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On the complexity of the isomorphism relation for finitely generated groups [1]

by

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ABSTRACT

Working within the framework of descriptive set theory, we show that the isomorphism relation for finitely generated groups is a universal essentially countable Borel equivalence relation. We also prove the corresponding result for the conjugacy relation for subgroups of the free group on two generators. The proofs are group-theoretic, and we refer to descriptive set theory only for the relevant definitions and for motivation for the results.

Introduction

Given a class $\mathcal{K}$ of structures for a fixed first order language $\mathcal{L}$, one may ask what kinds of complete invariants can be used to classify the elements of $\mathcal{K}$ up to isomorphism. For those classes consisting of the countable models of some $\mathcal{L}_{\omega_1,\omega}$-sentence, Friedman and Stanley [FS] proposed to use the methods of descriptive set theory to study their possible invariants and defined the notion of Borel reducibility between such classes of structures. In [HK], Hjorth and Kechris continued this study and situated it within the general theory of Borel equivalence relations. This provides tools for the analysis of the isomorphism relation and a framework for measuring the complexity of possible invariants.

For a countable first order language $\mathcal{L}$, let $X_{\mathcal{L}}$ denote the class of all $\mathcal{L}$-structures with universe $\mathbb{N}$. Then $X_{\mathcal{L}}$ is a Polish space with the natural topology. Each countable $\mathcal{L}$-structure is represented up to isomorphism within $X_{\mathcal{L}}$; and for every $\mathcal{L}_{\omega_1,\omega}$-sentence σ, the class $Mod(\sigma)$ of all models of σ is an isomorphism-invariant Borel subset of $X_{\mathcal{L}}$. We shall denote the isomorphism relation on $X_{\mathcal{L}}$ by $\cong$, and we shall denote the restriction of $\cong$ to $Mod(\sigma)$ by $\cong_\sigma$.

If X and Y are Borel sets and E and F are equivalence relations on X and Y respectively, then we say that E is *Borel reducible* to F and write $E \leq_B F$ if there is a Borel function $f : X \to Y$ such that $xEy \Leftrightarrow f(x)Ff(y)$. This means that elements of X can be effectively classified up to E-equivalence by invariants which are themselves F-equivalence classes. In the algebraic context, we are interested in Borel reducibility among equivalence relations of the form $\cong_\sigma$. In particular, if $\cong_\sigma$ is smooth, then we say that models of σ are *concretely classifiable*. Given two $\mathcal{L}_{\omega_1,\omega}$ sentences σ and τ, we write $\sigma \leq_B \tau$ if and only if $\cong_\sigma \leq_B \cong_\tau$. Intuitively, this means that τ has a more complex classification problem than σ. We call τ *Borel complete* if $\sigma \leq_B \tau$ for any other $\sigma \in \mathcal{L}_{\omega_1,\omega}$.

We can now rephrase several known interpretability results in this language. For example, the theory of graphs is Borel complete. Mekler [Me] showed that the class of nilpotent class 2 groups of exponent p is Borel complete. Fried and Kollár [FK] proved the same result for the class of fields of any characteristic $\chi \neq 2$ and Pröhle provided a proof for $\chi = 2$. This result was rediscovered in [FS], where the authors also show that the theory of linear orderings is Borel complete. Recently Hjorth [Hj] proved that the theory of torsion-free abelian groups is also Borel complete.

In general, $\cong_\sigma$ is analytic but not Borel. For example, this occurs if σ is Borel complete. However,there are theories σ for which $\cong_\sigma$ is Borel, and yet $Mod(\sigma)$ still has a complex classification problem. In particular, $\cong_\sigma$ is Borel for many theories whose models have "finite rank" in some sense. Examples of these are: the theory of finitely generated groups, or more generally finitely generated structures in a given language; the theory of finite rank torsion-free abelian groups; the theory of fields of finite transcendence degree, etc. In order to study these examples, it is useful to introduce the notion of an essentially countable equivalence relation. Recall that an equivalence relation E is said to be *countable* if each equivalence class of E is countable. Given a theory σ, we say that $\cong_\sigma$ is *essentially countable* iff $\cong_\sigma \leq_B E$ for some countable Borel equivalence relation E on a Borel set X. In [HK], Hjorth and Kechris show that the following are equivalent:

(i) $\cong_\sigma$ is essentially countable;

(ii) there is a countable fragment $F \subseteq \mathcal{L}_{\omega_1,\omega}$ containing σ such that if $\mathcal{M} = \langle \mathbb{N}, - \rangle$ is a model of σ, there exists $\bar{a} \in \mathbb{N}^{<\omega}$ such that $\mathrm{Th}_F(\langle \mathcal{M}, \bar{a} \rangle)$ is $\aleph_0$-categorical;

(iii) the same as in (ii) with $\langle \mathcal{M}, \bar{a} \rangle$ F-atomic.

Now for such σ, we can use the theory of countable Borel equivalence relations to study the possible invariants that can be used to classify models of σ up to isomorphism. Recall that a countable Borel equivalence relation E is called *universal* if $F \leq_B E$ for any other countable Borel equivalence relation F. Such an equivalence relation exists, and it is obviously unique up to Borel bireducibility. One particular realization of it is the equivalence relation given by the shift action of the free group F_2 on two generators on 2^{F_2}. Thus if we show that $\cong_\sigma$ is a universal essentially countable Borel equivalence relation, then we know that any kind of invariants used to classify models of $\cong_\sigma$ must necessarily be rather complex. For a more detailed discussion and many other examples, we refer the reader to [HK].

The main result of this paper is that the isomorphism relation for finitely generated groups (in fact, 5-generated groups) is a universal essentially countable Borel equivalence relation. This proves Conjecture 5.8 from [HK]. The same proof shows that this theory does not admit canonical models. (We shall define this notion in §3.) We shall also study the conjugacy relation on subgroups of F_2 and show that it is also universal. Our proofs are group theoretic, and we refer to the theory of Borel equivalence relations only for the relevant definitions and motivation for the results.

The paper is organized as follows. §1 contains definitions and preliminary results. None of these results are due to the authors. In §2 we study the holomorph action on subsets of F_2 and show that it produces a universal countable Borel equivalence relation. This is then used in §3 to prove the main theorem. In §4 we study the conjugacy relation on subgroups of a given countable group G, and prove the result referred to above.

1 Preliminaries

(I) Group actions. For a Polish group G, a *Polish G-space* is a Polish space X together with a continuous action $a : X \times G \to X$. A *Borel G-space* is a standard Borel space X together with a Borel action $a : X \times G \to X$. We denote the associated orbit equivalence relation by E_a. Thus xE_ay iff there is $g \in G$ such that $a(x, g) = y$. If the action a is clear from the context we shall often denote the induced equivalence relation by $E(G, X)$. A Borel function $\alpha : E_a \to G$ is called a *cocycle* if whenever xE_ayE_az, then $\alpha(x, z) = \alpha(x, y)\alpha(y, z)$. We say that the action a has the *cocycle property* if there exists a Borel cocycle α such that $a(x, \alpha(x, y)) = y$. For all undefined notions and basic facts concerning Polish group actions, we refer the reader to [BK].

(II) Polish spaces and Model theory. We now describe how to represent the class of countably infinite structures of a given first order language by the elements of a Polish space. For convenience, we shall work only with relational first order languagues. If there are function symbols, as is the case with the language of group theory, then we replace them by relation symbols representing the graphs of the associated functions. Given a language $\mathcal{L} = \{R_i : i \in I\}$, where R_i is an n_i-ary relation symbol, let

$$X_{\mathcal{L}} = \prod_{i \in I} 2^{\mathbb{N}^{n_i}}.$$

Thus the elements of $X_{\mathcal{L}}$ code $\mathcal{L}$-structures with universe $\mathbb{N}$. Given $x = (x_i)_{i \in I} \in X_{\mathcal{L}}$ the structure

$$\mathcal{M}_x = \langle \mathbb{N}, R_i^x \rangle_{i \in I}$$

represented by x is given by:

$$R_i^x(k_1, \ldots, k_{n_i}) \Leftrightarrow x_i(k_1, \ldots, x_{n_i}) = 1.$$

For $x \in X_{\mathcal{L}}$, we let $[x] = \{y \in X_{\mathcal{L}} : \mathcal{M}_y \cong \mathcal{M}_x\}$ denote the isomorphism class of x. The symmetric group S_∞ on $\mathbb{N}$ is a Polish group with the usual product topology inherited from $\mathbb{N}^{\mathbb{N}}$; and $X_{\mathcal{L}}$ together with the natural action of S_∞ is a Polish S_∞-space. Notice that x, $y \in X_{\mathcal{L}}$ lie in the same S_∞-orbit iff $\mathcal{M}_x \cong \mathcal{M}_y$. For this reason, the action is usually called the *logic action* of S_∞ on $X_{\mathcal{L}}$. Given an $\mathcal{L}_{\omega_1,\omega}$ sentence σ,

$$Mod(\sigma) = \{x \in X_{\mathcal{L}} : \mathcal{M}_x \models \sigma\}$$

represents the class of all countably infinite models of σ. Let $\cong_\sigma$ denote the isomorphism relation restricted to $Mod(\sigma)$. Then $Mod(\sigma)$ is an isomorphism-invariant Borel subset of $X_{\mathcal{L}}$. Moreover, by a theorem of Lopez and Escobar, every such set is of the form $Mod(\sigma)$ for some $\sigma \in \mathcal{L}_{\omega_1,\omega}$. We denote the restriction of $\cong$ to $Mod(\sigma)$ by $\cong_\sigma$.

In this paper, we shall be particularly interested in the class FGG of all finitely generated groups. Clearly this class is axiomatizable by an $\mathcal{L}_{\omega_1,\omega}$ sentence in the language $\mathcal{L} = \{R\}$ consisting of one ternary relation which represents the graph of the multiplication operation. Formally, in what follows we are required to produce groups with underlining set $\mathbb{N}$, but since the groups we define all have *canonical* isomorphisms to such groups, we can consider FGG as an isomorphism-invariant Borel subset of $X_{\mathcal{L}}$.

(III) Equivalence relations. If X and Y are Borel sets and E and F are equivalence relations on X and Y respectively, then we say that E is *Borel reducible* to F and write $E \leq_B F$ if there exists a Borel function $f : X \to Y$ such that $xEy \Leftrightarrow f(x)Ff(y)$. This means that elements of X can be effectively classified up to E-equivalence by invariants which are themselves F-equivalence classes. We say that E and F are *Borel bireducible* and write $E \sim_B F$ if $E \leq_B F$ and $F \leq_B E$. We say that E is *smooth* if $E \leq_B \Delta(Y)$, where $\Delta(Y)$ is equality on some Polish space Y. There are many examples of non-smooth Borel equivalence relations, the simplest of them being the *Vitali equivalence relation* E_0 on $2^{\mathbb{N}}$: defined by xE_0y iff there is n such that for all $m \geq n$ $x(m) = y(m)$.

A Borel equivalence relation E on a standard Borel space X is called *countable* if every equivalence class $[x]_E$ is countable. By a result of Feldman and Moore [FM], every such equivalence relation is of the form E_a for a Borel action a of a countable group G on X. A countable Borel equivalence relation E is called *universal* if $F \leq_B E$ for any other countable Borel equivalence relation F. One example of such an equivalence relation is given by the shift action of any free group on at least two generators. Recall that given any set X and any group G, the shift action of G on X^G is given by

$$x(h)g = x(hg).$$

Let $E(F_2, 2^{F_2})$ denote the equivalence relation induced by the shift action of the free group on two generators F_2 on 2^{F_2}. Then $E(F_2, 2^{F_2})$ is a universal countable Borel equivalence relation. We shall also denote $E(F_2, 2^{F_2})$ by E_∞. Clearly the permutation group $\left(F_2, 2^{F_2}\right)$ can be naturally identified with the permutation group $(F_2, \mathcal{P}(F))$, where F_2 acts on $\mathcal{P}(F)$ via right translations; and so $E(F_2, \mathcal{P}(F)) \sim_B E_\infty$. We say that an equivalence relation E is *essentially countable* if $E \leq F$ for some countable Borel equivalence relation F. For more on the general theory of such equivalence relations, see also the forthcoming paper [JKL].

(IV) Group theory. Most of our notation is fairly standard. In this paper, permutation groups always act on the right. If (G, Ω) is a permutation group, $\alpha \in \Omega$ and $g \in G$, then the image of α under g will be denoted by either αg or α^g. Consider the special case when $\Omega = F$ is a group. Then for each $x \in F$, the right and left translation maps by x will be denoted by x^ρ, x^λ respectively. We define $F^\rho = \{x^\rho : x \in F\}$ and $F^\lambda = \{x^\lambda : x \in F\}$. The *holomorph* of F is the group

$$Hol(F) = \langle F^\rho, Aut(F)\rangle \leqslant Sym(F).$$

It is easily checked that if $x \in F$ and $\pi \in Aut(F)$, then $\pi^{-1}x^\rho\pi = (x^\pi)^\rho$. Thus $Hol(F) = F^\rho \rtimes Aut(F)$. It follows easily that each $\varphi \in Hol(F)$ can be expressed as

$$\varphi = g^\rho\pi = \theta h^\lambda$$

for suitably chosen elements g, $h \in F$ and π, $\theta \in Aut(F)$.

Given groups G_1 and G_2 with a common subgroup H, we denote the free product of G_1 and G_2 with amalgamation over H by $G_1 *_H G_2$. If X is a set of right coset representatives of H in G_1 such that $1 \in X$ and Y is a set of right coset representatives of H in G_2 such that $1 \in Y$, then every element of $G_1 *_H G_2$ has a *unique normal form*

$$g = ha_1 \dots a_n$$

where $h \in H$, each $a_i \in (X \cup Y) \setminus \{1\}$, and a_i, a_{i+1} are not both in X or both in Y. We shall repeatedly use the fact that any element of $G_1 *_H G_2$ of finite order is conjugate to an element of either G_1 or G_2. For these and other basic results in combinatorial group theory, we refer the reader to [LS], [MKS], or [Se].

2 The holomorph action

Let $F = \langle a, b\rangle$ be the free group on two generators. Let (F, Ω) be the permutation group such that $\Omega = F$, and F acts on Ω via the right regular representation. In the next section, for each $S \subseteq \Omega$, we shall construct a finitely generated group G_S such that $G_S \cong G_T$ iff there exists an automorphism (π, φ) of the permutation group (F, Ω) with $S^\varphi = T$. Remember that the pair (π, φ) is an automorphism of (F, Ω) iff:

(i) $\pi \in Aut(F)$;

(ii) $\varphi : \Omega \to \Omega$ is a bijection; and

(iii) $(x^\varphi)^{g^\pi} = (x^g)^\varphi$ for each $x \in \Omega$ and $g \in F$.

There are two basic kinds of automorphism of (F, Ω).
(I) For each $a \in F$, let $i_a \in Inn(F)$ be the corresponding inner automorphism. Then (i_a, a^ρ) is an automorphism of (F, Ω).
(II) For each $\pi \in Aut(F)$, (π, π) is an automorphism of (F, Ω).
It turns out that every automorphism of (F, Ω) is the product of an automorphism of type (I) followed by an automorphism of type (II). (This is essentially proved in the next section.) Thus $G_S \cong G_T$ iff there exists $\varphi \in Hol(F)$ such that $S^\varphi = T$. Consequently, in order to show that the isomorphism relation for finitely generated groups is a universal essentially countable Borel equivalence relation, we must first show that $E(Hol(F), \mathcal{P}(F))$ is a universal countable Borel equivalence relation. For technical reasons, we shall work with $\mathcal{P}^*(F) = \mathcal{P}(F) \setminus \{\emptyset, F\}$ rather than with $\mathcal{P}(F)$. Of course, it is easily shown that $E(F, \mathcal{P}^*(F)) \sim_B E_\infty = E(F, \mathcal{P}(F))$.

Theorem 1 *$E(Hol(F), \mathcal{P}^*(F))$ is a universal countable Borel equivalence relation.*

PROOF: Clearly $E(Hol(F), \mathcal{P}^*(F))$ is a countable Borel equivalence relation. Thus it suffices to show that: $E(F, \mathcal{P}^*(F)) \leq_B E(Hol(F), \mathcal{P}^*(F))$. Let $[F, F]$ be the commutator subgroup of F. By [Serre, p.6], $[F, F]$ is a free group of infinite rank. Let $\{x_n : n \in \mathbb{N}\}$ be a set of free generators of $[F, F]$ and let $G = \langle x_0, x_1\rangle$.

Claim 1 *$N_F(G) = G$.*

PROOF: Let $N = N_F(G)$. By [MKS, Exercise 2.4.5], whenever H is a finitely generated subgroup of a free group, then H has finite index in its normalizer. In particular, $[N : G] < \infty$. Clearly $N \cap [F, F] = G$. Applying the Second Isomorphism Theorem, we see that

$$N/G = N/(N \cap [F, F]) \cong N[F, F]/[F, F] \leqslant F/[F, F].$$

Thus N/G is isomorphic to a finite subgroup of the torsion-free group $F/[F, F] \cong \mathbb{Z} \oplus \mathbb{Z}$. It follows that $N = G$.

□

For each subset $S \in \mathcal{P}^*(G)$, we define:

$$\widehat{S} = S \sqcup x_2 G \sqcup [F,F]a \sqcup [F,F]b \sqcup [F,F]a^2.$$

Notice that $S \sqcup x_2 G \subsetneq [F,F]$. Suppose now that $T \in \mathcal{P}^*(G)$ and that there exists $\pi \in Hol(F)$ such that $\widehat{S}^\pi = \widehat{T}$. We shall show that $\pi = g^\rho$ for some $g \in G$ such that $Sg = T$. This will complete the proof of Theorem 1.

Let $\bar{F} = F/[F,F] \cong \mathbb{Z} \oplus \mathbb{Z}$. Then under the natural mapping

$$\varphi : Aut(F) \to Aut(\bar{F})$$

we have that:

(a) φ maps $Aut(F)$ onto $Aut(\bar{F}) \cong GL(2,\mathbb{Z})$; and

(b) $\ker \varphi = Inn(F)$, the group of inner automorphisms of F.

(For example, see [LS, Proposition 4.5]). Notice that F^ρ also acts naturally on $\bar{F}$ by translations. Thus $Hol(F)$ also acts on $\bar{F} \cong \mathbb{Z} \oplus \mathbb{Z}$ as the group of affine transformations $(\mathbb{Z} \oplus \mathbb{Z}) \rtimes GL(2,\mathbb{Z})$. Since π permutes the cosets of $[F,F]$, and both $\widehat{S}$ and $\widehat{T}$ consist of three complete cosets and one nonempty proper subset of a coset, it follows that

$$(S \sqcup x_2 G)^\pi = T \sqcup x_2 G.$$

In particular, $[F,F]^\pi = [F,F]$. Hence if we identify $[F,F]a$ and $[F,F]b$ with the vectors $\mathbf{v} = (1,0)$ and $\mathbf{w} = (0,1)$ in $\mathbb{Z} \oplus \mathbb{Z}$, then π corresponds to an affine transformation $\bar{\pi}$ such that

(i) $(0,0)^{\bar{\pi}} = (0,0)$; and

(ii) $\{\mathbf{v}, \mathbf{w}, 2\mathbf{v}\}^{\bar{\pi}} = \{\mathbf{v}, \mathbf{w}, 2\mathbf{v}\}$.

Let $\pi = g^\rho \theta$, where $g \in F$ and $\theta \in Aut(F)$. By (i), $g \in [F,F]$; and so g^ρ acts trivially on $\bar{F}$. Consequently, $\bar{\pi}$ is the linear transformation induced by θ. By (ii), we must have that $\bar{\pi} = 1$. It follows that $\theta \in \ker \varphi = Inn(F)$; say, θ is conjugation by h. Thus for all $c \in F$, we have that $c^\pi = h^{-1} c g h$. Hence $\pi = y^\lambda z^\rho$, where $y = h^{-1}$ and $z = gh$. Thus:

$$\begin{aligned} T \sqcup x_2 G &= (S \sqcup x_2 G)^\pi \\ &= ySz \sqcup y x_2 G z \\ &= y z z^{-1} S z \sqcup y x_2 z z^{-1} G z \\ &= y z S^z \sqcup y x_2 z G^z. \end{aligned}$$

Claim 2 $x_2 G = y x_2 z G^z$.

PROOF: Let $A = T \sqcup x_2G$. Then A consists of a complete and a partial coset of G in F. First we shall explain how to group-theoretically recognise the complete coset in this decomposition. Let $a = x_2g \in x_2G$. Then for each element $b = x_2h \in x_2G$, we have that

$$ba^{-1}b = x_2hg^{-1}x_2^{-1}x_2h \in x_2G.$$

On the other hand, if $b \in T \leqslant G$, then

$$ba^{-1}b = bg^{-1}x_2^{-1}b \notin G \sqcup x_2G.$$

For each $a \in A$ define:

$$S(a) = \{b \in A : ba^{-1}b \in A\}.$$

Then we have just shown that if $a \in x_2G$, then $S(a) = x_2G$. It is easily checked that if $a \in T$, then $S(a) \subseteq T$. Thus $a \in x_2G$ iff there exists $g \in F$ with $A \subseteq S(a) \cup gS(a)$.

Now consider the decomposition $A = yzS^z \sqcup yx_2zG^z$. This time, we have expressed A as a complete and a partial coset of G^z in F. Arguing as above, we find that if $a \in yx_2zG^z$, then $yx_2zG^z \subseteq S(a)$; and hence there exists $g \in F$ with $A \subseteq S(a) \cup gS(a)$. Consequently, $yx_2zG^z \subseteq x_2G$; and so $G^z \leqslant G$. If we also had that $yzS^z \cap x_2G \neq \emptyset$, this would yield that $yzG^z \cap x_2G \neq \emptyset$, and so $yzG^z \subseteq x_2G$. But this implies that

$$A \subseteq yzG^z \sqcup yx_2zG^z \subseteq x_2G,$$

which is a contradition. Thus $yx_2G^z = x_2G$.

□

In particular, $G^z = G$ and so $z \in N_F(G) = G$. Thus $x_2G = yx_2zG^z = yx_2G$; and so $y \in x_2Gx_2^{-1}$. We also have that $ySz = T$, and so $yG = G$. Hence $y \in G \cap x_2Gx_2^{-1} = \{1\}$. But this means that $\pi = z^\rho$ for some $z \in G$ such that $Sz = T$, as desired. This completes the proof of Theorem 1.

□

3 Finitely generated groups

In this section, we prove the main theorem saying that $(FGG, \cong)$ is a universal essentially countable Borel equivalence relation. Throughout this section, we let $\mathbb{F}$ denote the 2-element field $GF(2)$ and $V = \bigoplus_{w \in F} \mathbb{F}\widetilde{w}$ denote the vector space over $\mathbb{F}$ with basis indexed by elements of F. Thus V is an elementary abelian 2-group which is isomorphic to $([F]^{<\omega}, \Delta)$, the group of all finite subsets of F with the symmetric difference operation Δ. Let $P = V \rtimes F$, where $g^{-1}\widetilde{w}g = \widetilde{wg}$ for each $w, g \in F$. Let $\mathcal{B} = \{\widetilde{w} : w \in F\}$; so that $\mathcal{B}$ is a basis of the vector space V.

Lemma 1 *$\mathcal{B}$ is the unique F-orbit on V such that $V = \langle \mathcal{B} \rangle$.*

PROOF: Suppose otherwise. Then for some $n > 1$, there exists an n-subset $\{w_1, \ldots, w_n\}$ of F such that

$$\sum_{i=1}^{m}(\widetilde{w_1 g_i} + \ldots + \widetilde{w_n g_i}) = \widetilde{1}$$

for some $g_1, \ldots, g_m \in F$. But then, working in the group ring $\mathbb{F}[F]$, we obtain the equation

$$(w_1 + \ldots + w_n)(g_1 + \ldots g_m) = 1.$$

Thus $(w_1 + \ldots + w_n)$ is a nontrivial unit in $\mathbb{F}[F]$. But it is well-known that the group ring $\mathbb{F}[F]$ has no nontrivial units. (For example, see [P, Chapter 13].)

□

Lemma 2 *Suppose that $\pi \in Aut(P)$. Then:*

(i) $\mathcal{B}^\pi = \mathcal{B}$; and

(ii) there exists $\varphi \in Hol(F)$ such that $\widetilde{w}^\pi = \widetilde{w^\varphi}$ for all $w \in F$.

PROOF: Clearly V is a characteristic subgroup of P; and so $V^\pi = V$. By Lemma 2, $\mathcal{B}$ is the set of elements v of V such that the conjugacy class v^P generates V. Hence $\mathcal{B}^\pi = \mathcal{B}$. Clearly there exists an automorphism $\theta \in Aut(F)$ and a function $w \mapsto v_w$ from F into V such that

$$w^\pi = v_w w^\theta$$

for all $w \in F$. Let $\widetilde{1}^\pi = \widetilde{x}$; and let $\varphi = \theta x^\lambda \in Hol(F)$. We claim that $\widetilde{w}^\pi = \widetilde{w^\varphi}$ for all $w \in F$. To see this, note that

$$\begin{aligned} \widetilde{w}^\pi &= (w^{-1}\widetilde{1}w)^\pi \\ &= (w^\theta)^{-1} v_w^{-1} \widetilde{x} v_w w^\theta \\ &= \widetilde{x w^\theta}. \end{aligned}$$

□

For each $S \in \mathcal{P}^*(F)$, we shall define an associated finitely generated group G_S such that $G_S \cong G_T$ iff there exists $\pi \in Hol(F)$ such that $S^\pi = T$. First we shall define an auxillary group H_S. It is almost certainly true that H_S already satisfies our requirements; but when we attempted to verify this, we found that our arguments were becoming unpleasantly complicated. So in order to keep the proofs as straightforward as possible, we decided to work with the slightly larger group G_S.

Definition 1 *For each $S \in \mathcal{P}^*(F)$, let*

$$H_S = \langle P, t | t^3 = 1, [t, \widetilde{w}] = 1 \text{ for } w \in S \rangle.$$

Lemma 3 *Let $\pi = \theta x^{\lambda} \in Hol(F)$, where $x \in F$ and $\theta \in Aut(F)$. If $S^{\pi} = T$, then we can define an isomorphism $f : H_S \to H_T$ by:*

$$\begin{aligned} \widetilde{w}^f &= \widetilde{w^{\pi}}, \quad w \in F \\ g^f &= g^{\theta}, \quad g \in F \\ t^f &= t. \end{aligned}$$

PROOF: This is an easy calculation. Simply note that $\widetilde{wg}^f = \widetilde{x(wg)^{\theta}}$ and that

$$\begin{aligned} (g^{-1}\widetilde{w}g)^f &= (g^{\theta})^{-1}\widetilde{xw^{\theta}}g^{\theta} \\ &= \widetilde{xw^{\theta}g^{\theta}} \\ &= \widetilde{x(wg)^{\theta}}. \end{aligned}$$

Thus f induces an automorphism of $P = V \rtimes F$ such that $\{\widetilde{w} : w \in S\}^f = \{\widetilde{w} : w \in T\}$. The result follows.

$\square$

Definition 2 *For each $S \in \mathcal{P}^*(F)$, let:*

$$G_S = \langle H_S, z | z^5 = 1, [z,t] = 1, [z,\widetilde{w}] = 1 \textit{ for all } w \in F \rangle.$$

Note that G_S is generated by $\{\widetilde{1}, a, b, t, z\}$. Thus it suffices to prove the following result.

Theorem 2 *$G_S \cong G_T$ iff there exists $\pi \in Hol(F)$ such that $S^{\pi} = T$.*

Given this, we see that the map which to each $S \in \mathcal{P}^*(F_2)$ associates G_S is a Borel reduction of $E(Hol(F_2), \mathcal{P}^*(F_2))$ to $(FGG, \cong)$; and thus, by Theorem 1, we obtain the following.

Theorem 3 *$(FGG, \cong)$ is a universal essentially countable Borel equivalence relation.*

We shall prove Theorem 2 via a sequence of lemmas.

Lemma 4 *If there exists $\pi \in Hol(F)$ such that $S^{\pi} = T$, then $G_S \cong G_T$.*

PROOF: Let $f : H_S \to H_T$ be the isomorphism defined in Lemma 3. Then we can extend f to an isomorphism $\hat{f} : G_S \to G_T$ by setting $z^{\hat{f}} = z$.

$\square$

From now on, fix some $S \in \mathcal{P}^*(F)$. We first define some important subgroups of G_S.

Definition 3 *(i)* $V_S = \oplus_{w \in S} \mathbb{F}\widetilde{w}$

(ii) $B_S = V_S \oplus \langle t \rangle$

(iii) $C_S = \langle V, t \rangle$

(iv) $D_S = C_S \oplus \langle z \rangle$.

Now we shall describe some important structural features of G_S. The following lemma is completely straightforward.

Lemma 5 *We have the following decompositions as free product with amalgamation:*

(*a*) $C_S = V *_{V_S} B_S$

(*b*) $H_S = P *_{V_S} B_S$

(*c*) $G_S = H_S *_{C_S} D_S$.

Lemma 6 $D_S = C_{G_S}(z)$*, the centralizer of* z *in* G_S*.*

PROOF: Let $X = \{1, z, z^2, z^3, z^4\}$, so that X is a set of right coset representatives of C_S in D_S. Let Y be a set of right coset representatives of C_S in H_S such that $1 \in Y$. Then each element of $G_S = H_S *_{C_S} D_S$ has a *unique normal form*

$$g = ca_1 \dots a_n$$

where $c \in C_S$, each $a_i \in (X \cup Y) \setminus \{1\}$, and a_i, a_{i+1} are not both in X or both in Y. Clearly every element of D_S commutes with z. For the other direction, suppose g commutes with z and let $g = ca_1 \dots a_n$ be its normal form. Then

$$cza_1 \dots a_n = ca_1 \dots a_n z.$$

It follows easily that $a_1, a_n \in X \setminus \{1\}$; and then a moment's thought shows that $n = 1$.Thus $g \in D_S$.

□

Lemma 7 $P = N_{H_S}(V)$*, the normalizer of* V *in* H_S*.*

PROOF: This time we work with the free product with amalgamation decomposition $H_S = P *_{V_S} B_S$. Choose $I = \{1, t, t^2\}$ as a set of right coset representatives of V_S in B_S; and let J be a set of right coset representatives of V_S in P such that $1 \in J$. Clearly $P \leq N_{H_S}(V)$. Let $g \in N_{H_S}(V)$ be any element and let $g = ac_1 \dots c_n$ be its normal form. Choose any $\widetilde{w} \in \mathcal{B}$. Then

$$c_n^{-1} \dots c_1^{-1} a^{-1} \widetilde{w} a c_1 \dots c_n = c_n^{-1} \dots c_1^{-1} \widetilde{w} c_1 \dots c_n \in V.$$

If $c_1 \in I \setminus \{1\}$, choose $\widetilde{w} \in \mathcal{B} \setminus \widetilde{S}$, where $\widetilde{S} = \{\widetilde{w} : w \in S\}$. Then $c_n^{-1} \dots c_1^{-1} \widetilde{w} c_1 \dots c_n$ is essentially in normal form, and hence is not an element of V. Thus $c_1 \in P$. If $n > 1$, choose $\widetilde{w}$ such that $c_1^{-1} \widetilde{w} c_1 \in \mathcal{B} \setminus \widetilde{S}$. Once again, we reach a contradiction. Thus $g = ac_1 \in P$.

□

Now we are ready to begin the proof of Theorem 2. So suppose that $\varphi : G_S \to G_T$ is an isomorphism for some $S, T \in \mathcal{P}^*(F)$. Let $u = z^\varphi$. Then u has order 5. By [MKS, Corollary 4.4.5], u is contained in a conjugate of either H_T or D_T. Hence, by adjusting φ by an inner automorphism of G_T, we can suppose that $z^\varphi \in \langle z \rangle$. Applying Lemma 6, we now find that

$$D_S^\varphi = C_{G_S}(\langle z \rangle)^\varphi = C_{G_T}(\langle z \rangle) = D_T.$$

Notice that

$$D_S = (V \oplus \langle z \rangle) *_{V_S \oplus \langle z \rangle} (B_S \oplus \langle z \rangle).$$

Hence, by [MKS, Corollary 4.5], $Z(D_S) = V_S \oplus \langle z \rangle$. It follows that

$$(V_S \oplus \langle z \rangle)^\varphi = V_T \oplus \langle z \rangle;$$

and hence $V_S^\varphi = V_T$. Choose any element $v \in V \setminus V_S$. Then v^φ is conjugate in D_T to an element of $V \setminus V_T$. (Note that any element $v \in V \setminus V_S$ is noncentral, and hence is not conjugate to an element of $V_S \leq Z(D_S)$. We will make repeated use of this observation.) Hence, after adjusting φ by an inner automorphism corresponding to a suitably chosen element of D_T, we can suppose that:

(i) $\langle z \rangle^\varphi = \langle z \rangle$,

(ii) $V_S^\varphi = V_T$, and

(iii) there exists $v \in V \setminus V_S$ such that $v^\varphi \in V \setminus V_T$.

Claim 3 $V^\varphi = V$

PROOF: Choose any $x \in V \setminus V_S$. Then:

(a) neither x^φ nor v^φ is contained in a conjugate of $V_T \oplus \langle z \rangle$;

(b) $v^\varphi \in V \oplus \langle z \rangle$;

(c) $[x^\varphi, v^\varphi] = 1$.

By [MKS, Theorem 4.5], $x^\varphi \in V \oplus \langle z \rangle$ and so $x^\varphi \in V$. Thus $V^\varphi \leq V$. Since V is clearly a maximal elementary abelian 2-subgroup of D_S, it follows that $V^\varphi = V$.

□

This implies that $N_{G_S}(V)^\varphi = N_{G_T}(V)$. In particular, $P^\varphi \leq N_{G_T}(V)$. Now let:

$$\psi : G_T = H_T *_{C_T} D_T \to H_T$$

be the homomorphism such that $h^\psi = h$ for all $h \in H_T$, and $z^\psi = 1$. Note that $N_{G_T}(V)^\psi \leq N_{H_T}(V) = P$; and so $N_{G_T}(V)^\psi = P$. Let $\pi = \psi \circ \varphi \restriction P$. Then:

(i) $V^\pi = V$,

(2) $V_S^\pi = V_T$, and

(3) $P^\pi \leq P$.

Claim 4 $\pi \in Aut(P)$.

PROOF: First we shall show that π is injective. Suppose $1 \neq x \in \ker \pi$. Let $x = vg$, where $v \in V$ and $g \in F$. Since $\pi \restriction V$ is an automorphism of V, we must have $g \neq 1$. But then

$$x^{-1}\widetilde{1}x = g^{-1}\widetilde{1}g = \widetilde{g} \neq \widetilde{1}.$$

Applying π, we obtain that $(x^\pi)^{-1}\widetilde{1}^\pi x^\pi \neq \widetilde{1}^\pi$. Hence $x^\pi \neq 1$, which is a contradiction. Thus $\ker \pi = \{1\}$.

Now we shall show that π is surjective. Let $v \in V$ be any element, and let $w \in V$ be the element such that $w^\pi = v$. Then there exist $g_1, \ldots, g_n \in F$ such that

$$w = g_1^{-1}\widetilde{1}g_1 + \ldots + g_n^{-1}\widetilde{1}g_n.$$

Applying π, we obtain that

$$v = w^\pi = (g_1^\pi)^{-1}\widetilde{1}^\pi g_1^\pi + \ldots + (g_n^\pi)^{-1}\widetilde{1}^\pi g_n^\pi.$$

Hence $\{h^{-1}\widetilde{1}^\pi h : h \in P^\pi\}$ generates V; and so $\{h^{-1}\widetilde{1}^\pi h : h \in P^\pi\} = \mathcal{B}$. In particular, there exists $a \in F$ such that $\widetilde{1}^\pi = \widetilde{a}$. Now let $g \in F$ be any element. Then there exists $h \in P^\pi$ such that

$$h^{-1}\widetilde{a}h = h^{-1}\widetilde{1}^\pi h = \widetilde{ag}.$$

It follows that $h = ug$ for some $u \in V$; and so $g \in P^\pi$. Hence π is surjective.

□

By Lemma 2, $\mathcal{B}^\pi = \mathcal{B}$ and there exists $\theta \in Hol(F)$ such that $\widetilde{w}^\pi = \widetilde{w^\theta}$, for all $w \in F$. Notice that

$$\{\widetilde{w} : w \in S\}^\pi = (V_S \cap \mathcal{B})^\pi = V_T \cap \mathcal{B} = \{\widetilde{w} : w \in T\}.$$

It follows that $S^\theta = T$. This completes the proof of Theorem 2.

□

A sentence $\sigma \in \mathcal{L}_{\omega_1,\omega}$ is said to *admit canonical models* if there exists a "Borel definable" function S which to each countable model $\mathcal{M}$ of σ assigns a countable model $S(\mathcal{M})$ of σ such that

$$\mathcal{M} \cong \mathcal{N} \text{ implies } S(\mathcal{M}) = S(\mathcal{N}) \cong \mathcal{M}.$$

(For an exact definition, see [HK, Section 7].) A sentence $\sigma \in \mathcal{L}_{\omega_1,\omega}$ is *concretely classifiable* if S can be chosen so that $S(\mathcal{M})$ has universe $\mathbb{N}$ for each countable model $S(\mathcal{M})$ of σ. In

[HK], Hjorth and Kechris showed that many interesting sentences σ admit canonical models but are not concretely classifiable, including the theories of rank 1 abelian torsion-free groups and of rigid locally finite graphs; and they also proved that the theory of locally finite trees does not admit canonical models.

In the remainder of this section, we shall prove that the theory of finitely generated groups does not admit canonical models. We shall make use of the following results of Hjorth and Kechris [HK].

Theorem 4 *If $\sigma \in \mathcal{L}_{\omega_1,\omega}$, then the following are equivalent.*

(i) σ admits canonical models.

(ii) The logic action of S_∞ on $Mod(\sigma)$ has the cocycle property.

Theorem 5 *The shift action of the free group $F = \langle a, b\rangle$ on $\mathcal{P}(F)$ does not have the cocycle property.*

It follows easily that the shift action of F on $\mathcal{P}^*(F) = \mathcal{P}(F) \setminus \{\emptyset, F\}$ also does not have the cocycle property.

Lemma 8 *The action of $Hol(F)$ on $\mathcal{P}^*(F)$ does not have the cocycle property.*

PROOF: We shall use the notation which was introduced in the proof of Theorem 1. Suppose that the Borel cocycle α witnesses that the action $(Hol(F), \mathcal{P}^*(F))$ has the cocycle property. Consider the shift action of G on $\mathcal{P}^*(G)$. Then whenever S, $T \in \mathcal{P}^*(G)$ lie in the same G-orbit, the cocycle α selects an element $\pi = \alpha(\widehat{S}, \widehat{T}) \in Hol(F)$ such that $\widehat{S}^\pi = \widehat{T}$. Now the proof of Theorem 1 shows that $\pi = g^\rho$ for some element $g \in G$ such that $Sg = T$. But then the Borel function β defined by

$$\beta(S,T) = \alpha(\widehat{S}, \widehat{T})$$

witnesses that the action $(G, \mathcal{P}^*(G))$ has the cocycle property, which is a contradiction.

□

Theorem 6 *Let σ be the theory of finitely generated groups. Then the logic action of S_∞ on $Mod(\sigma)$ does not have the cocycle property.*

PROOF: We shall use the notation which was introduced in the proof of Theorem 2. Suppose that the Borel cocycle α witnesses that the logic action of S_∞ on $Mod(\sigma)$ has the cocycle property. For each $S \in \mathcal{P}^*(F)$, there exists an element $\mathcal{M}_S \in Mod(\sigma)$ and a canonical isomorphism $\pi_S : G_S \to \mathcal{M}_S$. Thus whenever $G_S \cong G_T$, then the map

$$\beta(G_S, G_T) = \pi_S \alpha(\mathcal{M}_S, \mathcal{M}_T) \pi_T^{-1}$$

is an isomorphism from G_S onto G_T. Furthermore, if $G_S \cong G_T \cong G_U$, then

$$\beta(G_S, G_U) = \beta(G_S, G_T)\beta(G_T, G_U).$$

Now identify F with the subset $\mathcal{B} = \{\widetilde{w} : w \in F\}$; and define γ by

$$\gamma(S,T) = \beta(G_S, G_T) \restriction \mathcal{B} \in Sym(F).$$

Then the proof of Theorem 2 shows that $\gamma(S,T)$ is an element of $Hol(F)$ such that $S^{\gamma(S,T)} = T$. So γ witnesses that the action of $Hol(F)$ on $\mathcal{P}^*(F)$ has the cocycle property. But this contradicts Lemma 8.

□

Combining Theorems 4 and 6, it follows that the theory of finitely generated groups does not admit canonical models.

4 Conjugacy of subgroups

In this section, we shall show that another natural action of the free group F_2 gives rise to a universal countable Borel equivalence relation. For any group G, let $Sg(G)$ denote the set of all subgroups of G. If G is countable, then $Sg(G)$ is a closed subset of $\mathcal{P}(G)$ with the product topology and thus is a Polish space. Let $(G, Sg(G))$ denote the conjugacy action of G on $Sg(G)$. In [SZ, Lemma 3.9], Stuck and Zimmer proved that the Borel equivalence relation $E(F_2, Sg(F_2))$ is not smooth. The main theorem of this section is the following strengthening of their result.

Theorem 7 *$E(F_2, Sg(F_2))$ is a universal countable Borel equivalence relation.*

We begin by making some easy reductions of the problem. For each $n \geq 1$, F_n denotes the free group on n generators.

Definition 4 *A subgroup H of a group G is said to be* malnormal *iff whenever $g \in G \setminus H$, then $g^{-1}Hg \cap H = \{1\}$.*

Proposition 1 *$E(F_n, Sg(F_n))$ is Borel reducible to $E(F_2, Sg(F_2))$ for each $n \geq 1$.*

PROOF: By [KS, p.950], the set $\{a^\ell b^\ell a^\ell : \ell \geq 1\}$ freely generates a malnormal subgroup T of the free group $F = \langle a, b\rangle$. For each $\ell \geq 1$, let $x_\ell = a^\ell b^\ell a^\ell$. Then $G = \langle x_1, \dots x_n\rangle$ is clearly a malnormal subgroup of T; and hence G is also a malnormal subgroup of F.

Suppose that $H_1, H_2 \in Sg(G) \setminus \{1\}$ and that $g \in F$ satisfies $g^{-1}H_1 g = H_2$. Then $g^{-1}Gg \cap G \neq \{1\}$, and so $g \in G$. Thus $E(G, Sg(G))$ is Borel reducible to $E(F, Sg(F))$.

□

Let $(Aut(F_5), Sg(F_5))$ denote the natural action of $Aut(F_5)$ on $Sg(F_5)$. Most of our effort will go into proving the following result.

Theorem 8 *$E(Aut(F_5), Sg(F_5))$ is a universal countable Borel equivalence relation.*

Before proving Theorem 8, we shall use it to complete the proof of Theorem 7. First note that there is a natural embedding $F_5 \hookrightarrow Aut(F_5)$, obtained by sending each $w \in F_5$ to the corresponding inner automorphism $i_w \in Inn(F_5)$. Identify F_5 with $Inn(F_5)$ via this natural embedding. Then $F_5 \trianglelefteq Aut(F_5)$ and the natural action of $Aut(F_5)$ becomes identified with the conjugacy action of $Aut(F_5)$ on $Inn(F_5)$, since $\pi^{-1} i_w \pi = i_{w^\pi}$ for $w \in F_5$, $\pi \in Aut(F_5)$. Thus we can regard $(Aut(F_5), Sg(F_5))$ as a subaction of the conjugation action $(Aut(F_5), Sg(Aut(F_5)))$. Now recall that $Aut(F_5)$ is a finitely presented group, say arising from the surjective homomorphism

$$\varphi : F_n \twoheadrightarrow Aut(F_5).$$

For each subgroup H of $Aut(F_5)$, let $\widetilde{H} = \varphi^{-1}[H]$. Note that if H_1, H_2 are subgroups of $Aut(F_5)$ then the following statements are equivalent.

(a) There exists $\pi \in Aut(F_5)$ such that $\pi^{-1} H_1 \pi = H_2$.

(b) There exists $w \in F_n$ such that $w^{-1} \widetilde{H_1} w = \widetilde{H_2}$.

Thus $E(Aut(F_5), Sg(F_5))$ is Borel reducible to $E(F_n, Sg(F_n))$; and applying Proposition 1, we obtain Theorem 7.

Now we turn to the proof of Theorem 8. Let $F = \langle a, b \rangle$; and for each $S \in \mathcal{P}^*(F)$, let G_S be the group which was defined in §3. Note that G_S has the following presentation.

- **Generators**: $\widetilde{1}, a, b, t, z$, which we regard as generators of F_5.

- **Relators**:

  - $w^{-1}\widetilde{\widetilde{1}1}w$, for $w \in F$;
  - $[w^{-1}\widetilde{1}w, u^{-1}\widetilde{1}u]$, for $w, u \in F$;
  - $[t, w^{-1}\widetilde{1}w]$, for $w \in S$;
  - $[z, t]$
  - $[z, w^{-1}\widetilde{1}w]$, for $w \in F$.

For each $S \in \mathcal{P}^*(F)$, let N_S be the normal subgroup of F_5, *generated as a normal subgroup* by the relators for G_S. Thus $G_S \cong F_5/N_S$. Theorem 8 is an immediate consequence of the following result.

Lemma 9 *For $S, T \in \mathcal{P}^*(F)$, the following statements are equivalent:*

(a) There exists $\pi \in Aut(F_5)$ such that $N_S^\pi = N_T$.

(b) $G_S \cong G_T$.

PROOF: (a)$\Rightarrow$ (b) is clear, so we concentrate on proving (b)$\Rightarrow$ (a). Notice first that the proof in §3 shows that there exists $\varphi = \theta x^\lambda \in Hol(F)$, where $x \in F$ and $\theta \in Aut(F)$ such that $S^\varphi = T$. Define a homomorphism $\pi : F_5 \to F_5$ by:

- $a \mapsto a^\theta$
- $b \mapsto b^\theta$
- $t \mapsto t$
- $z \mapsto z$
- $\widetilde{1} \mapsto x^{-1}\widetilde{1}x$.

Notice that $\{a^\theta, b^\theta, t, z, x^{-1}\widetilde{1}x\}$ generates F_5 and hence is a basis of F_5. Thus $\pi \in Aut(F_5)$. It only remains to check that $N_S^\pi = N_T$. So we consider the action of π on each normal generator of N_S. First,

$$w^{-1}\widetilde{\widetilde{11}}w \mapsto (w^\theta)^{-1}x^{-1}\widetilde{\widetilde{11}}xw^\theta.$$

So π merely permutes these relators. Similarly π permutes the relators of the form $[w^{-1}\widetilde{1}w, u^{-1}\widetilde{1}u]$, and the relators of the form $[z, w^{-1}\widetilde{1}w]$; and fixes the relator $[z,t]$. Finally we see that

$$[t, w^{-1}\widetilde{1}w] \mapsto [t, (w^\theta)^{-1}x^{-1}\widetilde{1}xw^\theta] = [t, (xw^\theta)^{-1}\widetilde{1}xw^\theta],$$

and $T = \{xw^\theta : w \in S\}$. Hence $N_S^\pi = N_T$.

$\square$

The methods of this section clearly yield the following result.

Theorem 9 *Let G be a countable group. If G has a malnormal free nonabelian subgroup, then $E(G, Sg(G))$ is a universal countable Borel equivalence relation.*

In particular, if G is an arbitrary countable nonabelian free group, then $E(G, Sg(G))$ is a universal countable Borel equivalence relation. We hope to give some less trivial applications of Theorem 9 in a later paper.

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