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# How tight is the Bollobás-Komlós conjecture

by

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## ABSTRACT

The bipartite case of the Bollobás and Komlós conjecture states that for every $\Delta_0, \gamma > 0$ there is an $\alpha = \alpha(\Delta_0, \gamma) > 0$ such that the following statement holds: If $G$ is any graph with

$$\delta(G) \geq \frac{n}{2} + \gamma n,$$

then $G$ contains as subgraphs all bipartite graphs, $H$, satisfying

$$\Delta(H) \leq \Delta_0 \text{ and } b(H) \leq \alpha n.$$

Here $b(H)$, the bandwidth of $H$, is the smallest $b$ such that the vertices of $H$ can be ordered as $v_1, \ldots, v_n$ such that $v_i \sim_H v_j$ implies $|i - j| \leq b$.

This conjecture has been proved in [1]. In this note we show that this conjecture is tight in the sense that as $\gamma \to 0$ then $\alpha \to 0$. More precisely, we show that for any $\gamma \leq \frac{1}{100}$ there is a $\Delta_0$ such that that $\alpha(\Delta_0, \gamma) \leq 4\gamma$.

# 1 Introduction

An embedding conjecture of Bollobás and Komlós states that any $n$ vertex graph whose minimum degree exceeds $(1-\frac{1}{k})n+\gamma n$ contains all $k$ chromatic, bounded degree graphs which do not expand. The precise statement is as follows:

**Conjecture 1.1 (Bollobás-Komlós [2])** *For every $k, \Delta_0$ and $\gamma > 0$ there is an $\alpha = \alpha(k, \Delta_0, \gamma) > 0$ such that the following statement holds: If $G$ is any graph with*

$$\delta(G) \geq \left(1 - \frac{1}{k}\right) n + \gamma n,$$

*then $G$ contains as subgraphs all graphs, $H$, satisfying*

$$\chi(H) \leq k, \Delta(H) \leq \Delta_0 \text{ and } b(H) \leq \alpha n.$$

Here $b(H)$, the bandwidth of $H$, is the smallest $b$ such that the vertices of $H$ can be ordered as $v_1, \ldots, v_n$ such that $v_i \sim_H v_j$ implies $|i-j| \leq b$.

The bipartite case of this conjecture has been settled in [1]. In this note we show that this conjecture is tight in the sense that as $\gamma \to 0$ then $\alpha \to 0$. More precisely, we prove the following theorem.

**Theorem 1.1** *For any $0 < \gamma \leq \frac{1}{100}$ there is a $\Delta_0$ such that the following statement holds for infinitely many $n$: There is a graph $BK_{n,\gamma}$ on $n$ vertices with*

$$\delta(BK_{n,\gamma}) \geq \frac{n}{2} + \gamma n$$

*and a bipartite graph, $H$, on $n$ vertices with*

$$\Delta(H) \leq \Delta_0, b(H) < 4\gamma n.$$

*Yet $H$ is not a subgraph of $BK_{n,\gamma}$.*

Throughout this paper we let $\gamma$ be a fixed real number satisfying $0 < \gamma \leq \frac{1}{100}$. Define $s = \lfloor \frac{1}{2\gamma} \rfloor - 3$. Note that $s > \frac{3}{8\gamma}$. We choose, $\epsilon' = \frac{1}{10\gamma}$ and $\epsilon = \frac{\epsilon'}{20s}$.

The remaining of the paper is organized as follows: In the next section we construct the graph $H$. In section 3 we define the graph $BK_{n,\gamma}$ and prove Theorem 1.1. In the last section we give some concluding remarks.

# 2 A Controlled Expander

In this section we construct a graph such that the graph is a reasonably good expander and yet has low bandwidth. In oder to achieve this we will need bipartite graphs which are good expanders. The existence of these graphs follows from standard probabilistic arguments, therefore we omit the proof of the next lemma.

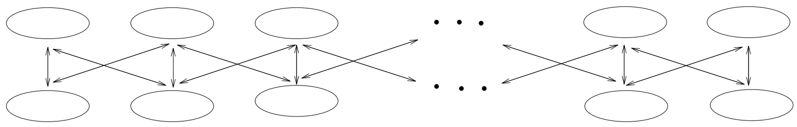

Figure 1: A lowbandwidth expander

**Lemma 2.1** *For every $\epsilon > 0$ there is $\Delta_1$ and $N_0$ such that the following holds for all $N \geq N_0$: There is bipartite graph $R_{N,N} = (V, W, E)$ with $|V| = |W| = N$ such that for any $A \subseteq V$ or $A \subseteq W$*

$$|A| < \frac{N}{5} \text{ implies } |\Gamma(A)| \geq 3|A|$$

*and*

$$|A| = \frac{N}{5} \text{ implies } |\Gamma(A)| \geq (1 - \epsilon)N.$$

*Furthermore, the maximum degree of $R_{N,N}$ is at most $\Delta_1$.* □

Let $n = 2sN$. We define a graph $H$ on of $n$ vertices. Let $V_1, \ldots, V_s$ and $W_1, \ldots, W_s$ be vertex disjoint sets of size $N$. For any $i, j \in \{1, \ldots, s\}$ with $|i - j| \leq 1$ there is a copy of $R_{N,N}$ between $V_i$ and $V_j$. $H$ is a bipartite graph as the two color classes of $H$ are given by $V = \bigcup_{i=1}^{s} V_i$ and $W = \bigcup_{i=1}^{s} W_i$. Figure 1 gives a pictorial representation of the graph $H$. It is also not hard to bound the bandwidth of $H$.

**Lemma 2.2** *The bandwidth of $H$ is less than $4\gamma n$.*

**Proof.** Order the vertices of $H$ of

$$\rho_1(V_1), \rho(W_1), \rho(W_2), \rho(V_2), \rho(V_3), \rho(W_3), \ldots$$

Here $\rho(V_i)$ is any arbitrary ordering of $V_i$ and $\rho(W_i)$ denotes any arbitrary ordering of $W_i$. It is not hard to see that this ordering shows that $b(H) \leq 3N$. Hence, $b(H) \leq \frac{3}{2s} n$. Since $s > \frac{3}{8\gamma}$, the lemma follows. □

Next, our task is to show that $H$ is a reasonably good expander. We prove the following lemma.

**Lemma 2.3** *Let $S \subset V$ or $S \subset W$ such that*

$$\frac{n}{4} - \gamma n \leq |S| \leq \frac{n}{4} + \gamma n,$$

*then*

$$|\Gamma(S)| \geq (1 - \epsilon)\,(|S| + N).$$

**Proof.** Set $|S| = t$ and without loss of generality assume that $S \subset V$. Let

$$A_i = S \cap V_i \text{ and } a_i = |A_i|.$$

For simplicity, we define $a_0 = a_{s+1} = 0$. We also let,

$$B_i = \Gamma(S) \cap W_i \text{ and } b_i = |B_i|.$$

Lemma 2.1 implies

$$b_i \geq \begin{cases} a_{i-1} + a_i + a_{i+1} & \text{if } \max\{a_{i-1}, a_i, a_{i+1}\} < \frac{N}{5}, \\ (1-\epsilon)N & \text{otherwise.} \end{cases} \tag{1}$$

Let us define

$$c_i = \begin{cases} a_{i-1} + a_i + a_{i+1} & \text{if } \max\{a_{i-1}, a_i, a_{i+1}\} < \frac{N}{5}, \\ (1-\epsilon)N & \text{otherwise.} \end{cases} \tag{2}$$

We have

$$|\Gamma(S)| = \sum_{i=1}^{s} |B_i| = \sum_{i=1}^{s} b_i \geq \sum_{i=1}^{s} c_i.$$

Now consider the following mathematical program:

$$\text{minimize } \sum_{i=1}^{s} c_i$$

subjected to:

$$\begin{aligned} c_i &= \begin{cases} a_{i-1} + a_i + a_{i+1} & \text{if } \max\{a_{i-1}, a_i, a_{i+1}\} < \frac{N}{5}, \\ (1-\epsilon)N & \text{otherwise.} \end{cases} \\ \sum_{i=0}^{s+1} a_i &= t \text{ and} \\ a_i &\geq 0, a_0 = a_s = 0, a_i \text{ integer.} \end{aligned}$$

A lower bound on the optimal solution of this program is a lower bound on the size of $\Gamma(S)$. Assume $a_1^*, \ldots, a_s^*$ minimizes the above program. Let $i$ and $j$ be indices such that $\frac{N}{5} \leq a_i* < N$ and $a_j* > 0$. We can replace $a_i^*$ with $a_i^* + 1$ and $a_j^*$ with $a_j^* - 1$ without increasing the value of the above program. To see this notice that increasing $a_i^*$ by one does not increase $c_{i-1}, c_i, c_{i+1}$ as they are already equal to $(1-\epsilon)N$. Whereas decreasing $a_j^*$ cannot increase $c_{j-1}, c_j, c_{j+1}$. A similar argument works if either one of $i$ or $j$ belongs to $\{1, s\}$.

Hence, we can assume that there is at most one index $i$ such that $\frac{N}{5} \leq a_i^* < N$. We have the trivial inequality,

$$c_i \geq (1-\epsilon)a_i^* \text{ for all } i.$$

To prove the lemma, it suffices to show that there are two indices $j$ and $k$ such that

$$c_j + c_k \geq (1-\epsilon)(a_j^* + a_k^* + N).$$

Let us call $a_i^*$ *full* if $a_i^* \geq \frac{N}{5}$. We consider two several cases:

**Case 1:** $a_1$ is full.

Let $j$ be the smallest index such that $a_j^*$ is not full. Hence, $a_1^*, a_2^*, \ldots, a_{j-1}^*$ are full. Note that $j \leq \frac{3s}{4}$, since otherwise there is at most one of the $a_1^*, \ldots, a_j^*$ is less than $N$ and we have,

$$\sum_{i=1}^{j-1} a_i^* \geq (j-2)N > t$$

We now discuss two subcases: if $a_{j+1}^*$ is not full then

$$c_j = (1-\epsilon)N \text{ and } c_{j+1} \geq a_j^* + a_{j+1}^*.$$

Hence $j$ and $j+1$ are the two indices we want.

The second subcase happens when $a_{j+1}^*$ is full. Then we let $k$ be the first index larger than $j$ such that $a_k^*$ is not full. Again, it is not hard to see that $k$ exists and is less than $s$. Now notice that

$$c_j = (1-\epsilon)N \text{ and } c_k = (1-\epsilon)N.$$

Whereas,

$$a_j^* + a_k^* \leq \frac{N}{5} + \frac{N}{5} < (1-\epsilon)N.$$

This yeilds,

$$c_j + c_k \geq a_j^* + a_k^* + (1-\epsilon)N.$$

**Case 2:** $a_s$ is full. By symmetry this is similar to Case 1.

**Case 3:** We assume that both of the above cases do not happen. It is easy to see that at least one of the $a_i^*$ is full. Let $j$ be the first index such that $a_j^*$ is full. Furthermore, let $k$ be the last index such that $a_k^*$ is full. Note that $2 \leq j \leq k \leq s-1$. It is easy to see that

$$c_{j-1} + c_{k+1} \geq (1-\epsilon)(a_{j-1}^* + a_{k+1}^* + N).$$

This concludes our proof. $\square$

# 3 The Bollobás-Komlós Graph

We let $BK_{n,\gamma}$ denote the graph with the vertex set $\{1, \ldots, n\}$. It consists of two cliques of size $\frac{n}{2} + \gamma n + 1$ sharing $\gamma n + 2$ vertices. More precisely, the edge set of $BK_{n,\gamma}$ is given by,

$$\{\{i,j\} : i,j \leq \frac{n}{2} + \gamma n + 1\} \cup \{\{i,j\} : i,j \geq \frac{n}{2} - \gamma n - 1\}.$$

The minimum degree of $BK_{n,\gamma}$ is

$$\frac{n}{2} + \gamma n.$$

Note that $BK_{n,\gamma}$ is not a very good expander. Hence, any subgraph of $BK_{n,\gamma}$ is also not a good expander. This fact is enough to prove our main theorem.

**Theorem 3.1** *$H$ is not a subgraph of $BK_{n,\gamma}$.*

Assume that $H$ is a subgraph of $BK_{n,\gamma}$, then there is a one to one mapping $\psi : V \cup W \rightarrow \{1, \ldots, n\}$ such that,

$$x \sim_H y \text{ implies } \psi(x) \sim_{BK_{n,\gamma}} \psi(y).$$

Let $X$ be the set of $H$ which are mapped to $\{1, \ldots, \frac{n}{2} - \gamma n - 2\}$. Define, $S_1 = X \cap V$ and $S_2 = X \cap W$. First we show that

$$|S_1|, |S_2| < \frac{n}{4} + \gamma n.$$

Suppose this is not the case and $|S_1| \geq \frac{n}{4} + \gamma n$. Let $S_1'$ be any subset of $S_1$ of size exactly $\frac{n}{4} + \gamma n$. Then $\Gamma(S_1') \subset \{1, \ldots, \frac{n}{2} + \gamma n + 1\}$. Hence,

$$|S_1'| + |\Gamma(S_1')| \leq \frac{n}{2} + \gamma n + 1.$$

This gives us

$$|\Gamma(S_1') \leq \frac{n}{2} + 1,$$

Contradicting Lemma 2.3. Similarly, $|S_2| \leq \frac{n}{4} + \gamma n$. This implies that

$$|S_1|, |S_2| \geq \frac{n}{4} - 2\gamma n.$$

Hence, we can apply Lemma 2.3 to both $S_1$ and $S_2$. From Lemma 2.3 we have,

$$\begin{aligned} |\Gamma(S_1)| + |\Gamma(S_2)| &\geq (1 - \epsilon)(|S_1| + |S_2| + 2N) \\ &\geq \frac{n}{2} - \gamma n + 2N - 2\epsilon' N \end{aligned}$$

Whereas, we must have $\psi(\Gamma(X)) \subseteq \{1, \ldots, \frac{n}{2} + \gamma n + 1\}$ and hence,

$$|\Gamma(S_1)| + |\Gamma(S_2)| \leq \frac{n}{2} + \gamma n + 1.$$

This implies that $2N(1 - \epsilon') \geq 2\gamma n$; that is

$$s \geq \frac{1}{2\gamma} - \frac{\epsilon'}{2\gamma}$$

a contradiction. $\square$

# 4 Conclusion

We have prove that the Bollobás-Komlós conjecture is tight in some sense. It is interesting to contrast the results of this note with those in [1]. In [1] it is shown that if $\gamma > 0$ then $\alpha(\Delta_0, \gamma) > 0$ for all $\Delta_0$. However, the proof uses of regularity lemma of Szemerédi[3]. Hence, the lower bounds on $\alpha$ are extremely poor. Although the bounds are not explicitly computed in [1] it is clear that the proof will gives bounds

$$\alpha(\gamma, 3) > \frac{1}{T(\frac{1}{\gamma})}$$

where $T$ grows at least as fast as the tower function. Hence, the gap between the upper bounds on $\alpha$ and lower bounds is huge. The author is inclined to believe that the upper bound is perhaps closer to the truth. In this case improving the lower bounds would mean finding a proof of the bipartite case of Bollobás-Komlós conjecture without using the regularity lemma. This seems to be a very challenging problem.

# 5 Acknowledgments

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