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CROSSING PROPERTIES OF GRAPH RELIABILITY FUNCTIONS

by

Alexander K. Kelmans
Rutgers University, New Brunswick, New Jersey
University of Puerto Rico, San Juan, Puerto Rico

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ABSTRACT

We consider undirected graphs, and assume that each edge of G exists with probability $p \in (0,1)$. The *all–terminal reliability function* $A(G,p)$ of such a random graph G is the probability that the spanning subgraph formed by the existing edges is connected. A *two–graph* is a graph with two distinguished vertices called *terminals*. The *two–terminal reliability function* $T(G,p)$ of a two–graph G is the probability that the subgraph of G induced by the existing edges contains a path connecting the terminals. Till recently no examples of pairs of graphs have been known whose all–terminal or two–terminal reliabilities cross more than twice and only one example of exactly two crossings. In [9] we proved that for every integer N there exist pairs of graphs (two–graphs) whose all–terminal (respectively, two–terminal) reliabilities cross more than N times. In this paper we give simple constructions that provide, for every sequence $(m_1, \ldots, m_k)$ of natural numbers, infinitely many pairs of graphs (two–graphs) (G_1, G_2) of the same size such that $A(G_1,p) - A(G_2,p)$ (respectively, $(T(G_1,p - T(G_2,p))$ have exactly k roots $x_1 < \ldots < x_k$ in $(0,1)$ such that x_i is of multiplicity m_i, $i = 1, \ldots, k$. These results are based on so called A– and T–*multiplying constructions* that are interesting in themselves.

Contents

1 Introduction

One of possible models of a non–reliable network is an undirected graph with randomly deleted elements. We assume that the nodes of the network are reliable but links are not. Thus our model is an undirected finite graph G with possible multiple edges and loops, and we assume that each edge of G exists independently (or is independently operational) with probability p (and so p is the *reliability* of links in the network). There are many different notions of reliability for networks with non-reliable links. One of the natural notions is the probability that G is connected. This notion is sometimes called *all–terminal reliability* of G. Another natural alternative is to specify two distinguished vertices (or *terminals*) x and y of G. As a result we obtain a *two–terminal graph* or simply a *2–graph*. In this case it is natural to ask what is the probability that the terminals x and y are connected in G. This is so called *two–terminal reliability* of G. Clearly all–terminal and two–terminal reliabilities depend on the reliability p of edges. For that reason, we will call them *all–terminal* and *two–terminal reliability functions.*

Many papers are devoted to the study of various properties of network reliability functions (see, for example, [3]).

For instance, Moore and Shannon [13] have studied properties of the two–terminal reliability function in order to find a construction that for every $t \in (0,1)$ provides a series of two–terminal graphs whose two–terminal reliability functions tend to a t–threshold function on $(0,1)$ ($f(x)$ is a *t-threshold function* if $f(x) = 1$ for $x > t$, and $f(x) = 0$ for $x < t$). The existence of a *t–threshold series of 2–terminal graphs* shows that by using many sufficiently non–reliable relays it is possible to build a circuit which functions as a sufficiently reliable relay.

In [4, 5] we have described a construction that for every $t \in (0,1)$ provides a so called *t–threshold series of asymptotically optimal graphs.* Such a series has the property that all–terminal reliability functions of its graphs tend to a t–threshold function, and, moreover, this series is a best possible t–threshold series in the sense that the graphs of this series have the asymptotically minimum number of edges (equal to $(n \log n)/2|\log(1-t)|$). As a byproduct, this construction and results provide an asymptotically optimal solution to the problem of finding a graph minimizing the number of edges among the graphs with a given number of vertices whose all–terminal reliability is at least a given function. It also yields a construction of a sufficiently reliable switching system from many non–reliable switching devices (relays), and this construction uses the asymptotically minimum number of such non–reliable devices.

Margulis [12] investigated the asymptotic behavior of all–terminal reliability functions of graphs $\{G_i : i = 1, \ldots\}$ as i tends to ∞ under condition that the edge connectivity $c(G_i)$ tends to ∞. He has shown that although the all–terminal reliability functions of G_i may not have limit, the length $l_\epsilon(G_i)$ of the interval of edge reliabilities for which the all–terminal reliability of G_i is in the interval $(\epsilon, 1-\epsilon)$ has a limit and tends to 0 for every $\epsilon \in (0,1)$ as i tends to ∞.

In this paper we are going to discuss and to study *crossing properties* of reliability functions. When comparing the reliability of two graphs of the same size, a natural question arises as to whether the result of the comparison (which one is better) depends on the edge reliability p or not. In other words, the question is whether the reliability functions of different graphs of the same size can cross. In [4] we observed that there are graphs with the same number of vertices and edges whose all–terminal reliability polynomials cross (i.e. the difference of the polynomials has a root in (0,1)). The same is true for two–terminal reliabilities of graphs.

In fact, we proved a much stronger result concerning so called optimal graphs [7, 8]. (A graph is *optimal for p* if it has the maximum all–terminal reliability among all graphs with the same number of vertices and the same number of p–reliable edges). We gave a construction [7, 8] that provided for every $n \geq 6$ a pair of graphs with n vertices and the same number, say $m(n)$, of edges such that one of them is optimal for $p's$ close to 0 and not optimal for $p's$ close to 1, while the other graph is optimal for $p's$ close to 1 and not optimal for $p's$ close to 0. In particular this means, that for every $n \geq 6$ there does not exist a so called *uniformly optimal graph* with n vertices and $m(n)$ edges (a graph is *uniformly optimal* if it is optimal for every $p \in [0,1]$).

In all these results reliability functions cross only once in $(0,1)$. Till recently no examples of pairs of graphs have been known whose all–terminal or two–terminal reliabilities cross more than twice and only one example (up to some trivial operations) of exactly two crossings. This example was found by the aid of computer [2]. (In **7.3** below we give a very simple construction that provides infinitely many pairs of graphs with exactly two crossings.)

A natural question is how many crossings can occur among all–terminal (respectively, two–terminal) reliability functions for graphs with the same number of vertices and edges. Equally natural questions are $(q1)$ whether there exists a pair of graphs of the same size whose reliability polynomials have a touching point (i.e. the difference of the polynomials has a root in $(0,1)$ of an even multiplicity), $(q2)$ how many touching points such polynomials may have, $(q3)$ how large the multiplicity of a touching point (if any) may be, etc.

In [9] we proved that for all integers M and N there exist pairs of graphs whose all–terminal reliabilities cross (touch) more than N times with the multiplicities greater than M. The same was shown to be true also for two–terminal reliability.

In this paper we give simple constructions that provide, for every sequence $(m_1, \ldots, m_k)$ of natural numbers, infinitely many pairs of two–graphs (G_1, G_2) of the same size such that $A(G_1, p) - A(G_2, p)$ (respectively, $(T(G_1, p - T(G_2, p))$ have exactly k roots $x_1 < \ldots < x_k$ in $(0, 1)$ such that x_i is of multiplicity m_i, $i = 1, \ldots, k$. These results are based on so called *A– and T–multiplying constructions* that are interesting in themselves. Our construction does not use any results found by computer.

The results of this paper were reported at the 25th Southeastern International Conference, Florida, March 94 and at the 7th International Conference on Random Structures and Algorithms, Atlanta, May, 1995, and were preliminary described in [10, 11].

2 Notions, notation, and preliminaries

We consider finite and undirected graphs with possible parallel edges and no loops [1]. Let $V(G)$ and $E(G)$ be the sets of vertices and edges of a graph G, respectively.

A *random graph* (G, p) is a graph G where every edge exists independently with probability p (and so $q = 1 - p$ can be interpreted as the *edge failure probability*).

The *all–terminal reliability* of such random graph (G, p), written $A(G, p) = A[G])$, is the probability that the subgraph of G induced by the existing edges is connected. It is easy to see that

$$A(G,p) \quad = \quad \sum_{i=n-1}^{m} A_i(G) p^i q^{m-i}, \tag{2.1}$$

where $m = |E(G)|$ is the number of edges of G, $A_i = A_i(G)$ is the number of spanning connected subgraphs of G with i edges, and in particular, $A_{n-1}(G)$ is the number of spanning trees of G. Let $c = c(G)$ be the edge connectivity of G. Then clearly $A_i = \binom{m}{i}$ for any $i \in \{m - c + 1, \ldots, m\}$, and G is disconnected if and only if $A(G, p) \equiv 0$.

A *two–terminal graph* (or simply a *2–graph*) xGy is a graph G with a pair x, y of specified vertices, called the *terminals* of G.

The *two–terminal reliability* of xGy, written $T(xGy, p) = T(G, p) = T[G])$, is the probability that the subgraph of G induced by the existing edges contains a path connecting the terminals x and y.

It is easy to see that

$$T(xGy,p) = \sum_{i=l}^{m} T_i(xGy) p^i q^{m-i}, \tag{2.2}$$

where $m = |E(G)|$ is the number of edges of G, $T_i = T_i(xGy)$ is the number of subgraphs of G with i edges that contain a path between x and y, and $l = l(xGy)$ is the distance between x and y in G (i.e. the number of edges of a shortest path in G connecting x and y). Clearly T_l is the number of shortest paths between x and y in G. An edge subset Z of G is called an *(x,y)-cut* of G if the vertices x and y are in different components of $G \setminus Z$. Let $c = c(xGy)$ be the size of a minimum (x, y)–cut of

xGy. Clearly $T_i = \binom{m}{i}$ for every $i \in \{m-c+1, \ldots, m\}$ and x, y belong to different components in G if and only if $T(G,p) \equiv 0$.

Obviously $A(F,p)$ and $T(F,p)$ are polynomial functions of p.

Given a graph H, let $s(H) = (|V(H)|, |E(H)|)$. Given a 2–graph xHy, let $s(xHy) = s(H) = (|V(H)|, |E(H)|, l(xHy))$. We call $s(H)$ the *size of a graph or 2–graph* H, respectively.

Clearly if G is connected then $n-1 = |V(F)|-1$ is the minimum non–negative integer s such that $A_s(G) \neq 0$, and $A_{n-1}(G)$ determines the behavior of $A(G,p)$ for p close to 0. Similarly if G has an (x,y)–path then $l = l(xGy)$ is the minimum positive integer s such that $T_s(xGy) \neq 0$, and $T_l(xGy)$ determines the behavior of $T(xGy,p)$ for p close to 0. Notice that the parameters $c(G)$ and $c(xGy)$ play similar role when p is close to 1 for $A(p,G)$ and $T(xGy)$, respectively. For that reason when discussing crossing properties of these polynomials we will consider pairs (G_1,G_2) of graphs or 2–graph of the same size.

A pair $H = (H_1,H_2)$ of graphs (2–graphs) is called *equisize* if H_1 and H_2 are of the same size: $s(H_1) = s(H_2)$.

A graph is *simple* if it does not have loops or parallel edges.

A connected graph G is *2–connected* if $|V(G)| \geq 3$ and $G-v$ is connected for every vertex $v \in V(G)$. A 2–graph G is *2–connected* if the graph obtained from G by adding a new edge between the terminals of G is 2–connected.

Let $H = (H_1,H_2)$ be an ordered pair of graphs (2–graphs). Let $L \in \{A,T\}$. Put

$$\Delta_L((H_1,H_2),p) = \Delta_L(H,p) = \Delta_L[H] = L(H_1,p) - L(H_2,p),$$

We call $\Delta_L[H]$ the *L–difference* of the pair H. Clearly $\Delta_L(H,p)$ is a polynomial functions of p.

Let ϕ be a polynomial and $\{x_1 < \ldots x_k\}$ be the set of all roots of ϕ in $(0,1)$ and x_i is of multiplicity m_i, $i = 1,\ldots,k\}$. Put $Prf(\phi) = (m_1,\ldots,m_k)$.

We call a root $r \in (0,1)$ of $\Delta_L[H])$ an *L–crossing point of* $H = (H_1,H_2)$ and $Prf(\Delta_L[H])$ the *L–crossing profile* of H.

3 Preliminary discussion

3.1 The main idea that allows to find a *T–construction*, i.e. a construction providing pairs of 2–graphs of any given crossing profile of 2–terminal reliability functions is the following.

Let $\mathcal{T}$ be the set of ordered pairs of 2–graphs. Let $M : \mathcal{T}^2 \to \mathcal{T}$ be an operation that gives a pair $M(X,Y)$ of 2–graphs for every two pairs of 2–graphs X and Y.

An operation M is called a *T–multiplying operation* if

$$Prf(\Delta_L[M(G,F)]) = Prf(\Delta_T[G] \cdot \Delta_T[F])$$

for every two pairs G and F of 2–graphs.

Suppose that M is a T–multiplying operation. Let $\mathcal{F}$ be an infinite set of ordered pairs of 2–graphs having the following properties:

$(f1)$ every pair F in $\mathcal{F}$ has exactly one T–crossing point $r(F)$ and this crossing point is of multiplicity one and

$(f2)$ if $X, Y \in \mathcal{F}$ and $X \neq Y$ then $r(X) \neq r(Y)$.

Suppose we want to find a pair G of 2–graphs having the T–crossing profile $P = (m_1, \dots m_k)$.

Let $X_i \in \mathcal{F}$, $i = 1, \dots, k$, and x_i be the T–crossing point of X_i, where $0 < x_1 < \dots < x_k < 1$. Now for each $i \in \{1, \dots, k\}$, M–multiply each X_i exactly m_i times to obtain a pair $X_i^{m_i}$ and after that M–multiply all $X_i^{m_i}$, $i = 1, \dots, k$, to obtain a pair Ω. Since M is a T–multiplying operation and $\mathcal{F}$ is an infinite set of pairs of 2–graphs with properties $(f1)$ and $(f2)$, clearly the T–crossing profile of Ω is precisely P.

Thus in order to find a T–construction, it is sufficient to find a T–multiplying operation M and an infinite set $\mathcal{F}$ with properties $(f1)$, $(f2)$.

It is not difficult to find an infinite set $\mathcal{F}$ with properties $(f1)$, $(f2)$ (see Section 6). The main problem is to find a T–multiplying operation M.

It turns out that such operation for the 2–terminal reliability function does exist (see Section 5). The construction we found is based on series and parallel operations on 2–graphs (see Section 4). One of the reasons why we succeed in finding a T–multiplying operation is the fact that the 2–terminal reliability is closed under series and parallel operations. In other words, we can find the reliabilities for 2–graphs $X \circ Y$ and $X \parallel Y$ if the only available information is the 2–terminal reliabilities for 2–graphs X and Y.

3.2 The situation with the all–terminal reliability is more complicated than that for the two–terminal reliability because the all–terminal reliability is not closed under parallel operation. In other words, in order to find the all–terminal reliability of $X \parallel Y$, it is not enough to know the all–terminal reliabilities of X and Y, and some additional information on 2–graphs X and Y is needed. This additional information is the probability $C(xFy, p)$ that the spanning subgraph of 2–graph xFy formed by the existing edges consists of two components, each containing exactly one terminal.

The vector–function $(A(F,p), C(F,p))$ of a 2–graph F turns out to be closed under both series and parallel operations (see **4.2** below). This fact, interesting in itself, is very useful for our purposes.

If we want to use the "2–terminal" strategy described in Subsection 3.1 for the all–terminal case we should find a multiplying operation for the A–difference

$$\Delta_A((H_1, H_2), p) = A(H_1, p) - A(H_2, p).$$

However it is not clear how to find such an operation (based on series and parallel or some other operations) and whether such an operation exists at all.

The key notion we introduce and use in all–terminal case is the *mixed difference* (or *B-difference*)

$$\Delta_B((H_1, H_2), p) = A(H_1, p)C(H_2, p) - A(H_2, p)C(H_1, p).$$

Also we should consider the *C–difference* $\Delta_C((H_1, H_2), p) = C(H_1, p) - C(H_2, p)$. We will see that the B–difference of a pair H of 2–graphs is related with the A–difference

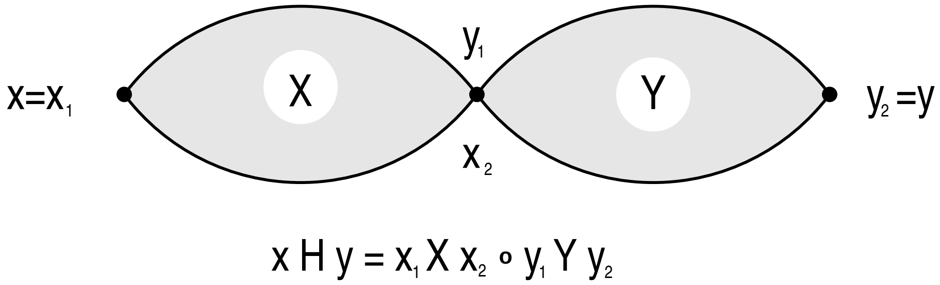


Figure 1

and C–difference of the pairs obtained from H by some operations. Moreover it turns out that the "2–terminal" strategy can be used in the all–terminal case if instead of the all–terminal difference $\Delta_A(H,p)$ (similar to the 2–terminal difference $\Delta_T(H,p)$) we consider a *vector–difference* $(\Delta_A(H,p),\ \Delta_B(H,p),\ \Delta_C(H,p))$.

The corresponding modification of T–construction provides pairs of graphs of any given A–crossing profile (see Section 9). It uses *B–multiplying* and *C–multiplying operations* (see Section 5) and an infinite set of pairs of 2–graphs each having exactly one B–intersecting point of multiplicity one and such that the B–intersecting points of different pairs are different (see Section 7).

4 Some operations on 2–graphs

Let x_1Xx_2 and y_1Yy_2 be disjoint 2–graphs.

A 2–graph xHy is said to be obtained from x_1Xx_2 and y_1Yy_2 by a *series operation*, written

$$xHy = x_1Xx_2 \circ y_1Yy_2,$$

if $x = x_1$, $y = y_2$, and G is obtained from X and Y by identifying x_2 and y_1 (see Figure 1).

A 2–graph xHy is said to be obtained from x_1Xx_2 and y_1Yy_2 by a *parallel operation*, written

$$xHy = x_1Xx_2 \parallel y_1Yy_2,$$

if $x = x_1$, $y = y_2$, and G is obtained from X and Y by identifying x_1 with y_1 and x_2 with y_2 (see Figure 2).

The first simple but important observation is that the 2–terminal reliabilities $T[X\circ Y]$, $T[X \parallel Y]$ of graphs obtained from two graphs X and Y, respectively, by series and parallel operations are functions of the 2–terminal reliabilities $T[X]$ and $T[Y]$ of the 2–graphs X and Y. More precisely, one can easily prove

4.1 *Let X and Y be 2–graphs. Then*

$$T[X \circ Y] = T[X] \cdot T[Y], \tag{4.1}$$

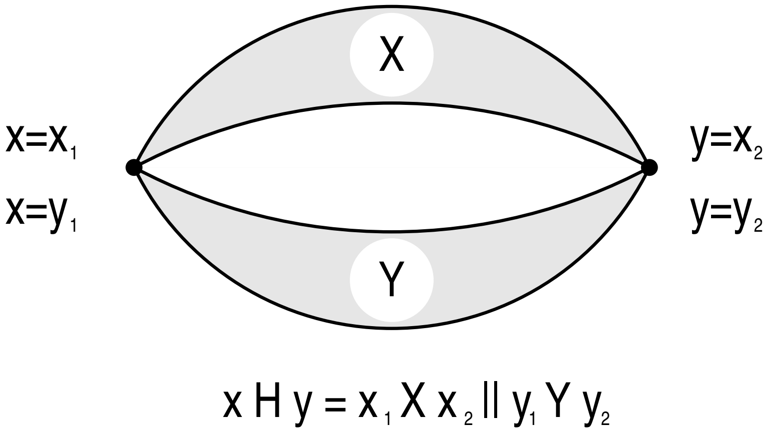


Figure 2

$$T[X \parallel Y] = T[X] + T[Y] - T[X] \cdot T[Y], \tag{4.2}$$

$$T(X(Y), p) = T(X, T(Y, p)) \tag{4.3}$$

The second important observation is that the vector–function $(A(F,p), C(F,p))$ is also closed under series and parallel operations of 2–graphs. More precisely it is easy to see that

4.2 *Let X and Y be 2–graphs. Then*

$$A[X \circ Y] = A[X] \cdot A[Y], \tag{4.4}$$

$$C[X \circ Y] = A[X] \cdot C[Y] + A[Y] \cdot C[X], \tag{4.5}$$

and

$$A[X \parallel Y] = A[X] \cdot A[Y] + A[X] \cdot C[Y] + A[Y] \cdot C[X], \tag{4.6}$$

$$C[X \parallel Y] = C[X] \cdot C[Y]. \tag{4.7}$$

5 Multiplying constructions

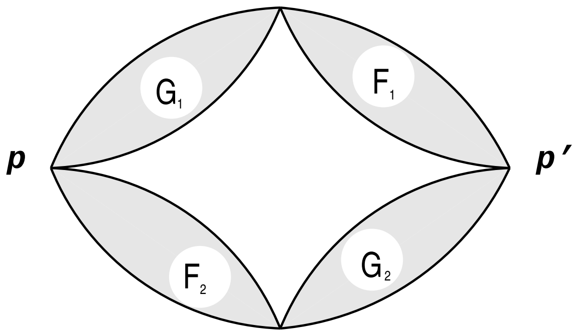
Let as above $\mathcal{T}$ be the set of ordered pairs of 2–graphs. Let $M : \mathcal{T}^2 \to \mathcal{T}$ be an operation that gives a pair $M(X,Y)$ of 2–graphs for every two pairs of 2–graphs X and Y.

Let $L \in \{A, B, C, T\}$. As above, an operation M is called a L*–multiplying operation* if

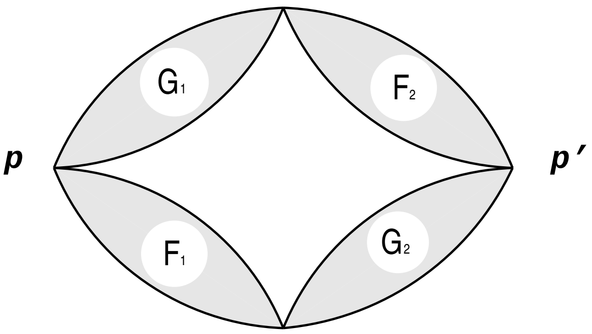
$$Prf(\Delta_L[M(G,F)]) = Prf(\Delta_L[G] \cdot \Delta_L[F])$$

for every two pairs G and F of 2–graphs.

In this section we describe operations M on pairs of 2–graphs that are L–multiplying for $L \in \{B, C, T\}$. These operations form a basis for our constructions providing pairs



$P_1(G, F)$



$P_2(G, F)$

$G = (G_1, G_2)$, $F = (F_1, F_2)$

Figure 3

of 2–graphs of any given A–crossing and T–crossing profiles (as well as B– and C–crossing profiles).

Let $G = (G_1, G_2)$ and $F = (F_1, F_2)$ be two ordered pairs of 2–graphs. We define an operation P that gives from G and F a new pair $P(G, F) = (P_1(G, F), P_2(G, F))$ of 2–graphs as follows (see Figure 3):

$$P_1(G, F) = (G_1 \circ F_1) \parallel (F_2 \circ G_2) \ \ \textit{and} \ \ P_2(G, F) = (G_1 \circ F_2) \parallel (F_1 \circ G_2). \tag{5.1}$$

Similarly we define an operation S that gives from G and F a new pair $S(G, F) = (S_1(G, F), S_2(G, F))$ of 2–graphs as follows (see Figure 4):

$$S_1(G, F) = (G_1 \parallel F_2) \circ (G_2 \parallel F_1) \ \ \textit{and} \ \ S_2(G, F) = (G_1 \parallel F_1) \circ (G_2 \parallel F_2). \tag{5.2}$$

5.1 *Let $G = (G_1, G_2)$ and $F = (F_1, F_2)$ be two ordered pairs of 2–graphs. Then*

$$\Delta_T[P(G, F)] = \Delta_T[S(G, F)] = \Delta_T[G] \cdot \Delta_T[F].$$

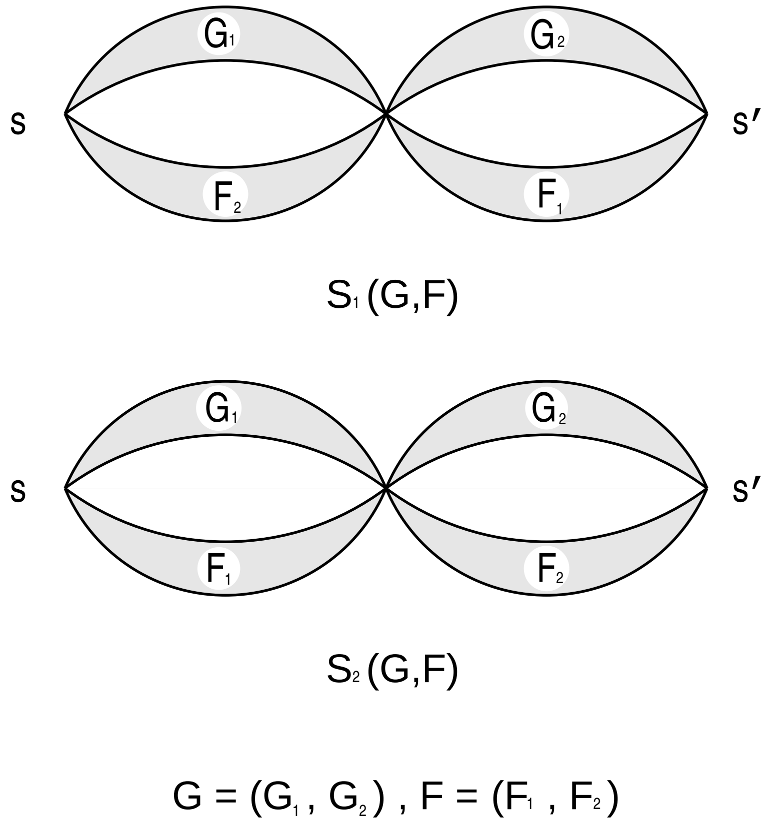


Figure 4

Proof (uses formulas (4.1) and (4.2) from **4.1**).

By (4.2), $T[P_1(G,F)] = T[G_1 \circ F_1] + T[F_2 \circ G_2] - T[G_1 \circ F_1] \cdot T[F_2 \circ G_2]$.

By (4.1), $T[G_1 \circ F_1] = T[G_1] \cdot T[F_1]$ and $T[F_2 \circ G_2] = T[G_2] \cdot T[F_2]$.

Therefore $T[P_1(G,F)] = T[G_1] \cdot T[F_1] + T[G_2] \cdot T[F_2] - T[G_1] \cdot T[F_1] \cdot T[G_2] \cdot T[F_2]$.

Similarly $T[P_2(G,F)] = T[G_1] \cdot T[F_2] + T[G_2] \cdot T[F_1] - T[G_1] \cdot T[F_2] \cdot T[G_2] \cdot T[F_1]$.

Therefore $\Delta_T[P(G,F)] = T[P_1(G,F)] - T[P_2(G,F)] =$

$T[G_1] \cdot T[F_1] + T[G_2] \cdot T[F_2] - T[G_1] \cdot T[F_2] - T[G_2] \cdot T[F_1] =$

$(T[G_1] - T[G_2]) \cdot (T[F_1] - T[F_2]) = \Delta_T[G]\Delta_T[F]$.

The relation $\Delta_T[S(G,F)] = \Delta_T[G] \cdot \Delta_T[F]$ can be proved similarly. □

By using formulas (4.4)–(4.7) in **4.2**, we also can prove (see [11]) that

5.2 *Let $G = (G_1, G_2)$ and $F = (F_1, F_2)$ be two ordered pairs of 2–graphs. Then*

$$\Delta_A[S(G,F)] = -\Delta_C[S(G,F)] = -\Delta_C[P(G,F)] = \Delta_B[G] \cdot \Delta_B[F]. \qquad (5.3)$$

and

$$\Delta_A[P(G,F)] = A[P_1(G,F)] - A[P_2(G,F)] = 0. \tag{5.4}$$

Both P and S are particular cases of the following more general operation.

5.3 Definition of operation H^* (see Figure 5).
Let $G = (G_1, G_2)$ and $F = (F_1, F_2)$ be two pairs of 2-graphs, and $h_1 H h_2$ be an arbitrary 2–graph (possibly, $h_1 = h_2$). Let x_{ij} be the common vertex of G_i and F_j in $G_i \circ F_j$. Let $H_1^*(G,F)$ be the 2–graph obtained from disjoint 2–graphs $P_1(G,F) = (G_1 \circ F_1) \parallel (F_2 \circ G_2)$ and $h_1 H h_2$ by identifying the vertices x_{11} with h_1 and x_{22} with h_2. Similarly let $H_2^*(G,F)$ be the 2–graph obtained from disjoint 2–graphs $P_2(G,F) = (G_1 \circ F_2) \parallel (F_1 \circ G_2)$ and $h_1 H h_2$ by identifying the vertices x_{12} with z_1 and x_{21} with h_2. Put $H^*(G,F) = (H_1^*(G,F), H_2^*(G,F))$.

In particular, if H has two vertices and no edges, then $H^*(G,F) = P(G,F)$, and if H has one vertex and no edges, then $H^*(G,F) = S(G,F)$.

It is easy to see that $T[H_i^*(G,F)] = T[H] \cdot T[S_i(G,F)] + (1 - T[H]) \cdot T[P_i(G,F)]$.

Therefore we have from **5.1**:

5.4 *Let $G = (G_1, G_2)$ and $F = (F_1, F_2)$ be two ordered pairs of 2–graphs, and H be a 2–graph. Then*

$$\Delta_T[H^*(G,F)] = \Delta_T[G] \cdot \Delta_T[F],$$

and so H^* is a T–multiplying operation. (Note that the difference $\Delta_T[H^*(G,F)]$ does not depend on H).

Let $i = 1, 2$ and $L \in \{A, C\}$. It is easy to see that

$$L[H_i^*(G,F)] = A[H] \cdot L[S_i(G,F)] + C[H] \cdot L[P_i(G,F)].$$

Therefore we have from **5.2**:

5.5 *Let $G = (G_1, G_2)$ and $F = (F_1, F_2)$ be two ordered pairs of 2–graphs, and H be a 2–graph. Then*

$$\Delta_A[H^*(G,F)] = A[H] \cdot \Delta_B[G] \cdot \Delta_B[F],$$

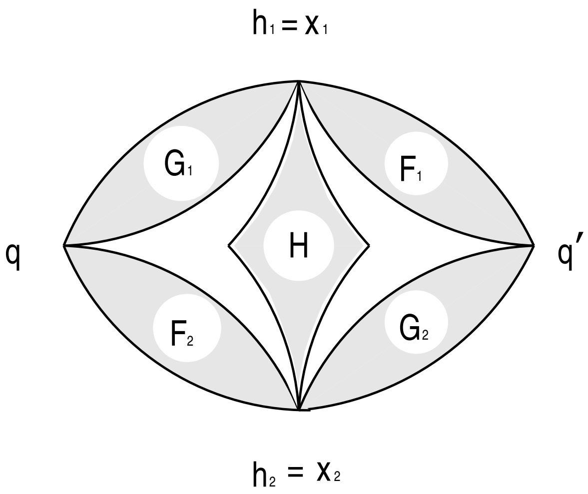
$$\Delta_C[H^*(G,F)] = -(A[H] + C[H]) \cdot \Delta_B[G] \cdot \Delta_B[F].$$

Now it is clear why the mixed difference of a pair of 2–graphs is so important.

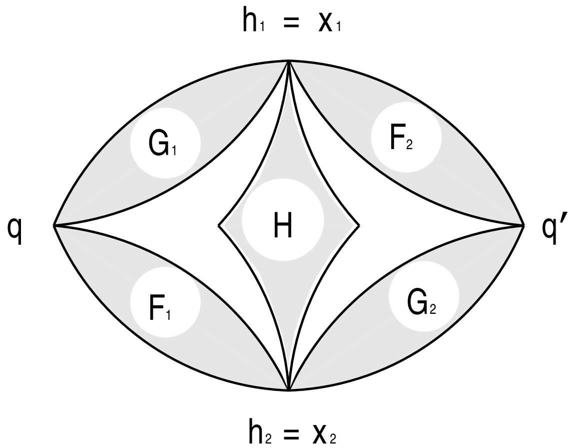
5.6 *Let $G = (G_1, G_2)$ and $F = (F_1, F_2)$ be two ordered pairs of 2–graphs. Suppose that G_i and F_i, $i = 1, 2$, are connected graphs. Then*

$$\Delta_B[P(G,F)] = -A[P_i(G,F)] \cdot \Delta_B[G] \cdot \Delta_B[F] \tag{5.5}$$

for $i = 1, 2$, and therefore P is a B–multiplying operation.



$H_1^*(G, F)$



$H_2^*(G, F)$

$G = (G_1, G_2)$, $F = (F_1, F_2)$

Figure 5

Proof (uses **5.2**).

Let $Z = (Z_1, Z_2)$. From the definition of $\Delta_B[Z]$ it follows that if $A[Z_1] = A[Z_2] = A$ then $\Delta_B[Z] = A \cdot \Delta_C[Z]$. By (5.4) in **5.1**, $A[P_1(G,F)] = A[P_2(G,F)]$. Since G_i and F_i, $i = 1,2$, are connected graphs, $P_j(G,F)$ is also connected graph and therefore $A[P_j(G,F)] \neq 0$. Hence (5.5) follows from (5.3) in **5.2**. □

¿From **5.2** we also have:

5.7 *Let $G = (G_1, G_2)$ and $F = (F_1, F_2)$ be two ordered pairs of 2–graphs, and H be a 2–graph. Suppose that $A[G_1] = A[G_2] = A_g$ and $A[F_1] = A[F_2] = A_f$. Then*

$$\Delta_C[H^*(G,F)] = -(A[H] + C[H]) \cdot A_g \cdot A_f \cdot \Delta_C[G] \cdot \Delta_C[F].$$

Therefore if G_i and F_i, $i = 1,2$, are connected graphs, and H is either connected or has exactly two components each containing a terminal of H and if $A[G_1] = A[G_2]$ and $A[F_1] = A[F_2]$ then H^* is a C–multiplying operation.

It is easy to see that

5.8 *Let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be two pairs of 2–graphs and $(Z_1, Z_2) = H^*(X,Y)$. Then $|V(Z_1)| = |V(Z_2)|$, $|E(Z_1)| = |E(Z_2)|$, and if $l(X_1) = l(X_2)$ and $l(Y_1) = l(Y_2)$ then $l(Z_1) = l(Z_2)$.*

6 Pairs of 2–graphs with exactly one T–crossing point

In this section we describe some constructions providing an infinite set $\mathcal{F}$ with the properties (see Subsection 3.1):

$(f0)$ if $F = (F_1, F_2) \in \mathcal{F}$ then F_1 and F_2 are of the same size, i.e. $s(F_1) = s(F_2)$,

$(f1)$ every pair F in $\mathcal{F}$ has exactly one T–crossing point $r(F)$ and this crossing point is of multiplicity one, and

$(f2)$ if $X, Y \in \mathcal{F}$ and $X \neq Y$ then $r(X) \neq r(Y)$.

A *multitriangle* $V(k_1, k_2, k_3)$ is a 2–graph with exactly three vertices v_1, v_2, and v_3, with the terminals v_1 and v_2 having k_3 parallel edges between the terminals v_1 and v_2, k_2 parallel edges between v_1, v_3, and k_1 parallel edges between v_2, v_3 (see Figure 6).

It is easy to see that

$$T(V(k_1,k_2,k_3),p) = (1 - q^{k_3}) + q^{k_3}(1 - q^{k_1})(1 - q^{k_2}). \tag{6.1}$$

It can be shown (see [10]) that

6.1 *Let $V = V(k,r,s)$ and $V' = V(k',r',s')$. Suppose that $k+r+s = k'+r'+s' = m$, $s < s'$, and $r' + s' < \min\{k+s, r+s\}$. Then the difference $\delta(q) = T(V(k',r',s'),1-q) - T(V(k,r,s),1-q)$ has exactly one root in $(0,1)$, and this root is of multiplicity one.*

In particular we have:

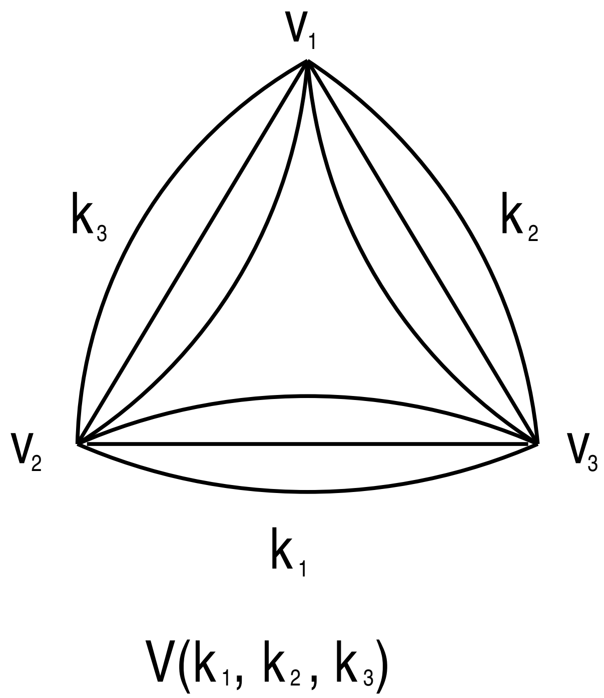


Figure 6

6.2 *Let $V^1_{k,s} = V(k,k,s)$ and $V^2_{k,s} = V(k+1, k-2, s+1)$ where $k \geq 3$ and $s \geq 0$. Then the 2–terminal reliabilities of 2–graphs $V^1_{k,s}$ and $V^2_{k,s}$ have the following properties:*

$(a1)$ $r_{k,s} = r = (5^{1/2} - 1)/2$ is the only root of $\delta_{k,s}(q) = T(V^2_{k,s}, 1-q) - T(V^1_{k,s}, 1-q)$ in $(0,1)$ (and so the root does not depend on k and s) and

$(a2)$ the multiplicity of the root r is one.

Proof Obviously

$T(V^1_{k,s}, 1-q) = (1-q^s) + q^s(1-q^k)^2$ and

$T(V^2_{k,s}, 1-q) = (1-q^{s+1}) + q^{s+1}(1-q^{k+1})(1-q^{k-2})$.

Therefore $\delta_{k,s}(q) = -q^{k+s-1}(1-q)(1-q-q^2)$. So $r = (5^{1/2}-1)/2$ is the only root of $\delta_k(q)$ in $(0,1)$ and this root is of multiplicity one. □

By using **6.2**, it is easy to show [10] that

6.3 *Let $V^1_{k,s}(t)$ and $V^2_{k,s}(t)$ be obtained from $V^1_{k,s}$ and $V^2_{k,s}$ in* **6.2**, *$k \geq 3$, $s \geq 0$, by replacing every edge by $t \geq 1$ parallel edges. Let $r = (5^{1/2} - 1)/2$. Then*

$(a1)$ $r(t) = r^{1/t}$ is the only root of $\delta_{k,s}(q) = T(V^2_{k,s}, 1-q) - T(V^1_{k,s}, 1-q)$ in $(0,1)$ (and so the root does not depend on k and s),

$(a2)$ the multiplicity of the root $r(t)$ is one, and

$(a3)$ $r(t_1) \neq r(t_2)$ if $t_1 \neq t_2$.

The next statement (see proof in [10]) provides another construction for an infinite set $\mathcal{F}$ with properties $(f0)$, $(f1)$, and $(f2)$.

6.4 *Let $W^1_{k,s} = V(2k, 2k, s)$ and $W^2_{k,s} = V(2k+1, k-1, k+s)$ where $k \geq 1$ and $s \geq 0$. Then the 2–terminal reliabilities of 2–graphs $W^1_{k,s}$ and $W^2_{k,s}$ have the following properties:*

$(a1)$ *$\delta_{k,s}(q) = T(W^1_{k,s}, 1-q) - T(W^2_{k,s}, 1-q)$ has exactly one root $r(k,s)$ in $(0,1)$,*

$(a2)$ *$r(k,s) = r(k)$ does not depend on s and $0 < r(k+1) < r(k) < r(5) = (5^{1/2}-1)/2$, and*

$(a3)$ *the multiplicity of the root $r(k)$ is one.*

7 Pairs of graphs with exactly one A–, B– , and C–crossing point

In this section we describe some constructions providing an infinite set $\mathcal{F}$ of 2–graphs each having exactly one A–, B– , and C–crossing point of multiplicity one and such that the corresponding crossing points of different pairs are different (see Subsection 3.2).

As in Section 6, we will use a multitriangle $V(k_1, k_2, k_3)$ (see Figure 6).

7.1 *Let $V_k = (V^1_k, V^2_k)$ where $V^1_k = V(k, k, 0)$, $V^2_k = V(k+1, k-2, 1)$ and $k \geq 3$. Let $r = (5^{1/2}-1)/2$. Then*

(s) *V_k is a pair of 2–graphs of the same size,*

(a) *$\Delta_A[V_k, p]$ has exactly one root a_k in $(0,1)$, this root is of multiplicity one, and $a_k > a_n$ if $k < n$,*

(b) *r is the only root of $\Delta_B(V_k, 1-q)$ in $(0,1)$ (and so the root does not depend on k), and the root of $\Delta_B(V_k, 1-q)$ is of multiplicity one.*

(c) *r is the only root of $\Delta_C(V_k, 1-q)$ in $(0,1)$ (and so the root does not depend on k), and the root of $\Delta_C(V_k, 1-q)$ is of multiplicity one.*

Proof

(psa) Statement (s) is obvious. A proof of (a) and a more general statement can be found in [11].

(pb) Let us prove (b). Put $A(V^i_k, p) = A^i_k(q)$ and $C(V^i_k, p) = C^i_k(q)$ for $i = 1, 2$, and $\Delta_k(q) = \Delta_B(V_k, 1-q) = A^1_k(q) \cdot C^2_k(q) - A^2_k(q) \cdot C^1_k(q)$ where $q = 1-p$. Obviously $A^1_k(q) = (1-q^k)^2$, $A^2_k(q) = (1-q)(1-q^{2k-1}) + q(1-q^{k+1})(1-q^{k-2})$, and $C^1_k(q) = 2q^k(1-q^k)$, $C^2_k(q) = q \cdot [q^{k+1}(1-q^{k-2}) + q^{k-2}(1-q^{k+1})]$.

¿From these formulas we have:

$\Delta_k(q) = q^{k-1}(1-2q+q^3+q^{2k}-q^{2k+3}-2q^{3k}+2q^{3k+1}) = q^{k-1}(1-q^k)(q^2+q-1)(1+q^k).$

Therefore $r = (5^{1/2} - 1)/2$ is the the only root of $\Delta_B(V_k, 1-q)$ in $(0,1)$, and this root is of multiplicity one.

(pc) Let us prove (c). From the above formulas for $C_k^1(q)$ and $C_k^2(q)$ we have: $\Delta_C(V_k, 1-q) = C_k^1(q) - C_k^2(q) = q^{k-1}(1-2q+q^3) = q^{k-1}(1-q)(q^2+q-1)$. Therefore r is the the only root of $\Delta_C(V_k, 1-q)$ in $(0,1)$, and this root is of multiplicity one. □

By using **7.1**, it is easy to prove [11] that

7.2 *Let $V_k(t) = (V_k^1(t), V_k^2(t))$ where $V_k^1(t)$ and $V_k^2(t)$ are obtained from V_k^1 and V_k^2 in* **7.1** *by replacing every edge by $t \ge 1$ parallel edges. Let $r = (5^{1/2} - 1)/2$. Then*

(s) $V_k(t)$ is a pair of 2–graphs of the same size,

(a) $\Delta_A(V_k(t), 1-q)$ has exactly one root $a_k(t)$ in $(0,1)$ equal to $(a_k)^{1/t}$ where a_k is the root of $\Delta_A[V_k, 1-q]$ in $(0,1)$, (see **7.1** *(a)), this root is of multiplicity one, and so $a_k(t) < a_k(s)$ if $t < s$,*

(b) $\Delta_B(V_k(t), 1-q)$ has exactly one root $b(k,t) = b(t)$ in $(0,1)$ equal to $r^{1/t}$, this root is of multiplicity one, and so $b(t) < b(s)$ if $t < s$, and

(c) $\Delta_C(V_k(t), 1-q)$ has exactly one root $c(k,t) = c(t)$ in $(0,1)$ equal to $r^{1/t}$, this root is of multiplicity one, and so $c(t) < c(s)$ if $t < s$.

In [2] a single pair of 2–graphs was found by the aid of computer whose all–terminal and two–terminal reliability functions have exactly two crossings. The next statement (see proof in [11]) describes a simple construction that provides infinitely many pairs of 2–graphs having exactly two different A–crossing points, exactly one B–crossing point and exactly one C–crossing point of multiplicity one each.

7.3 *Let $W_k = (W_k^1, W_k^2)$ where $W_k^1 = V(2k, 2k, 0)$ and $W_k^2 = V(2k+1, k-1, k)$ and $k \ge 2$. Then*

- *(s) the 2–graphs W_k^1 and W_k^2 are 2–connected and of the same size,*
- *(a) $\Delta_A(W_k, 1-q)$ has two roots $a_k^1 < a_k^2$ in $(0,1)$, both of these roots are of multiplicity one, and $a_k^1 < a_{k+1}^1 < a_{k+1}^2 < a_k^2$,*
- *(b) $\Delta_B(W_k, 1-q)$ has one root b_k in $(0,1)$, this root is of multiplicity one, and $b_i \ne b_j$ if $i \ne j$,*
- *(c) $\Delta_C(W_k, 1-q)$ has one root c_k in $(0,1)$, this root is of multiplicity one, and $c_i > c_j$ for $i < j$.*

8 Pairs of 2–graphs with a given T–crossing profile

In Subsection 3.1 we have described the main idea of a T–construction that provides pairs of 2–graphs of any given T–crossing profile. Now we are ready to formulate and prove the main result on T–crossings for 2-graphs.

8.1 *For any sequence of natural numbers $M = (m_1, \dots m_k)$ there exists an equisize pair Ω of 2–graphs such that the T–crossing profile of Ω is M.*

Proof (uses **5.4**, **5.8**, and **6.3**).

Theorem **6.3** provides an infinite set $\mathcal{F}$ of ordered pairs of 2–graphs such that:

$(f0)$ if $F = (F_1, F_2) \in \mathcal{F}$ then F_1 and F_2 are of the same size, i.e. $s(F_1) = s(F_2)$,

$(f1)$ every pair F in $\mathcal{F}$ has exactly one T–crossing point $r(F)$ and this crossing point is of multiplicity one, and

$(f2)$ if $X, Y \in \mathcal{F}$ and $X \neq Y$ then $r(X) \neq r(Y)$.

Let $X_i \in \mathcal{F}$, $i = 1, \dots, k$, and x_i be the T–crossing point of X_i, where $0 < x_1 < \dots < x_k < 1$.

Now we will use operation H^* on 2–graphs described in **5.3** (see Figure 5) where H is a 2–graph with possibly equal terminals (operations P and S are particular cases of H^*, see Figures 3 and 4).

Let us construct a pair Ω of 2–graphs as follows:

$(t1)$ H^*–multiply m_i copies of X_i to obtain a pair $X_i^{m_i}$, and

$(t2)$ H^*–multiply all $X_i^{m_i}$, $i = 1, \dots, k$, to obtain $\Omega = \Omega(H)$.

By **5.4**, H^* is a T–multiplying operation. Therefore from $(f1)$ and $(f2)$ it follows that $\{x_1 < \dots < x_k\}$ is the set of T–crossing points of Ω and x_i is of multiplicity m_i, and so the T–crossing profile of Ω is M. By **5.8**, the results of M–multiplication of two equisize pairs of 2–graphs is also an equisize pair of 2–graphs. Therefore from $(f0)$ it follows that Ω is an equisize pair. $\square$

9 Pairs of graphs with a given A–crossing profile

In this section we will formulate and prove the main result on A–, B– and C–crossings for 2–graphs.

9.1 *For any sequence $M = (m_1, \dots m_k)$ of natural numbers there exist equisize pairs Ω_a, Ω_b, and Ω_c such that M is the A–crossing profile of Ω_a, the B–crossing profile of Ω_b, and the C–crossing profile of Ω_c.*

Proof (uses **5.2**, **5.6**, **5.7**, **5.8**, **7.2**).

(p0) Theorem **7.2** provides an infinite set $\mathcal{F}$ of ordered pairs of 2–graphs having properties

$(f0)$ if $F = (F_1, F_2) \in \mathcal{F}$ then F_1 and F_2 are of the same size, i.e. $s(F_1) = s(F_2)$,

$(f1)$ for $L \in \{A, B, C\}$, every pair F in $\mathcal{F}$ has exactly one L–crossing point $r_L(F)$ and this crossing point is of multiplicity one,

$(f2)$ if $X, Y \in \mathcal{F}$ an $X \neq Y$, then $r_L(X) \neq r_L(Y)$.

Let $X_i \in \mathcal{F}$, $i = 1, \dots, k$, and l_i be the L–crossing point of X_i, where $0 < l_1 < \dots < l_k < 1$. Here $L \in \{A, B, C\}$ and $l \in \{a, b, c\}$. In particular, b_i is the B–crossing point of X_i of multiplicity 1 and $0 < b_1 < \dots < b_k < 1$.

(pb) We construct a pair Ω_b of 2–graphs by using operation P described in Section 5 (see Figure 3). The construction is similar to that for Ω in the proof of **8.1**, namely:

$(b1)$ P–multiply m_i copies of X_i to obtain a pair $X_i^{m_i}$, and

$(b2)$ P–multiply all $X_i^{m_i}$, $i = 1, \dots, k$, to obtain Ω_b.

By **5.6**, P is a B–multiplying operation. Therefore from $(f1)$ and $(f2)$ for $L = B$ it follows that $\{b_1 < \dots < b_k\}$ is the set of B–crossing points of Ω_b and b_i is of multiplicity m_i, and so M is the B–crossing profile of Ω_b. By **5.8** and $(f0)$, Ω_b is an equisize pair of 2–graphs.

(pc) Let us replace in **(pb)** operation P by operation H^* for a connected graph H (see Section 5 and Figure 5) and let Ω_c denote the resulting 2–graph. The arguments for $L = C$ similar to that in **(pb)** show that by **5.7** and **5.8**, Ω_c is an equisize pair with C–profile M.

(pa) Now we construct Ω_a by using operation P and H^* for a connected graph H.

$(a1)$ Suppose that $k = m_1 = 1$. Then let Ω_a be an arbitrary pair of 2–graphs from $\mathcal{F}$.

$(a2)$ Suppose that $k \geq 2$ and $m_i = 1$ for each $i = 1, \dots, k$. By using the above procedure for Ω_b in **(pb)**, we construct a pair Ω'_b such that $\{b_1 < \dots < b_{k-1}\}$ is the set of B–crossing points of Ω'_b and b_i is of multiplicity 1 for $i = 1, \dots, k-1$. Let $\Omega_a = \Omega_a(H)$ be an H^*–product of Ω'_b and X_k.

$(a3)$ Now suppose that $m_j \geq 2$ for some $j = 1, \dots, k$. Again by using the above procedure for Ω_b in **(pb)**, we construct a pair Ω''_b such that $\{b_1 < \dots < b_k\}$ is the the set of B–crossing points of Ω''_b, b_i is of multiplicity m_i if $i \neq j$, and of multiplicity $m_j - 1$ if $i = j$. Let $\Omega_a = \Omega_a(H)$ be an H^*–product of Ω''_b and X_j.

In case $(a1)$, Ω_a is a required pair of graphs, by $(f0)$ and $(f1)$ for $L = A$.

In case $(a2)$ and $(a3)$, by multiplying property **5.5** of operation H^*, $\{b_1 < \dots < b_k\}$ is the set of A–crossing points of Ω_a and b_i is of multiplicity m_i, and so M is the A–crossing profile of Ω_a. By **5.8** and $(f0)$, Ω_a is an equisize pair of 2–graphs. □

Remarks

(r1) Notice that each 2–graph $q_1 Q q_2$ in pairs Ω_a, Ω_b, and Ω_c, constructed in the proof of **9.1**, is 2–connected 2–graphs (i.e. the graph obtained from $q_1 Q q_2$ by adding an extra edge $q_1 q q_2$ is 2–connected).

(r2) Moreover if H is a 2–connected 2–graph, then the graphs in pairs Ω_a, Ω_b, and Ω_c, constructed in the above proof, are 2–connected unless $k = m_1 = 1$.

The following is a strengthening of **8.1** and **9.1**.

9.2 *Let $M = (m_1, \dots m_k)$ be a sequence of natural numbers and $L \in \{A, B, C, T\}$. Then there exists an infinite set $\mathcal{P}_L$ of equisize pairs of 2-connected graphs such that all pairs in $\mathcal{P}_L$ have the same list of L–crossing points, this list has exactly k numbers $0 < l_1 < \dots < l_k < 1$, and l_i is of multiplicity m_i, $i = 1, \dots, k$.*

Proof (uses **5.4**, **5.5**, **6.3**, **7.2**, **8.1**, and **9.1**).

Since **6.3** describes an infinite set of pairs of 2–graphs having the same T–crossing point of multiplicity 1, the construction in the proof of **8.1** provides infinitely many pairs Ω having the same list of T–crossing points of any given profile M.

Since **7.2** describes an infinite set of pairs of 2–graphs having the same L–crossing point of multiplicity 1 for $L \in \{B, C\}$, the construction in the proof of **9.1** provides infinitely many pairs Ω_l having the same list of L–crossing points of any given profile M and $L \in \{A, B, C\}$ except for the case when $M = (1)$ and $L = A$.

The above statements for $L \in \{A, T\}$ also follow from the constructions in **8.1** and **9.1** and the fact that the list of T–crossing points of $\Omega(H)$, the list of A–crossing points of $\Omega_a(H)$ and the corresponding multiplicities of these crossing points do not depend on H provided H is connected (see **5.4** and **5.5**).

Let $Y = (Y_1, Y_2)$ where Y_1 is the graph of two vertices and one edge and Y_2 is the graph of two vertices and two parallel edges. Then $\Delta_B(Y, p) = -p(1-p) < 0$ for $p \in (0, 1)$. Therefore our statements for $L \in \{A, C\}$ also follow from **5.5** and **9.1**.

¿From **Remark** above and the properties of operation H^* for a 2–connected graph H it follows that the graphs in the constructed pairs are 2–connected. □

Let $Z = (x_1 Z_1 y_1, x_2 Z_2 y_2)$ be a pair of 2–graphs and let $Z^* = (x_1 Z_1^* y_1, x_2 Z_2^* y_2)$ be a pair obtained from Z as follows:

($s1$) replace each edge of Z_i, $i = 1, 2$, by a copy of a given 2–graph $t_1 T t_2$ (for example, a (t_1, t_2)–path) to obtain Z_i' and

($s2$) put a 2–graph $r_1 R r_2$ (for example, an (r_1, r_2)–path) in parallel with $x_i Z_i' y_i$ to obtain a 2–graph $Z_i^* = Z_i' \parallel R$, $i = 1, 2$.

It is easy to see that

($c1$) if Z_i, R, and T are 2–connected (simple) 2–graphs then Z_i^* is a 2–connected (respectively, simple) graph,

($c2$) if Z_1 and Z_2 are of the same size, then Z_1^* and Z_2^* are also of the same size, and

($c3$) pairs Z and Z^* have the same L–crossing profile for $L \in \{A, B, C, T\}$.

Therefore from **8.1** and **9.1** we have:

9.3 *Let $L \in \{A, B, C, T\}$. For every sequence $M = \{m_1, \ldots m_k\}$ of natural numbers there exist infinitely many equisize pairs (G_1, G_2) of 2–graphs having the L–crossing profile M and such that both G_1 and G_2 are simple 2–connected graphs.*

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