - B) = 1$ holds for all $K \subseteq I$, $B \subseteq A$, or in other words, if $\mathcal{E} = \mathcal{E}^*$. The stable *-selfdual EFFs can be characterized as follows.

Proposition 10 (Abdou (1981)) *Stable *-selfdual EFFs are sub- and superadditive, furthermore *-selfdual, sub- and superadditive EFFs are stable.*

It is also know that *-selfdual EFFs form a subclass of convex EFFs.

Proposition 11 (Gurvich (1992)) *Stable *-selfdual EFFs are convex.*

Thus in the class of *-selfdual EFFs the classes of sub- and superadditive EFFs, stable EFFs and convex EFFs coincide.

It was conjectured by Moulin (1983, Problem 25) and by Danilov and Sotskov (1987) that stable and *-selfdual EFFs are exactly the maximal stable EFFs, i.e. for every stable EFF $\mathcal{E}$ there exists a stable and *-selfdual EFF $\mathcal{E}'$ such that $\mathcal{E}' \geq \mathcal{E}$. This conjecture would also provide a nice characterization of the stable EFFs, if it were true. We can show again that every EFF $\mathcal{E}$, such that it is stable and its dual $\mathcal{E}^d$ is not stable, provides a counterexample. Indeed, let us suppose that $\mathcal{E}$ is stable, $\mathcal{E}^d$ is not stable, and $\mathcal{E} \leq \mathcal{E}'$ for some stable and *-selfdual EFF $\mathcal{E}'$. Then $\mathcal{E}'$ is convex by Proposition 11 and thus $\mathcal{E}'^d$ is convex, too, consequently they are both stable by Theorem 7. Since we have $\mathcal{E}'^d \geq \mathcal{E}^d$, the stability of $\mathcal{E}^d$ is implied by Lemma 2, a contradiction again with our assumptions.

Furthermore, one could also conjecture that a stable EFF $\mathcal{E}$ is always majorized by a stable *-selfdual (or convex) one, in case both $\mathcal{E}$ and $\mathcal{E}^d$ are stable. This is however not true again, as the next example shows.

Example 4 (Gurvich (1990)) *Let us consider the EFF $\mathcal{E} =< K_j, B_j; J; I, A >$ with three players $I = \{i_1, i_2, i_3\}$, six outcomes $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and for which $\mathcal{E}(K, B) = 1$ iff $|K| \geq 2$ and B contains one (or more) of the subsets $\{a_1, a_3\}$, $\{a_2, a_4\}$, $\{a_3, a_5\}$, $\{a_4, a_6\}$, $\{a_5, a_1\}$, and $\{a_6, a_2\}$. One can show that EFFs $\mathcal{E}$ and $\mathcal{E}^d$ are stable, however, if $\mathcal{E}' \geq \mathcal{E}$ and $\mathcal{E}'$ is convex (or *-selfdual) then $\mathcal{E}'$ is not stable.*

Thus, it seems to be difficult to characterize maximal stable EFFs.

On the other hand, minimal unstable EFFs are easier to describe because every unstable EFF has a CARS. Let us consider a graph $G =< V, E >$ and a suborientation D of it, such that

(i) for every vertex $v \in V$ there exists an edge coming in and an edge going out,

(ii) orienting any of the non-directed edges of D would create a new clique-cycle.

One can show that in fact every such suborientation D is a CARS for some EFFs. Furthermore, among these EFFs there exists a unique minimum EFF $\mathcal{E}_D =< C_i, S_a; I, A; V >$, for which $\mathcal{C} = \{C_i, i \in I\}$ are all the maximal acyclic cliques of D, i.e. cliques of G without clique-cycles in D, and $\mathcal{S} = \{S_a, a \in A\}$ are $n = |V|$ maximal subsets $S_a \subseteq V$ such that S_a is rejected in D by a vertex $v = v(a)$, and the correspondance $a \leftrightarrow v(a)$ is a bijection between V and A, i.e. one can identify these sets and assume that $A = V$. It can be shown that in fact every minimal unstable effectivity function can be realized by some suborientation D in the above way.

7 Appendix. Perfect graphs.

Let $G =< V, E >$ be a graph, where V denotes the set of vertices and E denotes the set of edges. We denote the number of vertices and edges by $n = n(G)$ and $m = m(G)$, respectively, and call these numbers the *order* and the *size* of the graph G.

Given a graph $G =< V, E >$, we define the *complementary graph* as $G^c =< V, E^c >$, that is the set of vertices is the same and two vertices are adjacent in G^c iff they are not adjacent in G.

A *subgraph* $G' =< V', E' >$ of a graph $G =< V, E >$ is defined by a subset of vertices $V' \subseteq V$ and a subset of edges $E' \subseteq E$. We say that G' is an *induced subgraph* (or the *subgraph* of G *induced by the subset* $V' \subseteq V$) if *every* edge of E with both its vertices in V' belongs to E'.

A graph is called *complete* if every two of its vertices are *adjacent*, i.e. connected by an edge. The complement to a complete graph is called an *edge-free* graph. A complete subgraph is called a *clique*, and an edge-free *induced* subgraph is called a *stable set* (or *independent set.*) Each clique (stable set) which is not strictly contained in another clique (stable set) is called *maximal*, and each clique (stable set) of the maximal order is called *maximum*. The order of the maximum clique (stable set) is called *clique number* (*stability number*) of the graph G and is denoted by $\omega = \omega(G)$ (respectively, by $\alpha = \alpha(G)$). The relations $\alpha(G^c) = \omega(G)$, $\omega(G^c) = \alpha(G)$, $n(G^c) = n(G)$, and $m(G^c) + m(G) = n(G)(n(G) - 1)/2$ hold obviously.

The *chromatic number* $\chi = \chi(G)$ of a graph $G =< V, E >$ is the minimum possible number of stable sets the union of which is V. Obviously, $\chi \geq \omega$ holds for every graph.

A graph G is called *perfect* if $\chi(G') = \omega(G')$ for every induced subgraph G' of G including G itself. Perfect graphs were introduced by Berge (1961) who suggested the following two conjectures.

Conjecture 7 (Weak Perfect Graph Conjecture (WPGC)) *A graph G is perfect iff the complementary graph G^c is perfect.*

Lovasz (1972) proved that a graph G is perfect iff $\alpha(G') \times \omega(G') \geq n(G')$ for every induced subgraph G' of G including G itself. This result implies immediately the above conjecture.

A graph G is called $minimallyimperfect$ if G itself is not perfect but every induced subgraph G' of G, different from G, is perfect.

Which graphs are minimally imperfect? Let us consider an *odd hole* C_{2i+1}, $i \geq 2$, i.e. an odd chordless cycle of length 5 or longer. It is easy to compute that $2 = \omega(C_{2i+1}) < \chi(C_{2i+1}) = 3$. Thus, odd holes are imperfect, and it is not difficult to check that they are minimal in the sense that by deleting any of its vertices, we obtain a perfect graph.

Are there other minimal imperfect graphs? Let us consider an odd *antihole* C^c_{2i+1}, $i \geq 2$, i.e. the complement to the odd hole C_{2i+1}. It is easy to compute that $i = \omega(C^c_{2i+1}) < \chi(C^c_{2i+1}) = i + 1$. Thus, odd holes are imperfect, and it is not difficult to check that they are minimal again with respect to taking induced subgraphs. Are there more minimal imperfect graphs? The second famous conjecture of Berge (1961) claims that no other ones exist.

Conjecture 8 (Strong Perfect Graph Conjecture (SPGC)) *The only minimally imperfect graphs are the odd holes and odd antiholes, or in other words, every imperfect graph contains an induced odd hole or induced odd antihole.*

It is immediate to see that SPGC is stronger than WPGC, and as of today, SPGC is still open.

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