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QUASI-ISOMETRY CLASSIFICATION OF COARSE HADAMARD MANIFOLDS

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Abstract

Quasi-conformal mappings were first introduced by L. Ahlfors in 1935, in connection with open Riemann surfaces and value distribution theory, and then studied by O. Teichmüller in 1938. In 1956 A. Beurling and Ahlfors proved an extension theorem which states that a quasi-symmetric map $\phi : \mathbf{R} \to \mathbf{R}$ can be extended to a quasiconformal map of the upper half plane. In 1964 Ahlfors had proven a similar extension theorem for quasi-symmetric mappings of $\mathbf{R}^2$. Then in 1979 P. Tukia and J. Väisälä proved quasiconformal extension in dimensions ≥ 5. The case of extension from 3 to 4 dimensions is still unknown.

In this paper we prove three main quasi-conformal extension results. First we prove the Tukia-Väisälä theorem, which states that a quasiconformal map $f : \mathbf{R}^n \to \mathbf{R}^n$ can be extended to a quasiconformal map of the upper half space $\mathbf{R}^{n+1}_+ \to \mathbf{R}^{n+1}_+$, provided $n \geq 5$. The second result is a similar extension theorem for the strongly uniform domains introduced by Heinonen and Yang in 1993. Finally, we solve a problem posed by Pansu in 1989 regarding the existence of a quasiconformal extension theorem for complex hyperbolic spaces. We can in fact prove this result for manifolds of pinched negative curvature.

0. Introduction

Quasi-conformal mappings were first introduced by L. Ahlfors in 1935, in connection with open Riemann surfaces and value distribution theory, and then studied by O. Teichmüller in 1938. In 1956 A. Beurling and Ahlfors proved an extension theorem which states that a quasi-symmetric map $\phi : \mathbf{R} \to \mathbf{R}$ can be extended to a quasiconformal map of the upper half plane. In 1964 Ahlfors had proven a similar extension theorem for quasi-symmetric mappings of $\mathbf{R}^2$. Then in 1979 P. Tukia and J. Väisälä proved quasiconformal extension in dimensions ≥ 5. The case of extension from 3 to 4 dimensions is still unknown.

In this paper we prove three main quasi-conformal extension results. First we prove the Tukia-Väisälä theorem, which states that a quasiconformal map $f : \mathbf{R}^n \to \mathbf{R}^n$ can be extended to a quasiconformal map of the upper half space $\mathbf{R}^{n+1}_+ \to \mathbf{R}^{n+1}_+$, provided $n \geq 5$. The second result is a similar extension theorem for the strongly uniform domains introduced by Heinonen and Yang in 1993. Finally, we solve a problem posed by Pansu in 1989 regarding the existence of a quasiconformal extension theorem for complex hyperbolic spaces. We can in fact prove this result for manifolds of pinched negative curvature.

In addition we present a new proof of the topological invariance of Pontrjagin classes, which was first proven by S. Novikov in 1965, given an analytic proof by D. Sullivan using work of N. Teleman in 1977, and proven using controlled topology and operator K-theory by E.K. Pedersen, J.Roe and S. Weinberger in 1995. Our proof will be similar to the latter. The technique used for proving these results will be the controlled surgery theory developed by the author for classifying manifolds of bounded geometry up to smooth quasi-isometry.

A manifold of bounded geometry (bg) is a non-compact manifold whose geometric complexity is bounded. This paper is a sequel to [2] and [3], in which we initiated the classification of bg manifolds up to quasi-isometry, which is the bg analogue of homeomorphism. The Borel conjecture is that every homotopy equivalence of aspherical closed manifolds is homotopic to a homeomorphism, or equivalently that the topological structure of such a manifold is determined by its homotopy type. In [3] we established a *bg* analogue of surgery theory, and proved the *bg* analogue of the Borel conjecture for $\mathbf{R}^n$, namely that any *bg* homotopy equivalence $f : L \to \mathbf{R}^n$ is *bg* homotopic to a quasi-isometry. In fact, we computed the *bg* structure set of $M \times \mathbf{R}^n$ for a compact manifold M in terms of the uniformly finite homology $H_*^{uff}(\mathbf{R}^n; \mathcal{S}(M))$ with coefficients in the structure spectrum $\mathcal{S}(M)$. In this paper we replace $\mathbf{R}^n$ by any Hadamard manifold X (=complete, simply-connected, non-positively curved Riemannian manifold) and obtain the same formula for the *bg* structure set of $M \times X$. However, where X has strictly negative curvature, the uniformly finite homology of X is quite different from that of the Euclidean space $\mathbf{R}^n$ - this difference is already apparent with $\mathbf{Z}$ coefficients, since $H_0^{uff}(\mathbf{R}^n; \mathbf{Z})$ is large, while for the hyperbolic space $H_0^{uff}(\mathbf{H}^n; \mathbf{Z}) = 0$.

The goal of this article is threefold. First, we provide contrast with the Euclidean case by classifying coarsely hyperbolic manifolds, where coarse refers to coarse quasi-isometry in the sense of Gromov, whose behavior differs because of the vanishing of uf homology up to dimension $n-1$, where the manifold is coarsely $\mathbf{H}^n$. Secondly, and perhaps more importantly, we provide applications to quasiconformal mappings. These results improve the extension result of Tukia-Väisälä, and give a proof of it without Sullivan's theorem on the existence of Lipschitz structures on topological manifolds. Finally, we prove the same results for general Hadamard manifolds, along with even more general quasi-conformal mapping theorems.

A consequence of this will be that we will be able to answer a question posed by P.Pansu in [19], p.47. Tukia proved in [22] that every quasiconformal map of the sphere at infinity of hyperbolic space can be extended to a quasiisometry. Pansu asked whether this could be generalized to complex hyperbolic space. We show in this article that this is the case, and that the extension result holds in even more generality, namely the case of manifolds with pinched negative sectional curvature.

In [2], [3] we proved that the *bg* simple structure set of $M \times \mathbf{R}^n$ was given by $H^{uff}_*(\mathbf{R}^n; \mathcal{S}_*(M))$, where the precise definition of this group can be found in [2], [3]. In addition we used properties of uniformly finite homology to prove the following lifting theorem: If $f : M \to N \times T^k$ is a homotopy equivalence, then the free abelian cover of f, $\tilde{f} : \tilde{M} \to N \times \mathbf{R}^k$ is a smooth quasi-isometry if and only if f lifts to a diffeomorphism on a finite cover.

The main classification results proven here are:

Theorem 0.1. *Let M^k be a compact manifold. Then the bg simple structure set of $M \times \mathbf{H}^n$, $k + n \geq 5$ is*

$$\mathcal{S}^{bg,s}(M \times \mathbf{H}^n) = H^{uff}_0(\mathbf{H}^n; \mathcal{S}(M \times D^n, \partial)) \oplus ... \oplus H^{uff}_n(\mathbf{H}^n; \mathcal{S}^{1-n}(M))$$

where $\mathcal{S}^{-i}(M)$ denotes the fiber of the assembly map of Ranicki's lower L-theory.

Theorem 0.2. *Let M^k be a compact manifold, X^n a simply connected Hadamard manifold, $n + k \geq 5$. Then the bg simple structure set of $M \times X$ is*

$$\mathcal{S}^{bg,s}(M \times X) = H^{uff}_0(X; \mathcal{S}(M \times D^n, \partial)) \oplus ... \oplus H^{uff}_n(X; \mathcal{S}^{1-n}(M))$$

where $\mathcal{S}^{-i}$ is as in Theorem 0.1.

In the case where $X = \mathbf{R}^n$ this specializes to the main result of [3]. However, the result is considerably more general when stated in this way. For example, we recall the result of Gromov and Block-Weinberger [6]:

Theorem 0.3. *The uniformly finite homology groups of hyperbolic n-space, $\mathbf{H}^n$ vanish up to and including dimension $n - 2$.*

Using this Theorem, Theorem 0.1 implies the vanishing of most of the structure set terms, in contrast to the Euclidean case, where all of them are seen to be non-zero, provided the same is true rationally for the structure set of M. ¿From Theorem 0.1 we derive two results:

Theorem 0.4. *Let M^n, $n \geq 5$ be a uniformly contractible smooth manifold of bounded geometry so that there is a map*

$$f : M \to \mathbf{H}^n$$

so that f is EPL in the sense of [6] *or a coarse map of bounded geometry in the sense of* [2]*. Then M is quasi-isometric to $\mathbf{H}^n$.*

This gives another proof of an extension result of Tukia and Väisälä [23], which states that quasi-conformal mappings of $\mathbf{R}^n$ can be extended to quasi-conformal mappings of the half-space one dimension higher. In addition, we are able to derive a new result more general than the Tukia-Väisälä theorem.

Theorem 0.5. *If D is a strongly uniform domain in $\mathbf{R}^n$, $n \geq 5$ and if f is a quasi-symmetric homeomorphism from the boundary of the unit ball B^n in $\mathbf{R}^n$ onto ∂D, then f can be extended to a quasiconformal mapping from B^n onto D.*

We also have quasiconformal extension results for manifolds with boundary isometric to the compactification of a manifold whose sectional curvature is pinched between two negative constants.

We also have the following result, concerning the conditions under which a homotopy equivalence of compact manifolds lifts to a smooth quasi-isometry on the universal cover, when the fundamental group is the fundamental group of a hyperbolic manifold. The condition for such a lift is milder than in the case of a free abelian fundamental group, due to the vanishing of most of the terms in the structure set in Theorem 0.1:

Theorem 0.6. *Let M^k be a compact manifold and*

$$f : M \times X^n \to N^{k+n}$$

a homotopy equivalence, where X is a compact manifold with constant negative curvature, $k + n \geq 5$. Then the hyperbolic cover of f

$$\tilde{f} : M \times \mathbf{H}^n \to \tilde{N}$$

is bg homotopic to a quasi-isometry if and only if f represents the zero element in the group

$$H_{n-1}(\Gamma; \mathcal{S}(M \times I, \partial)) \otimes \mathbf{Q}$$

Where Γ is the fundamental group of X, and $\mathcal{S}(M \times I, \partial)$ is the structure set of $M \times I$ rel boundary.

I would like to thank Shmuel Weinberger for a number of discussions and suggestions, which were crucial to the development of this work. I would also like to thank Stephen Semmes for a number of important suggestions. In particular, the idea that the *bg* Borel conjecture for hyperbolic space implies the Tukia-Väisälä theorem paper, and the statement of the generalization of this theorem for strongly uniform domains are due to him. I would also like to thank John Roe for discussions of the topological invariance of Pontrjagin classes, and other discussions about this paper. Finally, I would like to thank M.Culler,R. Edwards, S.Hurder, H. Masur, G.Mess, Andrew Ranicki for criticisms.

1.Uniformly finite homology

The definitions of uniformly finite homology and fine uniformly finite homology can be found in [2], [6]. We first recall some fundamental facts about uniformly finite homology.

Definition 1.1. A normed abelian group is an abelian group G equipped with a function, the norm,

$$|\cdot|: G \to \mathbf{R}_{+}$$

so that the induced function on the Cayley graph of G is non-decreasing as one moves away from the identity element in G.

Definition 1.2. A simplicial complex has bounded geometry if there is a uniform bound on the number of simplices in the link of each vertex. A simplicial map is said to have bounded geometry if the inverse image of each simplex contains a uniformly bounded number of simplices.

Definition 1.3. Let X be a simplicial complex of bounded geometry. Define the group of chains $C_q^{uff}(X; G, |\cdot|)$ to be the group of formal sums of simplices in X,

$$c = \sum a_\sigma \sigma$$

so that there exists a $K > 0$ depending on c so that $| a_\sigma | \leq K$. The boundary is defined to be the natural linear extension of the singular boundary. We write the resulting homology

$$H_*^{uff}(X; G, |\cdot|)$$

We will adopt the conventions of [2], [3] and use the "quantized" norm given by taking each chain to have coefficients lying in a finite subset of G.

The following is a theorem of Gromov, Block and Weinberger (see [2], p.513):

Theorem 1.4. *Let X be a simply connected n-dimensional symmetric space of non-positive curvature of rank r. Then*

$$H_i^{uff}(X; \mathbf{Z}) = 0 \textit{ for } i \leq n - r - 1$$

$$H_i^{uff}(X; \mathbf{Z}) \neq 0, i \geq n - r$$

Definition 1.5. Let $\Omega_\beta^i(M)$ be the Banach space of smooth L^∞ i-forms ϕ on M, whose exterior derivative $d\phi$ is also in L^∞. Specifically ϕ is bounded in the norm

$$\| \phi \|_M = \sup_m \{ \| \phi \|_m, \| d\phi \|_m \}$$

where $\| \cdot \|_m$ is the norm on $\Lambda^i T^* M_m$ induced by the Riemannian metric on M. This gives rise to a complex $d_i : \Omega_\beta^i(M) \to \Omega_\beta^{i+1}(M)$. The bounded de Rham groups are defined by

$$H_\beta^p(M) = [\mathrm{Ker} d_p]/[\mathrm{Im} d_{p-1}]$$

where we are not taking the closure of Imd.

Definition 1.6. Let M and N be Riemannian manifolds, with metrics ds^2_M and ds^2_N respectively. Let w be a positive C^∞ function on M. Then the manifold $M \times N$ endowed with the metric

$$ds^2 = ds^2_{M,m} + [w(m)]^2 ds^2_{N,n}$$

is called a warped product, denoted $M \times_w N$. w will be called the warping function.

We will also need the analogue of the Eilenberg-Zilber theorem for warped products in the uniformly finite homology and cohomology theories. This theorem similar to the one due to the author and J.Block in the unwarped case which appears in [4].

Definition 1.7. Let X be a smooth manifold of bounded geometry. Let g be a positive smooth function on X. Let G be a normed group. Then $C^{uff}_q(X; gG)$ the group of simplicial q-chains on X,

$$c = \sum a_\sigma \sigma$$

where the a_σ satisfy

$$\mid a_\sigma \mid g(\sigma) \leq K(c)$$

and where $g(\sigma)$ is the average of g over the simplex σ.

Define $\Omega^i_\beta(X; gV)$, where V is a real vector space, to be the complex of bounded differential forms, where the norm on V is replaced by g times the norm on V.

The analogue of the following theorem for L^2 chains is due to Zucker [25].

Eilenberg-Zilber Theorem 1.8. *Let X and Y be smooth manifolds of bounded geometry. Let w be a warping function. Let Y have dimension m. Let $C^{uff}_n(X; \mathbf{Z})$ be the uniformly finite cellular n-chains with coefficients in $\mathbf{Z}$. Then*

$$C^{uff}_n(X \times_w Y; \mathbf{Z}) = \bigoplus_{i+j=n} C^{uff}_i(X; w^{m-j} C^{uff}_j(Y; \mathbf{Z}))$$

where $C_i^{uff}(X; C_j^{uff}(Y))$ means i-chains with values in the uniformly finite j-chains. More explicitly, we take uniformly finite chains with coefficients in the normed group $C_j^{uff}(Y)$, where the norm of a chain is defined to be the supremum of its coefficients. The boundary map is given by the Leibnitz rule.

Proof. We first note that we can assume that the uff homology groups of X and Y are torsion-free. This is so because the torsion part is isomorphic to locally finite homology, to which the usual Eilenberg-Zilber theorem applies.

Note that

$$dV = w^m dV_X dV_Y$$

for the volume forms, dV the volume form for $X \times_w Y$ and dV_X,dV_Y the volume forms of X and Y respectively. Note also that we are using a norm on $\Lambda^k T^* Y_m$ at each point m induced from the Riemannian metric. The L^∞ norm of an element of $\Omega_\beta^k(Y)$ pulled back to $X \times_w Y$ is w^{-k} times its L^∞ norm on Y. Thus the L^∞ norm of $\phi \otimes \psi$ is the norm

$$\| \phi \|_X \| \psi \|_Y \; w^{m-k}$$

Let $\Omega_\beta^{j,k}(M \times N) = \Omega_\beta^j(M) \otimes \Omega_\beta^k(N)$. Then

$$\Omega_\beta^{j,k}(X \times_w Y) = \Omega_\beta^j(X; w^{m-k} \Omega_\beta^k(Y))$$

We thus obtain the chain level description on differential forms:

$$\Omega_\beta^n(X \times_w Y) = \bigoplus_{i+j=n} \Omega_\beta^i(X; w^{m-j} \Omega_\beta^j(Y))$$

Since we are neglecting torision, we can then pass to simplicial chains using the de Rham theorem of [4].

$$\Omega_\beta^n(X \times_w Y) \to C^n_{uff}(X \times_w Y; \mathbf{Z})$$

$$\Omega_\beta^i(X; w^{m-j} \Omega_\beta^j(Y)) \to C^i_{uff}(X; w^{m-j} C^j_{uff}(Y; \mathbf{Z}))$$

The result then follows upon application of Poincaré duality in [4].

We use a description of hyperbolic space as a warped product following P.Pansu [18], along with one due to F.T.Farrell, L. Jones and A.Nicas [9] to carry out a codimension one splitting argument. We first recall some basic facts regarding the geometry of hyperbolic space.

Definition 1.9. The hyperbolic metric on $\mathbf{R}^n$ is given by the warped product metric $\mathbf{R}_+ \times_{sinhr} S^{n-1}$

Definition 1.10. Let $\mathbf{H}^{n-1}$ be considered as the interior of the disc D^{n-1}, and let $\mid z \mid$ be the Euclidean norm of $z \in D^{n-1}$. Define the associate $\mathbf{H}^1(\mathbf{H}^{n-1})$ as the warped product $\mathbf{R} \times_f \mathbf{H}^{n-1}$, where

$$f(z) = \frac{1+ \mid z \mid^2}{1- \mid z \mid^2}$$

Proposition 1.11. *$\mathbf{H}^n$ can be partitioned into spherical shells of any desired thickness.*

Proof. This is easy to see from the model above and the compactness of the sphere.

We will also make use of Toponogov's theorem, which we recall [7]:

Theorem 1.12. *Let M be a complete manifold with $K_M \geq H$. Let $(\gamma_1, \gamma_2, \gamma_3)$ determine a geodesic triangle in M. Suppose γ_1, γ_3 are minimal. Then in the space M^H of constant curvature H there is a geodesic triangle $(\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3)$ so that $L[\gamma_i] = L[\overline{\gamma}_i]$ and the angles $\overline{\alpha}_1 \leq \alpha_1$, $\overline{\alpha}_3 \leq \alpha_3$. Here α_i is the angle between γ_{i+1} and γ_{i+2}.*

2. Whitehead group calculation

We recall the definitions from [2].

Definition 2.1. A CW-complex of bounded geometry is defined to be a CW-complex with a uniformly bounded number of cells attached to each cell and a finite number of homeomorphism types of attaching maps. A *bg* n-cell is a discrete collection of n-cells $\Sigma \times I^n$, equipped with an attaching map $\psi : \Sigma \times I^n \to X$. Two attaching maps $\psi_1, \psi_2 : \Sigma \times I^n \to X$ are of the same homeomorphism type if there is a cellular homeomorphism $h : X \to X$ so that $h\psi_1 h^{-1} = \psi_2$.

Definition 2.2. Let X be a CW complex of bounded geometry. An expansion of bounded geometry is a *bg* CW complex Y so that

i. (Y, X) is a *bg* CW pair.

ii. $Y = X \cup_f (\Sigma \times I^r) \cup_g (\Sigma \times I^{r+1})$ for *bg* $(r+i)$-cells $\Sigma \times I^{r+i}$, $i = 0, 1$ and attaching f, g.

iii. There is a characteristic map $\psi_{r+1} : \Sigma \times I^{r+1} \to Y$ for the *bg* $(r+1)$-cell so that $\psi_{r+1} \mid \Sigma \times I^r : \Sigma \times I^r \to Y$ is characteristic for the *bg* r-cell.

A CW complex of bounded geometry Y is a *bg* collapse of X if X is a *bg* expansion of Y.

Definition 2.3. Let DR^{bg} be the collection of all pairs (Y, X) for which X is a *bg* strong deformation retract of Y. The collection of equivalence classes of such pairs under elementary expansions and collapses rel X is denoted $Wh^{bg}(X)$.

We refer to [1] for the definitions of controlled bases, controlled modules and $R\mathcal{P}G_1(X)$ modules.

Definition 2.4. Let X be a metric space, R a ring. A controlled basis is a pro-object defined as follows. The basis is a pair (S, σ) where S is a set and σ is a function from S to the collection of open sets in a control space X. A morphism from (S, σ) to (T, ρ) is given by a map $\alpha : S \to T$ along with a natural transformation $\rho\alpha \to C^n\sigma$ where C^n is the operation on metric balls in the control space which increases the radius by 1.

Definition 2.5. Let X be a metric space, R a ring. Let (S, σ) be a controlled basis. A controlled free $R\mathcal{P}G_1(X)$-module is defined so that given a controlled basis (S, σ),

i. For any metric ball B, $F(\sigma)(B)$ is the free R-module on the basis

$$\{(\beta, s) \mid s \in S, \beta \text{ a path from } \sigma(s) \text{ to } b\}$$

ii. The controlled fundamental groupoid $\mathcal{P}G_1(X)$ acts by composition of paths.

Definition 2.6. The category of controlled free *bg* $\mathbf{Z}\mathcal{P}G_1(X)$-modules is defined to be the category of controlled modules so that in any ball of fixed radius the modules fall into a finite number of types. Morphisms are defined to be morphisms of the modules so that if the control space is partitioned into neighborhoods of a fixed radius, the restrictions fall into a finite number of equivalence classes. By abuse of notation, we will denote this category by $\mathbf{Z}\mathcal{P}G_1(X)^{bg}$.

Definition 2.7. We define $K_1(\mathbf{Z}\mathcal{P}G_1(X)^{bg})$ to be the abelian group generated by $[F,\alpha]$ where F is a controlled free *bg* module and α is an automorphism of F so that

i. $[F,\alpha]=[F',\alpha']$ if there is an isomorphism $\phi : F \to F'$ so that $\phi\alpha = \alpha'\phi$.
ii. $[F\oplus F', \alpha\oplus\alpha'] = [F,\alpha]+[F',\alpha']$
iii. $[F,\alpha\beta]=[F,\alpha]+[F,\beta]$

Definition 2.8. $Wh^{bg}(\mathcal{P}G_1(X))$ is defined to be the quotient of $K_1(\mathbf{Z}\mathcal{P}G_1(X)^{bg})$ defined by taking the quotient by the subgroup of elements of the form

$$[F(\sigma), u_{F(\sigma)}] \text{ and } [F(\sigma), F(\alpha,\nu)]$$

where (S,α) is any *bg* basis over $\mathcal{P}G_1(X)$, $u_{F(\sigma)}$ is multiplication by a unit and $F(\alpha,\nu)$ is an automorphism of bases. It is shown in [2] that this group is isomorphic to $Wh^{bg}(X)$.

Definition 2.9. The bounded Whitehead group $Wh^{bdd}(X)$ is defined to be a quotient of K_1 of the category $\mathbf{Z}\mathcal{P}G_1(X)$. The objects of $\mathbf{Z}\mathcal{P}G_1(X)$ are locally finitely generated controlled free modules over X and morphisms are controlled morphisms. $Wh^{bdd}(X)$ is $K_1(\mathbf{Z}\mathcal{P}G_1(X))$ modulo multiplication by a unit and automorphisms of bases.

Remark 2.10. There is a natural forgetful map $\mathbf{Z}\mathcal{P}G_1(X)^{bg} \to \mathbf{Z}\mathcal{P}G_1(X)$ given by considering a *bg* module as a boundedly controlled module. This induces a map

$$K_1(\mathbf{Z}\mathcal{P}G_1(X)^{bg}) \to K_1(\mathbf{Z}\mathcal{P}G_1(X))$$

It is then clear that one can obtain a map

$$Wh^{bg}(X) \to Wh^{bdd}(X)$$

We will introduce a filtration of $Wh^{bg}(M \times \mathbf{H}^n)$ which is obtained by easing the restrictions on bounded geometry in various directions. This technical modification allows us to inductively prove the theorem using methods of [3]. This filtration was suggested by S. Cappell.

Definition 2.11. Let $p : X \to \mathbf{H}^n$ be a control map. Let $exp : \mathbf{R}^n \to \mathbf{H}^n$ be the exponential map. Since $\mathbf{H}^n$ is a Hadamard manifold, this is a diffeomorphism. p is said to be $bg(k)$, if the restriction of p to a regular neighborhood $D^r \times exp^{-1}\mathbf{R}^k$ of the inverse image of $\mathbf{R}^k$, in the decomposition of $\mathbf{R}^n$ as $\mathbf{R}^k \times \mathbf{R}^{n-k}, \mathbf{R} \times \mathbf{R}^k \times \mathbf{R}^{n-k-1}, ... \mathbf{R}^{n-k} \times \mathbf{R}^k$ is bg.

In the case of a general Hadamard manifold X we can define the same filtration, again making use of the diffeomorphism given by the exponential map. The construction is the same as for the hyperbolic case.

Definition 2.12. Define the category $\mathbf{Z}\mathcal{P}G_1(X)^{bg(k)}$ to be the category of $bg(k)$ controlled modules, in the sense of the previous definition. $K_1(\mathcal{P}G_1(X)^{bg(k)})$ has a quotient $Wh^{bg(k)}(X)$ defined analogously to $Wh^{bg}(X)$.

Definition 2.13. Let $\mathcal{A}$ be a small additive category. The idempotent completion of $\mathcal{A}$ is the category with objects morphisms $p : A \to A$ of $\mathcal{A}$ so that $p^2 = p$ and morphisms given by morphisms $\phi : A_1 \to A_2$ so that $\phi = p_1 \phi p_2$ where $p_i : A_i \to A_i$ are the source and target.

Definition 2.14. The bg projective class group of X, $\tilde{K}_0^{bg}(X)$ where X is a bg simplicial complex, is defined to be K_0 of the idempotent completion of the category $\mathbf{Z}\mathcal{P}G_1(X)^{bg}$. There is a homomorphism

$$\text{rank} : K_0^{bg}(X) \to H_0^{uff}(X; \mathbf{Z})$$

given as follows. Let $m \in K_0^{bg}(X)$ be given element. One can find a representative for m which has basis elements only at each vertex of the simplicial complex X. Thus to a given vertex one can naturally associate a free module. Define rank(m) to be the uniformly finite 0-chain rank$(m) = \sum r_x x$ where x ranges over the vertices of X and r_x is the rank of the free module constructed above. This clearly gives a map $K_0^{bg}(X) \to C_0^{uff}(X;\mathbf{Z})$, and we observe that by taking infinite process tricks into account which cancel ranks, we can pass to homology. The kernel of this map is the reduced bg projective class group of X and is denoted $\tilde{K}_0^{bg}(X)$. It measures the obstruction to a bg projective module being free.

Definition 2.15. The category $\mathcal{C}_n(R)$ of $\mathbf{Z}^n$-graded objects and bounded homomorphisms over a ring R is defined as in [20] as follows. An object is a collection of objects $A(J)$ and a morphism is a collection

$$\phi(J,K) : A(J) \to B(K)$$

so that $\phi(J,K) = 0$ unless $\| J - K \| \leq d$ for some d. Define $\mathcal{C}_n^{bg}(R)$ to be the subcategory of modules which are bg, i.e. fall into a finite number of types over each portion of the lattice of finite radius.

Definition 2.16. $H_*^{uff}(\mathbf{H}^n; Wh_*(M))$ is the abelian group

$$H_0^{uff}(\mathbf{H}^n; Wh(M)) \oplus ... \oplus H_n^{uff}(\mathbf{H}^n; K_{1-n}(M))$$

where we take the unique norm on $Wh(M)$ so that every chain has coefficients in a finite subset of $Wh(M)$ depending on the chain.

Theorem 2.17. $Wh^{bg}(M \times \mathbf{H}^n) = H_*^{uff}(\mathbf{H}^n; Wh_*(M))$

Proof. We proved this theorem for $n = 1$ in [2]. To do the general case, we first consider the case $n = 2$. Observe that there are forgetful maps

$$Wh^{bg}(M \times \mathbf{H}^2) \to Wh^{bg(1)}(M \times \mathbf{H}^2) \to Wh^{bdd}(M \times \mathbf{H}^2)$$

Here the control $bg(1)$ is defined in Definition 2.11. Now note that the last group is known, due to Pedersen [20] coupled with the following discussion, to be $K_{-1}(M)$.

Definition 2.18 [16]. Let X and Y be metric spaces in which closed and bounded sets are compact. A coarse map from X to Y is a continuous and proper map $f : X \to Y$ so that for every $R > 0$ there exists $S > 0$ with

$$d(x, x') \leq R$$

implying

$$d(f(x), f(x')) \leq S.$$

A coarse homotopy from X to Y is a continuous and proper map

$$h : X \times [0,1] \to Y$$

so that for every $R > 0$ there exists $S > 0$ with

$$d(x, x') \leq R$$

implying

$$d(h(x,t), h(x',t)) \leq S$$

for all $t \in [0,1]$. Two coarse $f_0, f_1 : X \to Y$ are coarsely homotopic if there is a coarse homotopy $h : X \times [0,1] \to Y$ so that

$$f_0(x) = h(x,0)$$

$$f_1(x) = h(x,1)$$

A coarse map $f : X \to Y$ is a coarse homotopy equivalence if there is a coarse map $g : Y \to X$ so that $f \circ g$ and $g \circ f$ are coarsely homotopic to the identity maps on Y and X respectively.

The following is Theorem 7.3 of [16].

Theorem 2.19. *Let X be a Hadamard manifold, and let $\mathbf{0}$ be a fixed point in X. Denote by T the tangent space to X at $\mathbf{0}$. Then the inverse of the exponential map*

$$log : X \to T$$

is a coarse homotopy equivalence.

It is more useful to have the following notion, however, due to Higson, Roe and Pedersen, of Lipschitz homotopy, which was suggested to the author by J. Roe.

Definition 2.21. Let X and Y be coarse spaces. A Lipschitz homotopy is a coarse map

$$h : X \times \mathbf{R}_+ \to Y \times \mathbf{R}_+$$

of the form $h(x,t) = (H(x,t),t)$ which has the property that for each compact $K \subseteq X$ there is $t_K \in \mathbf{R}_+$ so that $H(x,t)$ is constant in t for $x \in K$, $t \geq t_K$.

Definition 2.22. Let X be a metric space. Define the bounded higher Whitehead group $Wh_2^{bdd}(X)$ as follows. Let $E^{bdd}(X)$ be the group of elementary bounded automorphisms: multiplication by a unit

$$[F(\sigma), u_{F(\sigma)}]$$

and change of basis

$$[F(\sigma), F(\alpha, \nu)]$$

. Following Kervaire we define the boundedly controlled analogue of Milnor's K_2 group. Define $St^{bdd}(X)$ and $K_2^{bdd}(X)$ to be the universal covering

$$1 \to K_2^{bdd}(X) \to St^{bdd}(X) \to E^{bdd}(X) \to 1$$

Lemma 2.24. *Let X and Y be metric spaces in which closed and bounded sets are compact. Let $f : X \to Y$ be a Lipschitz homotopy equivalence. Then f_* : $Wh^{bdd}(X) \to Wh^{bdd}(Y)$ is an isomorphism.*

Proof. We need to show that if f and g are Lipschitz homotopic maps, then they induce the same map from $Wh^{bdd}(X) \to Wh^{bdd}(Y)$. We can use Pedersen's method for calculating the bounded Whitehead group to show that the bounded Whitehead group of $X \times \mathbf{R}$ is isomorphic to

$$H_0^{lf}(\mathbf{R}; Wh^{bdd}(X)) \oplus H_1^{lf}(\mathbf{R}; K_0^{bdd}(X))$$

Since the first term is zero, we obtain $K_0^{bdd}(X)$. Following Anderson and Munkholm, we can think of the group $Wh^{bdd}(X)$ as the group of h-cobordisms of X and then use the bounded version of pseudo-isotopy theory of Hatcher and Wagoner to define $Wh_2^{bdd}(X)$ geometrically in terms of boundedly controlled pseudo-isotopies, and then Pedersen's method will again yield

$$H_0^{lf}(\mathbf{R}; Wh_2^{bdd}(X)) \oplus H_1^{lf}(\mathbf{R}_+; Wh^{bdd}(X))$$

via geometric splitting of h-cobordisms. This yields the result, since $f_* = h(x,0)_*$ and $g_* = h(x,\infty)_*$ agree.

Remark. In the case where X is a Hadamard manifold, the methods of T.Chapman and F.Quinn can be used to show that a boundedly controlled simple homotopy equivalence is a controlled simple homotopy equivalence, via subdivision. This implies that $Wh^{bdd}(X) = Wh^{bdd}(\mathbf{R}^n)$, since work of Quinn shows that the category of controlled maps and spaces is independent of the metric. C.B.Hughes, L. Taylor and B. Williams have used this philosophy to show that the simplicial space of bounded homeomorphisms of a Hadamard manifold is homotopy equivalent to the simplicial space of controlled homeomorphisms.

Pedersen proved that $Wh^{bdd}(M \times \mathbf{R}^2)$ is isomorphic to $K_{-1}(M)$.

A $bg(1)$-controlled h-cobordism representing an element of $Wh^{bg(1)}(M \times \mathbf{H}^2)$ is said to be split along circles if it can be decomposed into compact h-cobordisms over each $M \times S^1 \times I$, glued together over their common boundary. We claim that if an h-cobordism, which serves as a geometrical representative of an element of $Wh^{bg(1)}(M \times \mathbf{H}^2)$ is in the kernel of the forgetful map to $K_{-1}(M)$, then it is bg split along circles, and conversely. For if an element is bg split along circles, then by the infinite process trick of [20], one can move each module at each lattice point out to infinity along the radii without violating the loosened boundedness condition. The converse goes by an analogue of [2] where a radius $\mathbf{R}_+$ replaces the real line $\mathbf{R}$. An element $[A, \alpha]$ is defined to be split at a point in $\mathbf{R}_+$ if the automorphism α preserves the two parts of $\mathbf{R}_+$ below and above the point. We define the splitting obstruction in one direction, $\phi([A, \alpha])$ by

$$\overline{A} = \bigoplus_{j=l-2k}^{l+2k} A(j)$$

$$\phi([A, \alpha]) = \sum_{l \in 4k\mathbf{Z}_+} ([\overline{A}, \alpha p_-^l \alpha^{-1}])x_l.$$

Here, α is a controlled automorphism with bound k, p_-^l is the projection onto the part of the half-line below $j = l$ and x_l is the l-th vertex in the triangulation of $\mathbf{R}_+$ by intervals of length $2k$. Elements of the kernel of this map are split by the argument of [20]. Moreover this map is the forgetful map to $K_{-1}(M)$. Furthermore, applying an infinite process trick due to Swan (see [20] for a description) yields a surjection onto $K_{-1}(M)$ and shows in fact that any element of $K_{-1}(M)$ can be represented by a $bg(1)$-controlled element over $\mathbf{H}^2$. One first sees that the representatives in $K_{-1}(M)$ are $bg(1)$, since they can be chosen to have bounded geometry in one direction at a time.

Swan's infinite process trick is as follows. Given an element $(B, p) \in K_{-1}(M)$, one constructs an element of $Wh^{bg(1)}(M)$ by defining

$$A(j_1, ..., j_s, ..., j_{i+1}) = B(j_1, ..., j_{s-1}, j_{s+1}, ..., j_{i+1})$$

and mapping B to itself by p and $1 - p$ successively. We use the version which is described on p. 472 of [20].

We next analyze the map $Wh^{bg}(M \times \mathbf{H}^2) \to Wh^{bg(1)}(M \times \mathbf{H}^2)$. We claim that the kernel of this map is zero. Elements of the kernel of the map are bg split simultaneously along circles and radii. By splitting along radii we include rays going out to infinity which do not originate at the origin, but at a circle centered at the origin. The split pieces are required to have equal area, and the splittings need to be a sufficient distance apart in order that Pedersen's method be applicable. One should observe that this means that an exponentially growing number of splittings along rays are performed as one goes out to infinity.

To see this, note that if such an element is $bg(1)$ one can move the modules along the lattice points to a radius and then out to infinity. This is because the complexity is allowed to increase along the radii.

The converse is by comparison with a splitting obstruction. Thus the kernel of the map is represented by elements of $C_0^{uff}(\mathbf{H}^2; Wh(M))$, modulo infinite process tricks which identify different elements of $Wh^{bg}(M \times \mathbf{H}^2)$, which yields $H_0^{uff}(\mathbf{H}^2; Wh(M))$, as in [2]. However, this is zero.

We will be finished with the calculation as soon as we have analyzed the kernel of the map $Wh^{bg(1)}(M \times \mathbf{H}^2) \to K_{-1}(M)$ and show that both maps considered are split surjective. We claim that the obstruction to splitting along concentric circles is

$$C_1^{uff}(\mathbf{H}^2; \tilde{K}_0(M))$$

We define a splitting obstruction in one direction given by $\phi([A, \alpha])$ as before, which yields an element in $C_0^{uff}(\mathbf{R}_+; C_1^{uff}(S^1; \tilde{K}_0^{bg}(M))$ and then after splitting in this direction one obtains splitting obstructions in the transverse direction given by $C_1^{uff}(\mathbf{R}_+; (sinhr)^{-1} C_0^{uff}(S^1; \tilde{K}_0(M))$. This yields the claim. Next, observe that we need to factor out by infinite process tricks over $\mathbf{H}^2$. Applying the Eilenberg-Zilber theorem gives the result that the kernel of $Wh^{bg}(M \times \mathbf{H}^2) \to Wh^{bg(1)}(M \times \mathbf{H}^2)$ is $H_1^{uff}(\mathbf{H}^2; \tilde{K}_0(M))$. The splittings back are given by Swan's infinite process trick, and we have the result for $n = 2$. The result for general n follows similarly.

Theorem 2.25. *Let X be any Hadamard manifold, then $Wh^{bg}(X)$ is given by*

$$Wh^{bg}(X) = H^{uff}(X; Wh_*(pt.))$$

Proof. We indicate how to modify the proof in the hyperbolic case to the case of a general Hadamard manifold. The theorem follows from an identical argument to the one above except that one uses polar coordinates pulled back by the exponential map in place of the polar coordinates in hyperbolic space. The Rauch comparison theorem, the Toponogov comparison theorem, and the Bishop-Gromov comparison theorem are used to obtain a bound on the number of splittings that must be performed to obtain a uniform cellulation of the space. The comparison theorems guarantee, in particular, that the splittings can be done a sufficient distance apart from each other and that the volumes of the resulting cells are equal. The rest follows identically as in the hyperbolic case.

3. Proof of the Main Theorem

We apply a version of the splitting theorem of [2].

Theorem 3.1. *Let $h : M \to N$ be a bg homotopy equivalence, where M is the Cartesian product of a compact manifold with $\mathbf{H}^n$, or a simply connected Hadamard manifold V^n, $n \geq 5$ and N is bg over $\mathbf{H}^n$, V respectively. If $X \subset N$ is a bg codimension 1 submanifold which is the transverse inverse image of $S^{n-1} \times \mathbf{Z}$, then h can be split along X if and only if an obstruction in a summand $\tilde{K}_0^{bg}(N)$ of $Wh^{bg}(N)$ vanishes, and the components of X are sufficiently separated from each other.*

Proof. The proof is like the Euclidean case in [2], except that the splitting is along compact submanifolds, but growing in diameter and volume. As in the Euclidean case, the controlled kernels are defined, and after handle trading they can be represented as projective $\mathbf{Z}\mathcal{P}G_1(N)$ modules and so can be represented as elements of the projective class group $\tilde{K}_0(N)$. If the class of this element vanishes, then the controlled kernel can be represented by embedded spheres, and so can be surgered away. This surgery uses the hypothesis that the components of X are sufficiently separated from each other to insure that the Nil obstructions vanish. This, along with the details of [2], completes the proof.

In addition to the splitting along spheres, we need to split along hyperplanes in the direction of the radii. By this we essentially mean hyperplanes in the normal direction to the spheres in the warped product decomposition of $\mathbf{H}^n$.

Theorem 3.2. *Let M, N, h, V be as in the previous lemma. Let X be the transverse inverse image of an infinite collection of half-planes normal to the spheres in the warped product decomposition of $\mathbf{H}^n$, V. Then h can be split along X if and only if an obstruction in a summand $\tilde{K}_0^{bg}(N)$ of $Wh^{bg}(N)$ vanishes, and the components of X are sufficiently separated from each other.*

Proof. The proof is completely analogous to the previous theorem and to the theorem in [2], and so will be omitted.

Definition 3.3. We denote by $\mathcal{S}^{bg,i}(M)$ the PL bg structure set with simple homotopy equivalences replaced by maps with torsion in $K_i^{bg}(M)$ equal to zero.

Proof of 0.1. We apply Theorem 3.1 in the following manner.

Let N be simple homotopy equivalent to $M \times \mathbf{H}^n$. We will apply the splitting theorems above to split N into pieces of equal volume, each of which is compact and contractible. To do this we express $\mathbf{H}^n$ as a warped product, $S^{n-1} \times_{sinhr} \mathbf{R}_+$ and split along the spheres S^{n-1}. Then we split simultaneously along half-hyperplanes going out to infinity, and along orthogonal half-hyperplanes going out to infinity, and so on, so that the annuli $S^{n-1} \times I$ are cut into contractible cubes of equal volume.

Let N be bg simple homotopy equivalent to $M \times \mathbf{H}^n$. We can apply the splitting theorem, since the splitting obstruction vanishes, to obtain a splitting of N along concentric spheres. Each split piece is simple homotopy equivalent to $M \times S^{n-1} \times I$ and each boundary piece is homotopy equivalent to $M \times S^{n-1}$ and furthermore is PL homeomorphic to the next one.

We perform a second splitting transverse to the first. For this we apply the second splitting theorem to each piece, and then observe that the separation of the splitting depends only on the complexity of the given homotopy equivalence, which is uniformly bounded by hypothesis. Thus the splittings can be aligned with each other and we obtain a transverse splitting. The successive splittings produce a splitting along a uniform triangulation of $\mathbf{H}^n$ which is defined by the hyperbolic polar coordinates. We thus obtain chains with coefficients in the structure set, as in the Euclidean case. There is a chain homotopy of these chains to chains calculated with respect to "rectangular" hyperbolic coordinates, i.e. with hyperbolic space viewed as the upper half plane.

This splits the hyperbolic space down to a cell decomposition of the space, which is appropriate for the calculation of uniformly finite homology. We then make the observation that the split map is a simple homotopy equivalence on each cell $M \times D^n$ rel ∂, a homotopy equivalence on each cell $M \times D^{n-1}$, and so on, until one reaches the 0-cells M on which f is a homotopy equivalence whose torsion in $K_{1-n}(M)$ is zero. As in the Euclidean case, this completes the proof.

We also claim that the splitting preserves the group structures on the various structure sets involved. This can be seen, for example by converting the whole argument into algebraic L-theory and using Ranicki's description of the structure set as an L-group [21]. One then imitates the argument for the Whitehead group to see that the group structures are preserved by the splitting.

Proof of Theorem 0.2. This follows from the first main theorem except for some low dimensional difficulties, which are taken care of by Siebenmann periodicity.

In addition, to go from a continuous EPL map to a smooth quasi-isometry, we make use of the fact that the Casson-Sullivan invariant is a modulo 2 invariant, so that every homeomorphism from a contractible manifold of bounded geometry to hyperbolic space is *bg* homotopic to a *bg* PL homeomorphism. (See [21]).

Proof of Theorem 0.6. We first observe that a map $f : M \times X^n \to N$ lifts to a *bg* simple homotopy equivalence on the hyperbolic cover if and only if it is simple on a finite cover. This follows from the theorem of Farrell-Jones [9]:

$$Wh(\Gamma \times G) = H_*(\Gamma; Wh_*(G)) \oplus Nils$$

along with the calculation of the Whitehead group in section 4, proposition 2.10 in [2] and the observation that any given element of the Nils vanishes on a finite cover. We then apply the calculation of the structure set, Theorem 0.1 above. Applying proposition 2.10 of [2], we obtain a rational injection of $H_*(\Gamma; \mathcal{S}(M)) \to H_*^{uff}(\mathbf{H}^n; \mathcal{S}(M))$ for the top two dimensions. The only term in the direct sum decomposition of $\mathcal{S}^{bg,s}(M \times \mathbf{H}^n)$ which can contain torsion is $\mathcal{S}^{1-i}(M)$. However, these structures are already detected by boundedly controlled surgery, so they inject.

We therefore need to check that an element of the kernel of the map, which is a torsion element in $H_{n-1}(X^n; \mathcal{S}(M \times I, \partial))$, becomes trivial on a finite cover. But to see this, simply observe that taking an n-fold cover of a split structure on $M \times X$ has the effect of adding the structure on $(M \times I, \partial)$ to itself n times.

4. Application to quasi-conformal mappings

In this section we apply techniques of bounded geometry to the study of quasi-conformal maps. An isomorphism in the *bg* sense is a quasi-isometry. An isomorphism in the quasiconformal/quasiregular sense is a quasiconformal map. These notions are not the same: most quasiconformal maps are not quasi-isometries and conversely. However, one can introduce a new metric on a space with a quasi-conformal structure–the extremal metric–which is functionally related to the original metric, but which has bounded geometry and for which every quasiconformal map becomes a quasi-isometry. For manifolds of bounded geometry with sectional curvature pinched between two negative constants, the extremal metric is quasi-isometric to the Riemannian one. Hence quasiconformal maps can be treated as quasi-isometries.

The main advantage of using bounded geometry is that extension problems can be solved first homotopically, using the obstruction theory for controlled maps due essentially to Anderson and Munkholm [1], [2]. The result is a *bg* homotopy equivalence, which needs to be converted into a quasi-isometry, or in the negatively curved context, into a quasiconformal map. This conversion is then achieved by using the *bg* Borel conjecture, which is proved in section 3 for hyperbolic n-space as Theorem 0.4 of this paper.

We use two definitions of quasiconformal mapping, one for domains, one for Riemannian manifolds of negative curvature.

Definition 4.1 [11]. Suppose D and D' are domains in $\mathbf{R}^n \cup \infty$ and that $f : D \to D'$ is a homeomorphism. We let

$$H_f(x) = \lim_{r \to 0} \sup H_f(x, r)$$

for $x \in D - \{\infty, f^{-1}(\infty)\}$ where for $0 < r < dist(x, \partial D)$

$$H_f(x, r) = \frac{max\{\mid f(x) - f(y) \mid : \mid x - y \mid = r\}}{min\{f(x) - f(z) \mid : \mid x - z \mid = r\}}$$

and we extend $H_f(x)$ to the points ∞ and $f^{-1}(\infty)$ by setting $H_f(\infty) = H_{f \circ g}(0)$ and $H_f(f^{-1}(\infty)) = H_{g \circ f}(f^{-1}(\infty))$, where $g(x) = \frac{x}{|x|^2}$. We call $K(f) = \infty$ if $\sup_{x \in D} H_f(x) = \infty$. We let

$$K(f) = ess \sup_{x \in D} H_f(x) \text{ if } \sup_{x \in D} H_f(X) < \infty$$

the linear dilatation of f in D. f is K-quasiconformal if $K(f) \leq K, 1 \leq K < \infty$.

Definition 4.2 [13]. Let A, B be metric spaces. Let $\mathcal{U}$ be a family of neighborhoods $\mathcal{U} \subset A$ of a fixed point $a \in A$. We denote by $inrad(\mathcal{U}, a)$ the infimum of the distance function $dist(a, u)$ over $u \in \mathcal{U}$ and let $outrad$ denote the supremum of $dist(a, u)$. Then we set

$$asph(\mathcal{U}, a) = outrad/inrad$$

and

$$asph(\mathcal{U}) = \lim_{diam \to 0} \sup asph(\mathcal{U}, a)$$

For a continuous map $f : A \to B$ we define the conformality of f at a by

$$conf_a f = asph(f(B(r)))$$

for the balls $B(r) \subset A$ around a for $r \in (0, \infty)$. We call f λ-quasiconformal if

$$conf_a f \leq \lambda < \infty$$

for all a.

The following definitions concerning strongly uniform domains are from [15].

Definition 4.3. Let $\rho : [0, \infty) \to [1, \infty)$ be an increasing function and $m < n$ a non-negative integer. Then a domain $D \subset \mathbf{R}^n$ is (m, ρ)-uniform if for each $0 \leq i \leq m$, each pair of i-spheres $\Sigma^i = (\Sigma^i, g)$ and $\tilde{\Sigma}^i = (\tilde{\Sigma}^i, \tilde{g})$ in D can be joined by a homotopy F in D so that

$$\mathrm{diam}(F) \leq \rho(t)\mathrm{diam}(\Sigma^i \cup \tilde{\Sigma}^i),$$

and

$$\mathrm{dist}(x, \Sigma^i \cup \tilde{\Sigma}^i) \leq \rho(t)\mathrm{dist}(x, \partial D)$$

for each $x \in F$, whenever

$$\max\{diam(\Sigma^i), diam(\tilde{\Sigma}^i)\} \leq t\mathrm{dist}(\Sigma^i \cup \tilde{\Sigma}^i, \partial D)$$

We say D is m-uniform if it is (m, ρ)-uniform for some ρ, and D is strongly uniform if it is m-uniform for all $m < n$.

Definition 4.4. If $X \subset \mathbf{R}^n$ is an embedding, then $f : X \to \mathbf{R}^n$ is η-quasisymmetric if

$$\frac{\mid f(a) - f(b) \mid}{\mid f(a) - f(c) \mid} \leq \eta(\frac{\mid a - b \mid}{\mid a - c \mid})$$

for some homeomorphism $\eta : [0,\infty) \to [0,\infty)$ and for all triples (a,b,c) of distinct points in X.

The following definition can be found in [24].

Definition 4.5. Let D denote a proper subdomain of $\mathbf{R}^n$. In D we define a weight function $w : D \to \mathbf{R}_+$ by

$$w(x) = \frac{1}{dist(x, \partial D)}$$

The quasihyperbolic distance between x and y in D is defined by

$$k_D(x,y) = \inf_{\alpha \in \Gamma_{xy}} \int_\alpha w(x) \mid dx \mid$$

where Γ_{xy} is the set of rectifiable curves in D connecting x and y.

Definition 4.6. A domain D is quasihyperbolically m-connected, or $QHC(m)$ if there is an increasing function $\psi : [0,\infty) \to [0,\infty)$ so that every i-sphere $\Sigma^i \subset D$, $0 < i \leq m$, is homotopic to a point in D through a homotopy F with

$$k_D(F) \leq \psi(k_D(\Sigma^i))$$

Here $k_D(E)$ denotes the quasihyperbolic diameter of E.

Finally, we give a generalization of the Tukia-Väisälä theorem [23] to strongly uniform domains. This application was suggested to the author by S.Semmes. We recall the statement of the Tukia-Väisälä theorem:

Theorem 4.7 (Tukia-Väisälä [23]). *Suppose $f : \mathbf{R}^n \to \mathbf{R}^n$ is a quasiconformal map. Then f can be extended to a quasiconformal map of $\mathbf{R}^{n+1}_+ \to \mathbf{R}^{n+1}_+$.*

In addition we have a new generalization of this theorem:

Theorem 4.8. *If D is a strongly uniform domain and f is a quasisymmetric mapping homeomorphism of ∂D to the boundary of the unit ball B^n in $\mathbf{R}^n$ then there is an extension of f to a quasiconformal mapping $D \to B^n$.*

In order to prove this result we need to recall the correspondance between quasiconformal structures and bg structures in the hyperbolic case. This correspondance is discussed by P.Pansu in [17]. Pansu also discusses in [17] generalizations of this correspondance which he proves elsewhere, and which we will make use of below. We first define the isoperimetric profile of a manifold.

Definition 4.9. For a real number $v \leq \mathrm{Vol}(M)$, let $I(v)$ be the infimum of the volumes of the hypersurfaces in M which bound a compact domain of volume v:

$$I(v) = \inf\{(\partial D) : D \text{ compact }, vol(D) = v\}$$

We next define the extremal distance, which is only a pseudo-distance, but turns out to be an actual distance in the hyperbolic case.

Definition 4.10. A condenser in a manifold is a triple (C, B_0, B_1) where C is open, B_0 and B_1, the "plates", are closed and contained in the closure of C. We shall admit $B_0 = \infty$ when C is unbounded. The conformal capacity of a condenser is the infimum of the volumes of the conformal metrics on C for which the plates are a distance 1 apart.

Definition 4.11. Let M be a Riemannian manifold. We define the conformal pseudo-distance β as follows:

$$\beta(x,y) = \inf\{cap(M, B, \infty) : B \text{ is compact, connected and contains x and y}\}$$

Observation 4.12. A quasiconformal map is bg with respect to β.

Theorem 4.13 (Pansu [17]). *Assume that an n-dimensional complete Riemannian manifold M of bounded geometry satisfies a strictly stronger isoperimetric inequality that Euclidean n-space, i.e the integral*

$$\int^{+\infty} I(v)^{-n/n-1}\, dv$$

is finite. Then the conformal pseudo-distance β is a distance. If one has an isoperimetric inequality of the type

$$vol(\partial D) \geq const.vol(D)^{1-\tau}$$

with $\tau < 1/n$, then for all $x, y \in M$, there exist constants A and B so that

$$\beta(x,y) \geq Ad(x,y)^{1-n\tau} - B$$

For spaces of bounded negative curvature the reverse inequality holds also.

Consequently, quasi-conformal mappings of hyperbolic n-space are the same as quasi-isometries. This holds also for Riemannian manifolds of pinched negative sectional curvature, as the number τ in the statement of the theorem above is also zero in this case.

We use the representation of the upper half plane as $\mathbf{H}^{n+1}$. This can be compactified into the hyperbolic ball with boundary the one point compactification of $\mathbf{R}^n$.

Before proving the theorem of Tukia-Väisälä, we need a few definitions.

(See [12] for the definitions, and [13], p.99, Example 18.A.(b) for the case of symmetric spaces).

Definition 4.12. Let X be a metric space, Y and Z two non-empty subsets of X, and H a real positive number. We call the set

$$\{x \in X : d(x,Y) \leq H\}$$

$\mathcal{V}_H(Y)$, the H-neighborhood of Y. We call the number defined by

$$\mathrm{Inf}\{H > 0 : Y \subset \mathcal{V}_H(Z), Z \subset \mathcal{V}_H(Y)\}$$

the Hausdorff distance between Y and Z.

Definition 4.15. A map F between two metric spaces X_0 and X is a (λ, c)-quasi-isometry if

$$\frac{1}{\lambda} \mid s-t \mid -c \leq \mid F(s) - F(t) \mid \leq \lambda \mid s-t \mid +c$$

Recall that a (λ, c)-quasi-geodesic in a metric space X is a (λ, c)-quasi-isometry $f : \mathbf{R} \to X$. A quasi-isometric map $f : \mathbf{R}_+ \to X$ is called a quasi-ray. Two quasi-rays are equivalent if their images are a finite Hausdorff distance apart. If X is a

δ-hyperbolic metric space then the Gromov boundary of X is the set of equivalence classes of quasi-rays in X.

We will use a particular conformal structure on the boundary. The following definitions are due to Gromov and Pansu:

Definition 4.16 [14]. Let X be a manifold with pinched negative sectional curvature. Every two points s_1 and s_2 in the sphere at infinity ∂X can be joined by a unique geodesic in X, denoted (s_1, s_2). Define $\Delta(s_1, s_2, s_3, s_4) = dist((s_1, s_2), (s_3, s_4))$, where the distance between subsets in X refers to the infimum of the distances between their points. A map $f : \partial X \to \partial X'$ is quasiconformal if

$$a^{-1}\Delta(s_1, s_2, s_3, s_4) - b \leq \Delta'(f(s_1), f(s_2), f(s_3), f(s_4)) \leq a\Delta(s_1, s_2, s_3, s_4) + b$$

where Δ is the function on ∂X and Δ' the corresponding function on $\partial X'$.

Definition 4.17 [14]. A coarse quasi-isometry $f : X \to Y$ between two metric spaces satisfies the conditions

i. There exist positive constants A and B so that every four points x_1 and x_2 in X satisfy

$$A^{-1}d(x_1, x_2) - B \leq d(f(x_1), f(x_2)) \leq Ad(x_1, x_2) + B$$

ii. The domain of definition of f is a net in X while the image of f is a net in Y.

Note that the term coarse used here differs from the term coarse as used by Higson and Roe [15], and used above in Definition 2.18, where a coarse map is required to be continuous.

We observe that every quasiconformal map of ∂X extends to a coarse quasi-isometry. See [14], where other conformal structures on ∂X are discussed, for which this extension property is not known in general. The converse was first observed for hyperbolic spaces in [8].

Proof of Tukia-Väisala Theorem. $\mathbf{R}^n$ can be viewed as the boundary of $\mathbf{H}^{n+1}$, which is $\mathbf{R}^{n+1}_+$ with the hyperbolic metric. A quasiconformal map of $\mathbf{R}^n$ to itself can be extended to a quasiconformal map of the 1-point compactification $\mathbf{R}^n \cup \infty$ to itself by fixing the point at infinity. This is equivalent to a coarse quasi-isometry of $\mathbf{H}^{n+1}$ to itself as can be seen by making use of the fact that a quasi-conformal map of the boundary S^n to itself is quasiconformal with respect to the structure Δ defined above ([13], p. 93). This reduces the theorem to showing that a coarse quasi-isometry of $\mathbf{H}^{n+1}$ to itself is bg homotopic to a smooth quasi-isometry of $\mathbf{H}^{n+1}$ to itself. A coarse quasi-isometry of $\mathbf{H}^{n+1}$ can be extended to a continuous bg homotopy equivalence of $\mathbf{H}^{n+1}$ to itself, as can be seen by applying the controlled homotopy theory of [1]. Theorem 0.4 then implies that this map is bg homotopic to a bg homeomorphism.

Proof of Theorem 0.5. Equip D with the quasi-hyperbolic metric defined by using the density $\operatorname{dist}(x, \partial D)^{-1} \mid dx \mid$. The following is a theorem of Heinonen and Yang [15]:

Theorem 4.18. *A domain D is m-uniform if and only if it is both uniform and $QHC(m)$.*

As a corollary to this theorem, we see that D is strongly uniform also in the quasi-hyperbolic metric. We first claim that a quasisymmetric map on the boundary of D can be extended naturally to a coarse quasi-isometry on the interior of D in the quasi-hyperbolic metric. This follows again from [13].

The result then follows from the uniform contractibility of D and the bg Borel conjecture for hyperbolic n-space.

Theorem 4.19. *Let X^n, $n \geq 5$ be a simply connected manifold of bounded geometry with sectional curvature pinched between two negative constants. Let $\overline{X}$ be the compactification of X obtained by adding on the sphere at infinity, consisting of the geodesics on X. Then a quasiconformal map $f : \partial X \to \partial X$ of the boundary of $\overline{X}$ to itself, can be extended to a quasiconformal map of $\overline{X}$ to itself.*

Proof. This follows in the same way as in the hyperbolic case. We use the last statement of theorem 4.13.

Remark 4.20. The final statement of Theorem 4.13 is false in the case of solvmanifolds of negative curvature. See [17].

5. Topological Invariance of Pontrjagin Classes

The topological invariance of Pontrjagin classes was proven by S. Novikov in 1965. Here we give an alternative to the method of Pedersen, Roe and Weinberger, using surgery on manifolds of bounded geometry, answering a question posed in that article.

Theorem. *Let $\mathcal{O}(N)$ be the open cone on N, with the metric*

$$dt^2 + \phi(t)^2 g_{ij} dx^i dx^j$$

where $\phi(t)$ is a smooth function with $\phi(t) = 1$ for $t < -1$ and $\phi(t) = t$ for $t > 1$. Then

$$L_*^{bg}(\mathcal{O}(N)) = H_*^{uff}(\mathcal{O}(N); L_*)$$

Using this theorem we give a proof of the following theorem:

Theorem 5.1. *Let $h; N \to N'$ be a homeomorphism, then the total Pontrjagin classes p, p' of M and M' respectively satisfy $p = h^*(p')$.*

Proof. Let M and M' be open cones on N and N' as above. Then the proof of [PRW] shows that M and M' are boundedly controlled homotopy equivalent. There is a map

$$H_*^{uff}(\mathcal{O}(N); L_*) \to H_*^{lf}(\mathcal{O}(N); L_*)$$

which is surjective.

We have maps

$$K_*(C^*(X)) \xleftarrow{\alpha} L_*^{bg}(X) \xrightarrow{\beta} L_*^{bdd}(X)$$

Any bg controlled module A can be thought of as a projection in $C^*(X)$ defined by the generating set of A and hence as a projective $C^*(X)$-module. An endomorphism of A respects the $C^*(X)$-module structure. This defines α. β is the forgetful map which associates a boundedly controlled module to every bg controlled module. The proof then goes as in [PRW] by showing that $Sign_X(M) = \alpha\beta^{-1}\sigma(M)$.

Theorem 5.2.

$$Wh^{bg}(\mathcal{O}(N)) = 0$$

Proof. The calculation is analogous to the one for hyperbolic n-space above. We introduce the filtration of the bg Whitehead group as follows. Let $p : \mathcal{O}(N) \to \mathcal{O}(S^n) = \mathbf{R}^{n+1}$ be the map induced by $N \to S^n$. Let $q : X \to \mathcal{O}(N)$ be a control map. Then q is said to be $bg(k)$ if the restriction of q to a neighborhood $D^r \times p^{-1}\mathbf{R}^k$ in the decomposition of $\mathbf{R}^{n+1}$ as $\mathbf{R}^k \times \mathbf{R}^{n-k+1}$, $\mathbf{R} \times \mathbf{R}^k \times \mathbf{R}^{n-k}$, ...,$\mathbf{R}^{n-k+1} \times \mathbf{R}^k$ is bg. There are forgetful maps

$$Wh^{bg}(\mathcal{O}(N)) \to Wh^{bg(1)}(\mathcal{O}(N)) \to ... \to Wh^{bdd}(\mathcal{O}(N))$$

A $bg(1)$-controlled h-cobordism is split along copies of N if and only if it is in the kernel of the map to $Wh^{bdd}(\mathcal{O}(N))$. The kernel of the map

$$Wh^{bg(2)}(\mathcal{O}(N)) \to Wh^{bg(1)}(\mathcal{O}(N))$$

consists of elements split simultaneously along hyperplanes and copies of N. Continuing this way, we see that the kernel of

$$Wh^{bg}(\mathcal{O}(N)) \to Wh^{bg(n)}(\mathcal{O}(N))$$

is represented by elements of $C_0^{uff}(\mathcal{O}(N); Wh(pt.))$. Similarly, using the Eilenberg-Zilber theorem for warped products above, we see that the kernel of

$$Wh^{bg(k)}(\mathcal{O}(N); Wh(pt.)) \to Wh^{bg(k-1)}(\mathcal{O}(N); Wh(pt.))$$

is $C_{n-k}^{uff}(\mathcal{O}(N); Wh(pt.))$. Taking infinite process tricks into account we obtain:

$$Wh^{bg}(\mathcal{O}(N)) = H_*^{uff}(\mathcal{O}(N); Wh(pt.)) = 0$$

Proof of Theorem 5.1. We use a splitting theorem which is identical to Theorem 3.2. We split along copies of N going out radially, and then split transversely to N. Thus we can split a bg homotopy equivalence $f : X \to \mathcal{O}(N)$ along a triangulation of $\mathcal{O}(N)$. Thus $\mathcal{S}^{bg}(\mathcal{O}(N)) = 0$. Using the surgery exact sequence , we obtain the result.

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