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Note on the Permanent Rank of a Matrix [*], [†]

by

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ABSTRACT

Define the perrank of a matrix A to be the size of a largest square submatrix of A with nonzero permanent. Motivated in part by the Alon-Jaeger-Tarsi Conjecture [3], we prove several results on perranks.

1 Introduction

Recall that the permanent of an $n \times n$ matrix is defined like the determinant, but with all $n!$ terms having positive signs (see [9]). Define the *perrank* of a matrix A to be the size of a largest square submatrix of A with nonzero permanent. Throughout this paper F denotes a field, p denotes char(F) and Z_p denotes the finite prime field of p elements. All matrices have entries in F. We assume $p \neq 2$ throughout. Of course when $p = 2$ permanent and determinant coincide.

Lemma 1.1 *Suppose A is an $n \times m$ matrix with $perrank(A) = m < n$. Let $(A \;\; x)$ be the $n \times (m+1)$ matrix formed by A following with the column vector x. Then $dim\{x : perrank(A \;\; x) = m\} \leq m$.*

Proof. It is clear that $L := \{x : \mathrm{perrank}(A \;\; x) = m\}$ is a subspace. We may assume that the submatrix of A formed by its first m rows has nonzero permanent. Then since for every $x \in L$ the perrank of $(A \;\; x)$ is not full, the projection of L to the first m coordinates is one to one. Hence $\dim(L) \leq m$. ∎

In this lemma if we replace perrank by rank, then $\dim(L) = m$; so by induction on m we have

Corollary 1.2 *Let $m \leq n$ and A be an $n \times m$ random matrix with each entry chosen independently and uniformly from a finite field F. Then $Pr(perrank(A) = m) \geq Pr(rank(A) = m)$.*

In particular, if $m = n$ we have $\Pr(\mathrm{per}(A){=}0) \leq \Pr(\det(A){=}0)$. As $n \to \infty$, $\lim(\Pr(\det(A) = 0))$ is well-known (see e.g. [5]). But as far as we know, $\lim(\Pr(\mathrm{per}(A) = 0))$ is not known.

Theorem 1.3 *For any matrix A, $perrank(A) \geq rank(A)/2$.*

Proof. We may assume A is an $n \times n$ nonsingular matrix and let $m = \mathrm{perrank}(A)$. There exists an $n \times m$ submatrix B with full perrank. Lemma 1.1 gives $n - m \leq \dim\{x : \mathrm{perrank}(B \;\; x) = m\} \leq m$. ∎

Theorem 1.3 is tight as shown by $\begin{pmatrix} I & I \\ I & -I \end{pmatrix}$. This is essentially the only tight example we know at this point.

On the other hand, for $p > 0$ perrank is also bounded above in terms of rank: it is easy to see that $\mathrm{perrank}(A) \leq (p-1)\,\mathrm{rank}(A)$. The key is, if a square matrix has a column (or row) repeating p times, then its permanent is 0. This simple and known observation also gives

Lemma 1.4 *Suppose A is a matrix with a column repeating $p-1$ times, $p > 0$. Then adding a multiple of this column to any other column doesn't change $perrank(A)$.*

This lemma suggests that we can use elementary operations to study the permanent. (This idea has also been discovered independently by Kogan and Makowsky; see section 3.5 of [8].) The following result (true for $p = 3$ only) is an application of this idea.

Theorem 1.5 *If $p = 3$, then $perrank(A) = perrank(A^{-1})$ for any nonsingular A.*

Proof. Let $T_1, T_2, T_3, \cdots$ be a sequence of elementary column operations that change $A_{n\times n}$ to I. Consider the sequence of operations $T_1, T_1', T_2, T_2', T_3, T_3', \cdots$ on $\begin{pmatrix} I & I \\ A & A \end{pmatrix}$, where T_i is applied to the left n columns, and T_i' is the corresponding operation applied to the right n columns. We do operations in pairs so that every column in each intermediate matrix is repeated.

By Lemma 1.4, $\text{perrank}\begin{pmatrix} I & I \\ A & A \end{pmatrix} = \text{perrank}\begin{pmatrix} A^{-1} & A^{-1} \\ I & I \end{pmatrix}$. Set $C = \begin{pmatrix} I & I \\ A & A \end{pmatrix}$. It suffices to show $\text{perrank}(C) = n + \text{perrank}(A)$.

On the one hand, if B is an $r \times r$ submatrix of A with nonzero permanent, then the submatrix of C generated (in the obvious sense) by the left upper I and the right lower B has nonzero permanent $2^r \text{per}(B)$. So $\text{perrank}(C) \geq n + \text{perrank}(A)$.

On the other hand let D be a square submatrix of C with nonzero permanent. Suppose D has $r + s$ rows from $(I \;\; I)$ with r of them having two ones, s of them having a single one; and D has t rows from $(A \;\; A)$. The general form of D is:

$$D = \begin{pmatrix} I_r & I_r & 0 & 0 \\ 0 & 0 & I_s & 0 \\ U_{t\times r} & U_{t\times r} & V_{t\times s} & W_{t\times(t-r)} \end{pmatrix}$$

Here U, V are submatrices of A and W is a submatrix of $(A \;\; A)$. Write $W = (W_1 \;\; W_2)$ where W_1, W_2 are submatrices of A and W_1 has at least $(t-r)/2$ columns. It is clear that $(U \;\; V \;\; W_1)$ is a submatrix of A, so in particular $n \geq r+s+(t-r)/2$, or equivalently $(t+r)/2 \geq r+s+t-n$. Since $\text{per}(U \;\; W) = 2^{-r}\text{per}(D) \neq 0$, we have $\text{perrank}(U \;\; W_1) \geq (t+r)/2 \geq r+s+t-n$. So $\text{perrank}(C) \leq n + \text{perrank}(A)$. ∎

2 The Alon-Jaeger-Tarsi Conjecture

The Alon-Jaeger-Tarsi Conjecture, first proposed in 1981, is

Conjecture 2.1 *For any field F with $|F| \geq 4$ and any nonsingular matrix A over F, there is a vector x such that both x and Ax have only nonzero entries.*

This was proved by Alon-Tarsi [3] for every non-prime finite field. The case $|F| = \infty$ is trivial, but the case $F = Z_p$ $(p \geq 5)$ remains open. See [4] for a matroid approach. Motivated by Conjecture 2.1, Jeff Kahn (personal communication) proposed:

Conjecture 2.2 *For any $n\times n$ nonsingular matrix A over any field, $perrank(A \;\; A) = n$.*

Conjecture 2.2 implies Conjecture 2.1 via the polynomial argument of [3]. It also immediately implies Theorem 1.3. Actually Theorem 1.3 was motivated by Conjecture 2.2. Conjecture 2.2 is true when $p = 3$ (see [3]) and we also verified it for $n \leq 4$.

Theorem 2.3 *For any $n \times n$ nonsingular matrices $A_1, A_2, \cdots, A_l$,*

$$perrank(A_1 \;\; A_2 \;\; \cdots \;\; A_l) \geq (1 - 2^{-l})n$$

Proof. Theorem 1.3 gives $\text{perrank}(A_1) \geq n/2$, and Lemma 1.1 implies $\text{perrank}(A_1 \;\; A_2 \;\; \cdots \;\; A_{k+1}) \geq (\text{perrank}(A_1 \;\; A_2 \;\; \cdots \;\; A_k) + n)/2$. The theorem follows by induction on l. ∎

This theorem implies a small result on Conjecture 2.1. Choose $l = p - 2$ and $A_1 = A_2 = \cdots = A_l = A$. Then the argument of [3] gives

Proposition 2.4 *For any $n \times n$ nonsingular matrix A over Z_p, there is a vector x without zero entries such that Ax has at most $n/2^{p-2}$ zero entries. In particular, Conjecture 2.1 is true if $n < 2^{p-2}$.*

We don't know examples of equality in Theorem 2.3 if $l \geq 2$ and we believe the theorem is not tight. In fact, we don't know any example with $\text{perrank}(A_1 \;\; A_2) < n$. Motivated by the additive basis problem, the authors of [2] conjectured that $\text{perrank}(A \;\; B) = n$ for any nonsingular $n \times n$ matrices A, B over Z_3. (*Remarks*: this statement can be shown by Lemma 1.4 to be equivalent to Conjecture 4.3 (in the case $p = 3$) of [2]. The additive basis problem was motivated by the nowhere-zero flow problem, see section 3.3 of [6] for more details.)

Noga Alon (personal communication) has shown that for A chosen uniformly from the nonsingular $n \times n$ matrices over F, Conjecture 2.1 is true almost surely as $n \to \infty$. We will show that Conjecture 2.2 is true almost surely in the same sense.

Lemma 2.5 *For an $m \times n$ matrix A with $perrank(A \;\; A) = m < 2n$,*

$$dim(L := \{x : perrank \begin{pmatrix} x & x \\ A & A \end{pmatrix} = m\}) \leq \frac{m}{2}$$

(where x runs over row vectors of length n).

Proof. We regard the columns of A as distinct even though A may have identical columns, so that we can label the column set of A by $[n]$. Then the column set of $(A \;\; A)$ is naturally the multiset $W := [n] \cup [n]$. For any $U \subseteq W$, define $|U|$ to be the cardinality of U, $\|U\|$ to be the number of repeated elements in U and $A(U)$ to be the $m \times |U|$ submatrix of $(A \;\; A)$ consisting of the columns in U. For example, $|W| = 2n$, $\|W\| = n$, $A([n]) = A$.

Choose $V \in \{U : U \subseteq W, \; |U| = m, \; \text{per}(A(U)) \neq 0\}$ such that $\|V\|$ is maximum. We may assume that $V = [r] \cup [s]$ (multiset union), where $r = \|V\|$, $0 \leq r \leq s$, $r + s = m$.

It suffices to show that the projection of L to the first r coordinates is one to one, since $r \leq m/2$. So assume $x \in L$ and $x_1 = x_2 = \cdots = x_r = 0$. It is in fact enough to show that

$x_{r+1} = x_{r+2} = \cdots = x_s = 0$, since $x = 0$ is then immediate (this is as in the proof of Lemma 1.1).

Assume $r < s$ (otherwise we are done). Let $V' = V \cup \{r+1\}$. Since

$$\|V' \setminus \{r+2\}\| = \|V' \setminus \{r+3\}\| = \cdots = \|V' \setminus \{s\}\| = r+1 > \|V\|,$$

we have $\mathrm{per}(A(V' \setminus \{r+2\})) = \cdots = \mathrm{per}(A(V' \setminus \{s\})) = 0$ by our choice of V.

Since $x \in L$, the $(m+1) \times (m+1)$ submatrix of $\begin{pmatrix} x & x \\ A & A \end{pmatrix}$ consisting of the columns in V' has zero permanent. This means exactly that $x_{r+1} = 0$ (using $p \neq 2$). Similarly $x_{r+2} = \cdots = x_s = 0$. ∎

The next result gives Conjecture 2.2 almost surely for a uniformly chosen random nonsingular matrix, since $\lim(\Pr(\det(A) \neq 0)) > 0$ (again see [5]).

Corollary 2.6 *Suppose A is an $n \times n$ random matrix with each entry chosen independently and uniformly from a finite field F, then $Pr(perrank(A\ \ A) = n) \to 1$ as $n \to \infty$.*

Proof. Let A_i be an $i \times n$ random matrix and $q = |F|$. It is clear that $\Pr(\mathrm{perrank}(A_1\ \ A_1) = 1) = (1 - q^{-n})$. By Lemma 2.5 we have $\Pr(\mathrm{perrank}(A_2\ \ A_2) = 2) \geq (1 - q^{-n})^2$. By induction (and again Lemma 2.5) we have for $i \leq n$,

$$\Pr(\mathrm{perrank}(A_{2i}\ \ A_{2i}) = 2i) \geq (1 - q^{-n})^2 (1 - q^{-n+1})^2 \cdots (1 - q^{-n+i-1})^2$$

Applying this with $i = [n/2] + 1$. yields the corollary. ∎

We mention that Lemma 2.5 can be generalized. Suppose $(A\ \ A\ \cdots\ A)$, with A repeated k times, has full perrank $m < kn$, and $p > k$ or $p = 0$, then with L defined in analogy with Lemma 2.5, $\dim(L) \leq m/k$. We can use this and Lemma 1.4 to give an alternative proof of (essentially) the bound in Lemma 3.6 of [2].

Finally we have a strange result for fields of characteristic 0.

Lemma 2.7 *Suppose the i–th row of $A_{m \times n}$ has at least $2i - 1$ nonzero elements and $p = 0$, then $perrank(A) = m$.*

Proof. We proceed by induction, the lemma being trivial when $m = 1$. We label the column set of A by $[n]$ and let B be the submatrix of A formed by its first $m-1$ rows. Any $(m-1)$-set U of $[n]$ corresponds to an $(m-1) \times (m-1)$ submatrix of B and we use $\mathrm{per}(U)$ to denote its permanent. Since dividing a column by a nonzero number or permuting columns doesn't affect the problem, we may assume that the last row of A is $(1, 1, \cdots, 1, 0, 0, \cdots, 0)$ with k ones.

Now suppose $\mathrm{perrank}(A) < m$. For any fixed $(m-1)$-set $U \subseteq [n]$, let $R = U \cap [k]$, $r = |R|$ and $V = U \setminus R$. For any $(r+1)$-set $T \subseteq [k]$, the $m \times m$ submatrix of A corresponding to $T \cup V$ has zero permanent, that is

$$\sum_{S \subseteq T,\ |S| = r} \mathrm{per}(S \cup V) = 0$$

We have an unknown per$(S \cup V)$ for each r-set $S \subseteq [k]$ and we have an equation for each $(r+1)$-set $T \subseteq [k]$. The coefficient matrix of this linear system is exactly the inclusion matrix $M(k, r, r+1)$. Kantor [7] showed that inclusion matrix always has full rank over field of characteristic 0. Since $k \geq 2m-1 > 2r$, the number of equations is at least the number of unknowns, so all unknowns are zero. In particular, per(U)=0, that is, perrank$(B) < m-1$, a contradiction. ∎

This lemma gives the first part of the following theorem, and the second part then follows via the argument of [3] and Theorem 1.2 of [1].

Theorem 2.8 *Suppose the i–th row of* $A_{n\times n}$ *has at least* i *nonzero elements and* $p = 0$, *then* (1) *perrank* $(A \ \ A) = n$;

(2) *for any* $S_1, S_2, \cdots, S_n \subseteq F$ *with* $|S_i| = 3$, *there are* $x_i \in S_i$ *such that* Ax *has no zero entries.*

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