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Poincaré Duality for L^p Cohomology and Characteristic Classes

by

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ABSTRACT

We prove Poincaré Duality for L^p cohomology, $1 \leq p \leq \infty$ We study the pairings between L^p and L^∞ and construct characteristic classes.

1 Introduction

The purpose of this article is to prove Poincaré duality for L^p cohomology. This relates the amenability criteria of Gromov (see [J]) and Block-Weinberger [BW], and of [ABW] to the criterion described in [A1]. Finally, we will give a complete proof of the statement in [1] relating the structure set of $M \times \mathbf{R}$ to the Pontrjagin classes in uff cohomology.

In the following we introduce a fine version of this coarse homology theory and prove that it is Poincaré dual to bounded de Rham cohomology described by Gromov [G], Januszkiewicz [J] and Roe [R1]. This theory has also been studied by S. Gersten, who showed that a group is hyperbolic if and only if its second cohomology vanishes [Ge].

In a later paper the first author will use the Poincaré duality theorem to prove a Lefschetz fixed point theorem, similar to the theorem of Heitsch and Lazarov [HL]. We will prove here, however, an analogue of the Hopf index theorem, which characterizes the Poincaré duals of Januszkiewicz' classes as elements of homology.

Jürgen Eichhorn [E] has previously defined characteristic classes for L^p-cohomology, and J.Dodziuk [D] has proven the de Rham theorem for L^2-cohomology, but the general Poincaré duality theorem which we prove here does not appear to be in the literature, and it has some interesting consequences.

2 Uniformly finite homology

We first recall the definition of the coarse theory given in [BW1], where only the case $p = \infty$ was considered and was called uniformly finite homology.

Definition 1 *A normed abelian group is an abelian group G equipped with a "norm" function*

$$|\cdot|: G \to \mathbf{R}_+$$

which satisfies the triangle inequality.

Definition 2 *Let X be a metric space and X^{i+1} the $i+1$-fold cartesian product with the metric*

$$d((x_0, ..., x_i), (y_0, ..., y_i)) = \max_{0 \le j \le i} d(x_j, y_j).$$

Also, write Δ for the multidiagonal in X^{i+1}. Let $CX_i^{(p)}(X; G, |\cdot|)$, $(G, |\cdot|)$ a normed group, denote the group of infinite formal sums

$$c = \sum a_{\overline{x}} \overline{x}$$

$\overline{x} \in X^{i+1}$, $a_{\overline{x}} \in G$ satisfying

i. Given $r > 0$ there exists K_r so that for all $\overline{y} \in X^{i+1}$

$$\#\{\overline{x} \in B(\overline{y}, r) \mid a_{\overline{x}} \neq 0\} | \le K_r$$

ii. There exists $R > 0$ depending on c so that $a_{\overline{x}} = 0$ if $d(\overline{x}, \Delta) > R$.

iii. $\sum | a_{\overline{x}} |^p$.

Let $\mathcal{U}_{-\infty}(\mathcal{M})$ be the algebra of uniformly smoothing operators introduced by Roe in [R] (we will assume the operators in $\mathcal{U}_{-\infty}(\mathcal{M})$ have bounded propagation speed). These are defined to be operators whose propagation speed is bounded and whose kernels are uniformly bounded in C^∞ norm.

The following theorem of [BW2],[Yu] along with the results in [CM] generalizes index theorem in J.Roe's Ph.D. thesis:

Theorem 1 *There is a character map*

$$\chi : HC_*(\mathcal{U}_{-\infty}(\mathcal{X})) \to \mathcal{H}\mathcal{X}_*^\infty(\mathcal{X}; \mathbf{R})$$

There is also a "fine" homology corresponding to the "coarse" theory. It is defined as follows:

Definition 3 *A simplicial complex X is said to be a simplicial complex of bounded geometry, if there is a finite uniform bound on the number of simplices in the link of each vertex of X. This bound is called the complexity of X. Here a locally finite simplicial complex is considered to have the natural metric derived from barycentric coordinates. A simplicial map $f : X \to Y$ between simplicial complexes of bounded geometry is said to have bounded geometry if there is a finite uniform bound on the number of simplices in the inverse image of a simplex under f.*

Definition 4 *Define the group of chains $C_q^{(p)}(X; G, |\cdot|)$ to be the group of formal sums of simplices in X, $c = \sum a_\sigma \sigma$ so that $\sum |a_\sigma|^p < \infty$. The boundary is defined to be the natural linear extension of the singular boundary. We write the resulting homology $H_*^{(p)}(X; G, |\cdot|)$.*

A uniformly contractible space is a metric space for which the metric ball of radius r is contractible within the ball of radius $f(r)$ for some function $f : \mathbf{R}_+ \to \mathbf{R}_+$.

The following is an easy corollary of a Weil style double complex argument.

Proposition 1 *Let X be a uniformly contractible space, $(G, |\cdot|)$ a normed group. Then $HX_*^{(p)}(X; G |\cdot|) \simeq H_*^{(p)}(X; G |\cdot|)$.*

We can in fact introduce a "dual" cohomology theory for which there are two versions, one which involves differential forms, and the other which is simplicial. We first introduce the theory in terms of differential forms.

Definition 5 *Let $\Omega_{(p)}^i(M)$ denote the Banach space of differential i-forms bounded in the norm*

$$\| \alpha \|_p^p = \int_M (| \alpha(x) |^p + | d\alpha(x) |^p) dvol$$

Then the maps $d_i : \Omega_{(p)}^i(M) \to \Omega_{(p)}^{i+1}(M)$ define a complex. We define the bounded de Rham cohomology by

$$H_{DR(p)}^i(M) = [Ker(d_i)]/[Im(d_{i-1})]$$

The simplicial theory is defined in the following manner.

Definition 6 *Let $H^i_{(p)}(X)$, for X a simplicial complex of bounded geometry denote the cohomology group whose cochains are simplexwise the same as the cochains of ordinary simplicial cohomology, but satisfying conditions $\sum \mid c(\sigma) \mid^p < \infty$*

Theorem 2 *De Rham Theorem Let M be a smooth n-dimensional manifold of bounded geometry. Then there is an isomorphism*

$$H^i_{(p)}(M; \mathbf{R}) \simeq H^i_{DR(p)}(M)$$

To prove this we need the following result proven in [CMS] section 7 and [A].

Theorem 3 *Let M be a manifold of bounded geometry. Then M admits a triangulation as a simplicial complex of bounded geometry. The resulting simplicial complex is quasi-isometric to the original manifold.*

Remark 1 *This theorem yields a simplicial proof of the finiteness theorem of [C]. To see this, simply take the class of manifolds with curvature bounded in absolute value, volume bounded below and diameter above and take the connected sum of all of them. This can be done in such a way as to give rise to a manifold of bounded geometry, by gluing along the boundary sphere in at most a finite number of ways. The triangulation theorem then says that each ball of fixed diameter in this manifold can be covered by a fixed number of simplices. Hence any manifold in the class above can be covered by this same number of simplices. This then implies that there are a finite number of PL types in this class.*

In addition to this we need a notion of subdivision, which we call regular uniform subdivision. This is a refinement of the notion of uniform subdivision in [A]. We first recall that simpler notion.

Definition 7 *A subdivision of a simplicial complex of bounded geometry is said to be uniform if*

i. Each simplex is subdivided a uniformly bounded number of times on its n-skeleton, where the n-skeleton is the union of n-dimensional sub-simplices of the simplex.

ii. The distortion

$$\sup(length(e), length(e)^{-1})$$

of each edge e of the subdivided complex is uniformly bounded in the metric given by barycentric coordinates of the original simplex.

Definition 8 *A regular uniform subdivision is defined in the following manner. Let $\sigma = [p_0, ..., p_m]$ be a simplex in $\mathbf{R}^k$, $k \geq m$. The vertices of the standard subdivision of σ are the points*

$$p_{ij} = \frac{1}{2}(p_i + p_j), i \geq j.$$

Define a partial ordering of the vertices $S\sigma$ by setting

$$p_{ij} \leq p_{kl},$$

if $i \geq k$, $j \geq l$. The simplices of $S\sigma$ are the increasing sequences of vertices with respect to the above ordering.

We define a regular uniform subdivision to be any uniform subdivision which is a sequence of standard subdivisions.

Uniform subdivision induces a map $s : C^*_{(p)}(K;G) \to C^*_{(p)}(K';G)$ which has bounded norm.

Proof 1 *The idea is to use Whitney and de Rham maps, with the triangulation above serving to show that these maps are well-defined. The proof here imitates the ones found in [D],[W]. Define the de Rham map*

$$\int : \Omega^i_{(p)}(M) \to C^i_{(p)}(M;\mathbf{R})$$

by

$$\int(\omega)\cdot\sigma = \int_\sigma \omega.$$

Integration commutes with uniform subdivision as can be shown in the following manner.

To define the Whitney map, which is the chain homotopy inverse of the de Rham map, let c_σ be the cochain which assigns the value 1 to σ and the value 0 to every other simplex.

For each point q in M write q in barycentric coordinates as $q = \sum \nu_\alpha(q) q_\alpha$, where q_α runs over the vertices of the triangulation, and α is the corresponding index. For each α, let Q_α and Q'_α be subsets of M so that Q_α is the set of all p with $\nu_\alpha(p) \geq \frac{1}{n+1}$; Q'_α is the set of all q with $\nu_\alpha(q) \leq \frac{1}{n+2}$. Then $Q_\alpha \subset star(q_\alpha)$, $Int(Q'_\alpha) \supset M - star(q_\alpha)$. Let $\phi'_\alpha(p)$ be a smooth non-negative real function in M which has all of its derivatives uniformly bounded over α and so that each is positive in Q_α and zero in Q'_α. Construct the partition of unity

$$\phi_\alpha(p) = \frac{\phi'_\alpha(p)}{\sum_\beta \phi'_\beta(p)}$$

We will choose normalizations of the ϕ'_α below, so we will assume ϕ'_α normalized suitably for the moment.

Take any $p \in M$; since p has at most $n+1$ non-zero barycentric coordinates at least one of these, say $\nu_\beta(p)$ is $\geq \frac{1}{n+1}$. Hence $p \in Q_\beta$, $\phi'_\beta(p) > 0$ and $\phi_\alpha(p)$ is defined for all α.

Define the Whitney map W on each c_σ by

$$W(c_\sigma) = r! \sum_{i=0}^{r} (-1)^i \phi_{\alpha_i} d\phi_{\alpha_0} \wedge \ldots \wedge d\hat{\phi}_{\alpha_i} \wedge \ldots \wedge d\phi_{\alpha_r},$$

where $\sigma = q_{\alpha_0} \ldots q_{\alpha_r}$ and ϕ_α is defined above. We can then extend by linearity. Note that

$$suppW(c_\sigma) \subseteq \sigma.$$

Furthermore, because the triangulation is uniform, the resulting form is bounded. In fact, the map W defines a bounded map on chains in the norm defined by taking the supremum of the coefficients of the chain.

We now choose normalizations so that the de Rham theorem will be true. Choose normalizations of the ϕ'_α so that

$$\int W(c_{\sigma_\alpha})\sigma_\beta = \int_{\sigma_\beta} W(c_{\sigma_\alpha}) = \delta^\beta_\alpha,$$

since $supp W(c_\alpha) \subset \sigma_\alpha$*. Using this normalization, we have inductively for σ' a face of σ,*

$$\int_\sigma W(c_\sigma) = \int_\sigma W(c_{d\sigma'}) = \int_{\partial\sigma} W(c_{\sigma'}) = \int_{\sigma'} W(c_{\sigma'}) = 1.$$

This normalizes all of the lower dimensional cochains.

Define $\int^* : H^i_{DR(p)}(M) \to H^i_{(p)}(M;\mathbf{R})$ to be the map sending ω to the cohomology class defined by the linear functional $\sigma \mapsto \int_\sigma \omega$ on each simplex. Define $W^* : H^i_{(p)}(M;\mathbf{R}) \to H^i_{DR(p)}(M;\mathbf{R})$ to be the map induced by the Whitney map W defined above.

We claim that $\int^*$ and W^* are inverses of eachother. In fact, by Stokes' theorem we can observe that $\int$ and W are chain maps.

That $\int W = Id$ is proved by observing that the normalization conditions prove the result for each c(σ) *and then extending by linearity.*

That $W^* \int^* = Id$ follows from a series of easy estimates.

Let ω be a closed differential form in $\Omega_{DR(p)}(M)$. Suppose the cohomology class $[\int\omega]$=0 in $H_i^{(p)}(M;\mathbf{R})$. Since integration commutes with uniform subdivision, we have $[\int'\omega] = 0$, where $\int'$ denotes the integration map over a uniformly subdivided triangulation. Fix $\epsilon_1 > 0$. We claim that there exists a uniform subdivision so that

$$\| \omega - W \cdot \int \omega \| < \epsilon_1$$

Note that we have the estimate

$$| \omega(x) - W \cdot \int \omega | \leq C \cdot diam\tau \cdot \sup_{x\in\tau} | \frac{\partial\omega}{\partial x} |$$

over each simplex τ of the triangulation, where C can be chosen independently of ω, τ. One then obtains an estimate in terms of the mesh of the triangulation h

$$\| \omega - W \cdot \int \omega \|_0 \leq \sum_{\tau\in K} \int_\tau | \omega(x) - W \cdot \int \omega(x) | \, dV$$

$$\leq 4C \cdot h^N m \| \omega \|$$

where m denotes the multiplicity of the intesections of the bg coordinate charts of M. Since $\int\omega$ is a boundary, we can find a cochain f so that

$$\| \int \omega - \delta f \| \leq \epsilon_2$$

Then

$$\| \omega - dWf \| \leq \| \omega - W \cdot \int \omega \| + \| W \| \cdot \| \int \omega - \delta f \|$$

$$\leq \epsilon_1 + \| W \| \cdot \epsilon_2$$

Since ϵ_2 is chosen independently of ϵ_1 and both can be made arbitrarily small, we obtain that the cohomology class $[\omega] = 0$. This proves the de Rham theorem.

We have the corollary:

Theorem 4 *There is an isomorphism*

$$H^i_{DR(p)}(M) \simeq H^{(p)}_{n-i}(M; \mathbf{R})$$

for any orientable n-manifold M of bounded geometry.

Our proof will be based on that in [Mau]. The following is also a consequence of the proof of the previous theorem:

Theorem 5 *There is an isomorphism*

$$H^{n-i}_{(p)}(M; \mathbf{R}) = H^{(p)}_i(M; \mathbf{R})$$

between L^p homology and L^p cohomology $1 \leq p \leq \infty$.

Definition 9 *A bg block dissection of a uniform triangulation of a bg simplicial complex is defined to be a decomposition of the triangulation into polyhedra $(e_i, \cdot e_i)$ so that*

i. $H_r(e_i, \cdot e_i) = 0$, for $r \neq n$ and $H_n(e_i, \cdot e_i) = \mathbf{Z}$.

ii. Every simplex of the triangulation is completely contained in some e_i.

iii. The boundary of e_i consists entirely of simplices of dimension m, for $m < n$.

iv. The polyhedra $(e_i, \cdot e_i)$ fall into a finite number of isometry types up to a fixed amount of distortion.

Proof 2 *We use the existence of a bg block dissection of the triangulation of M which is dual to the uniform triangulation, in the sense that there is a one-to-one correspondance between the r-blocks and the $n - r$-simplices.*

Suppose K is a bg simplicial complex, K' a uniform subdivision of K. For each simplex σ of K define subcomplexes $e(\sigma) =$ all simplices of K' of the form $(\hat{\sigma}_n, ..., \hat{\sigma}_0)$ with $\sigma_n > ... > \sigma_0 > \sigma$ and $\cdot e(\sigma) =$ all simplices of $e(\sigma)$ not having $\hat{\sigma}$ as vertex.

Proposition 2 *Let K be a uniform triangulation of a manifold of bounded geometry. Then for each r-simplex σ of K, $(e(\sigma), \cdot e(\sigma))$ is an $(n - r)$-block and the set of all $(e(\sigma), \cdot e(\sigma))$ forms a block dissection of K'.*

Definition 10 *We need the analogue of the cap product. Let*

$$\sigma = \sum_i [b_i^0, ..., b_i^n]$$

be a generator of $C_n^{(\infty)}(K)$ *Let* $\sigma_i' = [b_i^0, ..., b_i^{n-r}]$, *and* $\sigma_i^{''} = [b^{n-r}, ...b^n]$. *We define the cap product by*

$$\alpha \cap \sigma = \sum_i \alpha(\sigma_i^{''})\sigma'$$

Note that this defines a product

$$H_n^{(\infty)}(K) \otimes H_\infty^r(K) \to H_{n-r}^\infty(K)$$

Definition 11 *To calculate uniformly finite homology from the blocks* $e(\sigma)$ *we must identify the corresponding block chain complex with a subcomplex of* $C_*^{(p)}(K)$*: this is done by choosing generators of each group of cycles* $Z_{n-r}(e(\sigma), \cdot e(\sigma))$. *If* n*-simplices of* K *are identified with elements of* $C_n^{(p)}(K)$, *then* $z \in C_n^{(p)}(K)$, *the sum of the* n*-simplices of* K, *is a representative cycle for the generator of* $H_n^{(p)}(K) = \mathbf{Z}$. *We totally order the vertices of* K' *so that* $\hat{\sigma} < \hat{\tau}$ *if* $dim\sigma > dim\tau$*; let* $s : C_*^{(p)}(K) \to C_*^{(p)}(K')$ *be the subdivision chain map,* $h : K' \to K$ *a bg simplicial approximation to the identity, and* h^* *the map induced by* h *on* ∞f *cochains. Finally, for each* r*-simplex* σ *of* K, *let* $\phi_\sigma \in C_{(p)}^r(K; \mathbf{Z})$ *be the cochain sending* σ *to 1 and all other* r*-simplices to 0.*

Definition 12 *Define*

$$z(\sigma) = h^*(\phi_\sigma) \cap s(z) \in C_{n-r}^{(p)}(K')$$

Proposition 3 $z(\sigma)$ *is a generator of* $Z_{n-r}(e(\sigma), \cdot e(\sigma))$.

Proof 3 *Each generator* $[\hat{\sigma}_n, ..., \hat{\sigma}_0]$ *of* $C_n^\infty(K')$ *occurs in* $s(z)$ *with coefficient* ± 1. *Now*

$$h^*(\phi_\sigma) \cap [\hat{\sigma}_n, ..., \hat{\sigma}_0] = [\hat{\sigma}, ..., \hat{\sigma}_r]\phi_\sigma h[\hat{\sigma}_r, ...\hat{\sigma}_0]$$

and this is zero except for just one simplex

$$(\hat{\sigma}_r, ...\hat{\sigma}_0)$$

contained in σ, *where it is*

$$\pm[\hat{\sigma}_n, ..., \hat{\sigma}_r].$$

Hence

$$z(\sigma) \in C_{n-r}(e(\sigma)).$$

Moreover

$$\partial z(\sigma) = (-1)^{n-r}\delta h^*(\phi_\sigma) \cap s(z)$$

since $\partial s(z) = s\partial(z) = 0$, *so*

$$\partial z(\sigma) = (-1)^{n-r} h^*\delta(\phi_\sigma) \cap s(z)$$

which is an element of $C_{n-r-1}(\cdot e(\sigma))$. *Lastly,* $z(\sigma)$ *is a generating cycle since each simplex in* $z(\sigma)$ *has coefficient* ± 1.

Definition 13 *We define the duality map*

$$D : C^r_{(p)}(K) \to C^{(p)}_{n-r}(K')$$

by $D(\phi_\sigma) = z(\sigma)$.

If d *is a boundary homomorphism for* $C^{(p)}_*$ *then*

$$dD(\phi\sigma) = \partial D(\phi_\sigma)$$

$$= (-1)^{n-r} h^* \delta(\phi_\sigma) \cap s(z)$$

δ *being the boundary homomorphism in* $C^*_{(p)}$. *Thus*

$$dD(\phi_\sigma) = \sum (-1)^{n-r}(\tau)$$

where the sum is taken over those $r+1$*-simplices* τ *so that* $\partial(\tau) = \sigma + \ldots$ *But the interior of* $e(\tau)$ *is in* K' *if and only if* τ *is in* K, *so*

$$dD(\phi_\sigma) = (-1)^{n-r} D\delta(\phi_\sigma)$$

So D *induces an isomorphism, which proves Poincaré duality.*

3 Products

We now set out to prove the Lefschetz theorem.

Definition 14 *Let* c *be an element of* $HX^\infty_i(X; \mathbf{R})$. *We can define the corresponding mean cohomology class to* c *as follows. Define the mean cochains of* X *to be the dual space* $C^\infty_i(X; \mathbf{R})$

$$C^i_{\mathcal{M}}(X; \mathbf{R}) = (C^\infty_i(X; \mathbf{R}))^*$$

with the coboundary defined by $\delta\mu(c) = \mu(\partial c)$. *We denote the corresponding cohomology groups by* $H^i_{\mathcal{M}}(X; \mathbf{R})$. *Mean homology is defined similarly in terms of uniformly finite cochains.*

Definition 15 *There are cap products:*

$$H^n_{\mathcal{M}}(X; \mathbf{R}) \otimes H^\infty_r(X; \mathbf{R}) \to H^{\mathcal{M}}_{n-r}(X; \mathbf{R})$$

$$H^n_\infty(X; \mathbf{R}) \otimes H^{\mathcal{M}}_r(X; \mathbf{R}) \to H^\infty_{n-r}(X; \mathbf{R})$$

These cap products are defined as follows. Let $\alpha \in C^n_{\mathcal{M}}(X; \mathbf{R})$ *be a mean cochain and let* $\sigma \in C^\infty_r(X; \mathbf{R})$ *be a uniformly finite chain. Then we can write* $\alpha = \sum \alpha_i$ *and* $\alpha_i = \alpha'_i \alpha''_i$. *Now let* $\alpha' = \sum \alpha'_i$, $\alpha'' = \sum \alpha''_i$. *As an* L^∞ *cochain,* σ *can be paired naturally with* α'' *to yield a real number. Thus we have a mean homology class*

$$\beta = \sigma(\alpha'')\alpha'$$

The other cap product is similar to this.

Proposition 4 *Let M be a smooth manifold of dimension n. Then there is a non-degenerate pairing*

$$H^n_{\mathcal{M}}(X;\mathbf{R}) \otimes H^\infty_0(X;\mathbf{R}) \to H^\infty_n(X;\mathbf{R}) = \mathbf{R}$$

Hence, given any class $x \in H^\infty_0(X;\mathbf{R})$, one can associate a dual class $\mu \in H^n_{\mathcal{M}}(X;\mathbf{R})$.

Proof 4 *This will follow from the proof of a second Poincaré duality theorem, in which we will use the cap product introduced above in place of the one used in the proof of the first Poincaré duality theorem. This second cap product will be called the global cap product.*

We triangulate M via a uniform triangulation and introduce the dual bg block dissection. Now let $\phi_\sigma \in C^n_{\mathcal{M}}(K;\mathbf{R})$ be the mean cochain evaluating to 1 on the ∞f chain σ. Define $z(\sigma) = h^*(\phi_\sigma) \cap s(z)$, where we now use the global cap product. This gives the duality map and proves that the cap product pairing is non-degenerate.

Definition 16 *We define an intersection product*

$$H^\infty_*(M;\mathbf{Z}) \otimes H^\infty_{n-*}(M;\mathbf{Z}) \to H^\infty_0(M;\mathbf{Z})$$

by taking the transverse intersection of cycles. More precisely, let C and D be simplicial complexes of bounded geometry representing cycles $[C] \in H^\infty_r(M)$ and $[D] \in H^\infty_s(M)$. The Poincaré dual of $[C]$ is then an $n-r$ form ψ_C and the Poincaré dual of $[D]$ is an s form ψ_D. Take the wedge product of these two forms. Then we define $[C]\cdot[D]$ to be the Poincaré dual of $\psi_C \wedge \psi_D$.

Claim: The class of $[C]\cdot[D]$ is the same as the class of the transverse intersection of C and D.

Proof 5 *We first make note of the following fact. The Poincaré duality theorem proved above associates to each simplex σ the form $\phi(c_{\sigma^*})$. This was achieved via the cap product defined above in the proof of Poincaré duality. We call this cap product the local cap product.*

The local cap product can be utilized to show that over a given compact subcomplex σ

$$\int_\mu \phi = \#(\sigma\cdot\mu)$$

where ϕ is the form constucted by applying the cap product to σ. Making use of local projections of neighborhoods from $M \times M$ to M, and applying the local intersection formula to the diagonal yields the result. Thus:

$$\int_{\mu\times\nu} \pi_1^*\phi \wedge \pi_2^*\psi = \#(\sigma\cdot\mu)\#(\tau\cdot\nu)$$

where ψ is obtained by a local cap product with τ and therefore, locally

$$\#(\sigma\cdot\tau) = \int \phi\wedge\psi.$$

Adding this up on both sides as a cycle in $H^\infty_0(M)$ gives the result.

Theorem 6 *There is an isomorphism*

$$C_n^{\infty}(M \times N; \mathbf{Z}) \simeq \bigoplus_{i+j=n} C_i^{\infty}(M; C_j^{\infty}(N; \mathbf{Z}))$$

Proof 6 *Let $M = \{\sigma_\alpha^k\}_{\alpha,k}$ and $N = \{\sigma_\alpha'^k\}_{\alpha,k}$ be simplicial complexes of bounded geometry. The Cartesian products of the simplices give a cell decomposition of the product space $M \times N$, with boundary operator*

$$\partial(\sigma_\alpha^k \times \sigma_\beta'^l) = \partial\sigma_\alpha^k \times \sigma_\beta'^l + (-1)^k \sigma_\alpha^k \times \partial\sigma_\beta'^l.$$

The Cartesian product

$$A \times B = \sum a_\alpha b_\beta \sigma_\alpha^k \times \sigma_\beta'^l$$

of two cycles $A = \sum a_\alpha \sigma_\alpha^k$ and $B = \sum b_\beta \sigma_\beta'^l$ in M and N is a cycle. We thus have a map

$$C_n^{\infty}(M \times N; \mathbf{Z}) \to \bigoplus_{i+j=n} C_i^{\infty}(M; C_j^{\infty}(N; \mathbf{Z}))$$

which we will claim to be an isomorphism. This follows from an Eilenberg-Zilber argument, namely that the cycles in $M \times N$ are seen to be generated by the product of cycles in M with cycles in N. We have a double complex $C_{p,q}(M \times N; \mathbf{Z}) = C_p(M; C_q(N; \mathbf{Z}))$ which is used to compute the uniformly finite homology, after the introduction of L^∞ estimates. Note that $l^\infty(X \times Y) = l^\infty(X; l^\infty(Y))$. This implies that $C_\cdot^\infty(X \times Y) = C_\cdot^\infty(X; C_\cdot^\infty(Y))$. There is already a norm on uniformly finite chains given by the supremum of the coefficients.

4 Chern-Weil Theory

Next we discuss Chern-Weil theory for manifolds of bounded geometry. This is taken directly from [J].

Given a Riemannian metric g on a compact manifold M one may compute, using second derivatives of components of the tensor g the matrix of the curvature form, and after evaluating a suitable $O(n)$-invariant polynomial on it, obtains the characteristic form. These are closed, and using the homotopy between two Levi-Civita connections coming from metrics g, g' one shows that the cohomology class depends only on the quasi-isometry type.

We therefore have the following theorem of Januszkiewicz

Theorem 7 *Any C^2 bounded metric on M defines a Chern-Weil homomorphism from the ring of polynomials on the dual Lie algebra of the group $O(n)$ invariant under the Ad-action into $H^*_{DR\infty}(M)$. It is depends only on the quasi-isometry type.*

The following discussion is based on [GH].

Definition 17 *Let $E \to M$ be a vector bundle of bounded geometry and*

$$\sigma = (\sigma_1, ..., \sigma_k)$$

be k global C^∞ sections of E, where the rank of E is k. Then the degeneracy set $D_i(\sigma)$ is the set of points $x \in M$ where the $\sigma_1, ..., \sigma_i$ are linearly independent, i.e.,

$$D_i(\sigma) = \{x : \sigma_1(x) \wedge ... \wedge \sigma_i(x) = 0\}.$$

The collection of sections σ is generic if for each i, σ_{i+1} intersects the subspace of E spanned by $\sigma_1, ..., \sigma_i$ transversely and if integration over $D_{i+1}(\sigma) - D_i(\sigma)$ forms a closed current. Then D_i represents a cycle in $H_i^\infty(M)$.

Definition 18 *A vector bundle of bounded geometry is a vector bundle ξ equipped with a Hermitian connection ∇ so ∇ and its associated curvature form Ω are bounded in the norm $\| \cdot \|$ on differential forms defined by*

$$\| \alpha \| = \{| \alpha(x) | + | d\alpha(x) |\}$$

Chern-Weil theory for vector bundles of bounded geometry works the same way as for the tangent bundle, so we will leave the details out.

Theorem 8 *Let $c_r(\xi) \in H^{2r}_{DR\infty}(M)$ be the r-th Chern class of a bounded geometry vector bundle ξ. Then its Poincaré dual in $H^\infty_{n-r}(M)$ is equal to the class of the degeneracy cycle D_{n-r}.*

Proof 7 *This follows by transversality and the corresponding fact for the universal bundle on BU. Suppose $\sigma_1, ..., \sigma_k$ are generic sections of E. Using a partition of unity, construct sections $\sigma_{k+1}, ..., \sigma_n$ of E such that $\sigma_1(x), ..., \sigma_n(x)$ span the fiber of E over each point x. These sections define a map into the Grassmannian. The image of this map then meets the Schubert cycle $\sigma_{1,...,1}$ exactly in the degeneracy set. However, we have that the Chern class $c_r(\omega)$ of the universal bundle ω is Poincaré dual to the Schubert cycle $\sigma_{1,...,1}$.*

Finally, we can deduce the following corollary for bg structure sets

Corollary 1 *Let M and N be two PL manifolds of bounded geometry which are bg homotopy equivalent $S^2 \times S^3 \times \mathbf{R}$ Then M and N are PL bg homeomorphic if and only if they have the same Pontrjagin classes in $H^*_{DR\infty}(M; \mathbf{R})$.*

Proof 8 *We use the computation of the bg structure set in [A1] to find that an element of the structure set of $S^3 \times S^2 \times \mathbf{R}$ is bg PL homeomorphic to the union of an infinite number of copies of a compact manifold homotopy equivalent to $S^3 \times S^2 \times I$ which are classified up to ordinary PL homeomorphism by their Pontrjagin classes, hence taking the Poincare duals, by their degeneracy cycles. These degeneracy cycles, glued together along the boundary of $S^3 \times S^2 \times I$ yields the result.*

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