DIMACS Technical Report 98-04
February 1998

Measurements of Uncolorability

by

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[1]supported by Deutsche Forschungsgemeinschaft

DIMACS is a partnership of Rutgers University, Princeton University, AT&T Labs, Bellcore, and Bell Labs.

DIMACS is an NSF Science and Technology Center, funded under contract STC–91–19999; and also receives support from the New Jersey Commission on Science and Technology.

ABSTRACT

Cubic bridgeless graphs with chromatic index four are called uncolorable. We introduce parameters measuring the uncolorability of those graphs and relate them to each other.

For $k = 2, 3$ let c_i be the maximum size of a k-colorable subgraph of a cubic graph $G = (V, E)$. We consider $r_3 = |E| - c_3$ and $r_2 = \frac{2}{3}|E| - c_2$. We show that on one side r_3 and r_2 bound each other, but on the other side that the difference between them can be arbitrarily large.

We also compare them to the oddness ω of G, the smallest possible number of odd circuits in a 2-factor of G. We construct cyclically 5-edge connected graphs where r_3 and ω are arbitrarily far apart, and show that for each $1 \leq c < 2$ there is a snark such that $\omega \geq cr_3$.

For $k = 2, 3$ let ζ_k denote the largest fraction of edges that can be k-colored. We give best possible bounds for these parameters, and relate them to each other.

1 Introduction

This paper considers connected graphs. The *chromatic index* χ' of a graph is the minimum number of colors needed to color its edges so that adjacent edges receive different colors. Johnson [6] proved that the chromatic index of a cubic multigraph (no loops) is either 3 or 4. The latter ones are also called *uncolorable* graphs while the others are called *colorable*, cf. [5]. We define a *snark* to be a bridgeless uncolorable graph.

This paper deals with parameters measuring their uncolorability. Let G be a graph with maximum vertex degree 3. We define
$c_3(G) := max\{|E(H)||H$ is subgraph of G and $\chi'(H) = 3\}$, $r_3(G) := |E| - c_3(G)$ and $\zeta_3(G) := \frac{c_3(G)}{|E(G)|}$. Parameter r_3 will be called the *resistancy* of G. (In [8], $r_3(G)$ is called *color number* and denoted by $c(G)$. But resistancy seems to be more intuitive.)

Obviously, $r_3(G) = 0$ and $\zeta_3(G) = 1$ if and only if G is 3-colorable, and if G is uncolorable, then $r_3(G) \geq 2$ (cf. [8]) and $\zeta_3(G) < 1$. Note that an uncolorable graph is cubic.

Throughout this paper the set of colors is $\mathcal{K} = \{0, 1, 2, 3\}$, and considering a fixed 4-coloring γ of a graph we will always assume that $\gamma^{-1}(0)$ is a smallest color class.

A 4-coloring having a color class of cardinality r_3 is called a *minimal* 4-coloring. It is easy to see that $\gamma^{-1}(0)$ of a minimal 4-coloring γ is the union of three pairwise disjoint sets A_1, A_2 and A_3, with $e \in A_i$ if and only if exactly two of the edges adjacent to e are colored i $(i = 1, 2, 3)$.

Lemma 1.1 ([8]) *If γ is a minimal 4-coloring of an uncolorable graph G with minimum color class $\gamma^{-1}(0) = A_1 \cup A_2 \cup A_3$, then $|A_1| \equiv |A_2| \equiv |A_3| \equiv r_3(G) \pmod 2$.*

Let γ be a minimal 4-coloring of a snark and $e = vw \in \gamma^{-1}(0)$, say $e \in A_1$. Then there is path with start vertex v and whose edges are colored 2 and 3 alternately. In [8] it is shown that this path ends in w. Hence e is contained in a circuit C_e of odd length, and we say that e *admits* C_e. The following lemma implies that each minimal 4-coloring of a snark induces a subgraph consisting of r_3 pairwise disjoint odd circuits.

Lemma 1.2 ([8]) *If e and f are two different edges of a minimum color class of an uncolorable graph, then C_e and C_f are disjoint.*

This rises the question about the relation of the resistancy r_3 to the *oddness* ω of a cubic graph G, which is the smallest possible number of odd circuits in a 2-factor of G. A 2-factor of G with precisely $\omega(G)$ odd components is called *minimum*.

This parameter, which is a structural term, also measures the uncolorability of G, since $\omega(G) = 0$ if and only if G is colorable, and if G is uncolorable, then $\omega(G) \geq 2$.

Let $\mathcal{F}$ be a 2-factor of G. A 4-coloring of G that colors precisely one edge of each odd component of $\mathcal{F}$ with color 0 and the edges of the complementary 1-factor with color 1 is called *corresponding to* $\mathcal{F}$ and denoted by $\phi_{\mathcal{F}}$. Note that $\phi_{\mathcal{F}}^{-1}(0) = A_1$.

For any minimum 2-factor $\mathcal{F}$ of G we have $|\phi_{\mathcal{F}}^{-1}(0)| = \omega(G)$ and therefore $r_3(G) \leq \omega(G)$. Furthermore it is well known that $r_3(G) = 2$ if and only if $\omega(G) = 2$, cf. [8].

In Section 2 we prove that these two parameters can be arbitrarily far apart even in cyclically 5-edge connected snarks. We also show that there is no constant c such that $1 \leq c < 2$ and $\omega(G) \leq cr_3(G)$ for all snarks G.

In Section 3 we consider ζ_3. An upper bound for ζ_3 is 1, and it cannot be replaced by a smaller number, since for any even number n there is a snark on n vertices $r_3 = 2$, see Thm. 2.9 [8]. We also give new lower bounds for ζ_3 which improve previous known bounds and show that they are best possible.

While Section 3 deals with 3-colorable subgraphs of cubic bridgeless graphs, Section 4 deals with 2-colorable subgraphs. Let G be a graph with maximum vertex degree 3. We define $c_2(G) := max\{|E(H)||H$ is subgraph of G and $\chi'(H) = 2\}$, $\zeta_2(G) := \frac{c_2(G)}{|E(G)|}$, and $r_2(G) = \frac{2}{3}|E(G)| - c_2(G)$.

Parameters r_2, ζ_2 play the same role with respect to 2-colorable subgraphs of a cubic graph G as r_3, ζ_3 with respect to 3-colorable. It holds $r_2(G) = 0$ and $\zeta_2(G) = \frac{2}{3}$ if and only if G is 3-colorable, and if G is uncolorable, then $r_2(G) \geq 1$ and $\zeta_2(G) < \frac{2}{3}$. Thus these parameters measure the uncolorability of G, too.

We prove that $r_2 = 1$ if and only if $r_3 = \omega = 2$, and that r_2 and r_3 bound each other. It follows $r_2 < r_3 \leq \omega$. Consequently parameters r_2 and ω and also r_2 and r_3 can be arbitrarily far apart. From the results for r_3 we deduce that there is no constant $1 \leq c < 4$ such that $\omega \leq cr_3$ for all snarks.

We prove that ζ_3 and ζ_2 bound each other and solve the problem to determine the best possible lower bound for ζ_2. We close with relating our results to recent results on the edge chromatic difference sequence of cubic graphs.

In Section 5 we construct some classes of snarks to which we will refer several times.

2 The Relation Between Resistancy and Oddness

We start with generalizing concepts due to Kochol [7]. A *multipole* $M = (V, E, S)$ consists of a set of vertices $V = V(M)$, a set of edges $E = E(M)$ and a set of semiedges $S = S(M)$. Each semiedge is incident to precisely one vertex v and denoted by (v) if there is no possibility of confusion. We say that M' is obtained from M by *identifying semiedges* (v) and (w) if these two semiedges are replaced by an edge vw.

If $|S(M)| = k$, then M is also called a *k-pole*, and it is called a $(k_1, k_2, \ldots, k_n)$*-pole* if $S(M)$ is the union of pairwise disjoint sets $S_1, \ldots, S_n$ and $|S_i| = k_i$, for $i = 1, \ldots, n$. We also write $M = (V, E, S_1, S_2, \ldots, S_n)$ in the latter case. All multipoles we consider are cubic, that is each vertex is incident to precisely three edges or semiedges. Let G be a cubic graph and $X \subseteq V(G)$ a set of two (or three) vertices v_1, v_2 (or v_1, v_2, v_3) of G. Then G_X denotes the cubic $(3,3)$-pole (V, E, S_1, S_2) (or $(3,3,3)$-pole (V, E, S_1, S_2, S_3)) obtained from $G - X$ by adding semiedges S_i to the neighbors of v_i, $i = 1, 2$ (or $= 1, 2, 3$).

A *4-coloring* of $M = (V, E, S)$ is a mapping ϕ from $E \cup S$ into $\mathcal{K}$ such that

- $\phi(e_1) \neq \phi(e_2)$ for all adjacent (semi-) edges e_1, e_2, and

- $\phi(e) \neq 0$ for all semiedges $e \in S$.

To obtain a group structure on $\mathcal{K}$ we define $0+i=i+0=i$, $i+i=0$, for each $i \in \mathcal{K}$, and $1+2=2+1=3$, $1+3=3+1=2$, $2+3=3+2=1$. Using the addition on $\mathcal{K}$ we define for any $X \subseteq S$

$$\overline{\phi}[X] = \sum_{e \in X} \phi(e).$$

A 4-coloring ϕ of M is called 3-*coloring* if $\phi^{-1}(0) = \emptyset$.

Lemma 2.1 **([7])** *If $M = (V, E, S)$ is a multipole and ϕ a 3-coloring of M, then $\overline{\phi}[S] = 0$.*

A 4-coloring ϕ of a multipole M is *locally forced* if for each $e \in \phi^{-1}(0)$ there is precisely one $i \in \{1, 2, 3\}$ such that $e \in A_i$. A 4-coloring ϕ of M is *minimal* if $|\phi^{-1}(0)| \leq |\gamma^{-1}(0)|$ for any other 4-coloring γ of M. Clearly, each minimal 4-coloring is locally forced, but not vice versa.

Let ϕ be a locally forced 4-coloring of M and $e \in \phi^{-1}(0) = A_1 \cup A_2 \cup A_3$. For $i = 1, 2, 3$ we define

$$\overline{\phi}(e) = i \text{ if } e \in A_i.$$

Let G be a graph and $X \subset V(G)$, then $\partial(X)$ denotes the set of edges with exactly one end in X. The following lemma generalizes Lemmas 1.1 and 2.1.

Lemma 2.2 *Let $M = (V, E, S)$ be a multipole or a subgraph of a snark with $S = \partial(M)$. If ϕ is a locally forced 4-coloring of M and $S \cap \phi^{-1}(0) = \emptyset$, then*

$$\overline{\phi}[S] + \sum_{e \in \phi^{-1}(0)} \overline{\phi}(e) = 0.$$

Proof. It suffices to prove the statement for multipoles M, since, if M is subgraph of a cubic graph, then cutting the edges of S yields the colored multipole $M = (V(M), E(M), S)$.

Obtain a new multipole M' from M by subdividing each edge e of $\phi^{-1}(0)$ by a vertex v_e and adding semiedge (v_e). Coloring ϕ on M can be modified to a 3-coloring ϕ' on M' such that $\phi'((v_e)) = \overline{\phi}(e)$. Now the statement follows from Lemma 2.1.

•

If G has maximum vertex degree k ($k \geq 2$), $\chi'(G) = k + 1$ and $\chi'(G - e) \leq k$ for each edge $e \in G$, then G is k-*critical.*

Theorem 2.3 *For all $k \geq 1$ there is a cyclically 5-edge connected snark G_k such that $r_3(G_k) = 6k$ and $\omega(G_k) \geq 8k$.*

Proof. Let H and H' be two 3-critical graphs. It is well known that they have divalent vertices. Let $e = vw$ be an edge of H and u a divalent vertex of H', with neighbors u_1, u_2. The *Hajós-union* of H and H' is the graph G obtained from $H - e$ and $H' - u$ by adding edges vu_1 and wu_2. It is well known that G is 3-critical, cf. [9] Thm. 4.6.

Let P_1, P_2, P_3 be copies of the Petersen graph minus a vertex. These graphs are 3-critical, hamiltonian and P_i has three divalent vertices u_i, v_i, w_i $(i = 1, 2, 3)$. Let e_i be an edge of an hamiltonian circuit in P_i, $i = 1, 2, 3$.

Let P' be obtained by applying Hajós-union on $P_1 - e_1$ and $P_2 - u_2$. Graph P' is 3-critical. Let T be obtained by applying Hajós-union on $P' - v_2$ and $P_3 - e_3$. Graph T is 3-critical, hamiltonian and has seven divalent vertices, namely u_1, v_1, w_1, u_3, v_3, w_3, and w_2. It is shown in Figure 1. Add a semiedge to each of the divalent vertices to obtain the cubic multipole T'.

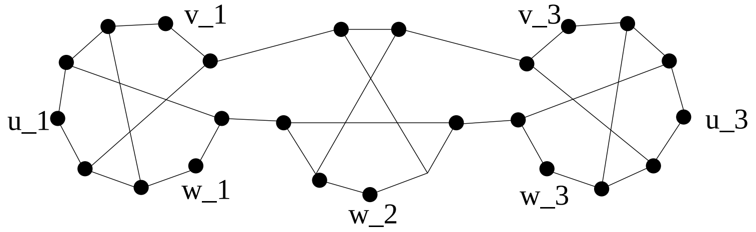

Figure 1. T

There are minimal 4-colorings γ_1, γ_2, and γ_3 of T' such that $\gamma_1(e) = 1$ for each semiedge e, $\gamma_2((w_2)) = 2$ and $\gamma_2(e) = 1$ for each semiedge $e \neq (w_2)$, and $\gamma_3((w_2)) = 3$ and $\gamma_3(e) = 1$ for each semiedge $e \neq (w_2)$.

Let G be the graph obtained from three copies T^1, T^2, T^3 of T' by adding a new vertex x with three semiedges (x_1), (x_2), (x_3) and identifying identically colored semiedges to edges $u_1^1u_1^2$, $v_1^1v_1^2$, $w_1^1w_1^2$, $u_3^1u_3^3$, $v_3^1v_3^3$, $w_3^1w_3^3$, $u_3^2u_1^3$, $v_3^2v_1^3$, $w_3^2w_1^3$ and $w_2^1x_1$, $w_2^2x_2$, $w_2^3x_3$. Colorings γ_1, γ_2, and γ_3 on T^1, T^2 and T^3, respectively, extend to a 4-coloring ϕ on G and henceforth $r_3(G) \leq 3$. Since G contains three pairwise disjoint 3-critical subgraphs, it follows that $r_3(G) = 3$ and ϕ is a minimal 4-coloring of G. We may assume that $\phi^{-1}(0) = \{e_1, e_2, e_3\}$ with $e_1 = ab \in T^1$, $e_2 = vw \in T^2$ and $e_3 = xy \in T^3$. Furthermore, G is cyclically 5-edge connected.

Take four copies M^j of G with fixed minimal 4-coloring ϕ^j and ${\phi^j}^{-1}(0) = \{a_jb_j, v_jw_j, x_jy_j\}$, $j = 0, \ldots, 3$. Then $M^0_{\{a_0,v_0,x_0\}}$ is 3-colorable and $\overline{\phi^0}[S_i] = i$, for $i = 1, 2, 3$. By renaming the colors we may assume that there is a minimal 4-coloring ψ^i of $M^i_{\{a_i\}} = (V, E, S^i)$ such that $\overline{\psi^i}[S^i] = i$, for $i = 1, 2, 3$.

We may further assume that the three semiedges in S_i receive the same colors as the three semiedges in S^i, $i = 1, 2, 3$.

Obtain a new cubic graph B' by identifying identically colored semiedges of S_i and S^i to the set D_i of three edges, for $i = 1, 2, 3$. Per construction colorings ϕ^0 and ψ^j $(j = 1, 2, 3)$ combine to a 4-coloring γ' on B' with ${\gamma'}^{-1}(0) = \{v_jw_j, x_jy_j | j = 1, 2, 3\}$. Since B' contains six pairwise disjoint 3-critical subgraphs it follows that γ' is a minimal 4-coloring, i.e. $r_3(B') = 6$. Recall that each of the $M^i_{\{a_i\}}$ contains two 3-critical subgraphs.

Let $B = B' - \{v_1, v_2, v_3\}$ and γ be the restriction of γ' on B. Then follows that $\gamma^{-1}(0) = \{x_1y_1, x_2y_2, x_3y_3\}$. The shape of B' and B is shown in Figure 2.

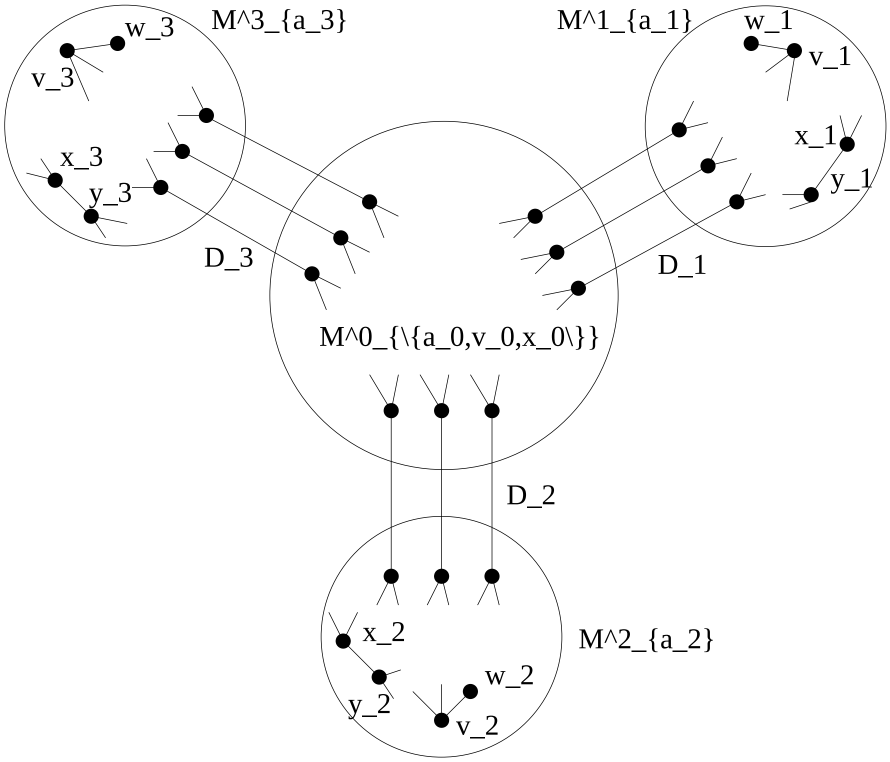

Figure 2. B', $(B = B' - \{v_1, v_2, v_3\})$

Claim 2.3.1 *If B is subgraph of a cubic graph G, then B contains at least four odd components of any 2-factor $\mathcal{F}$ of G.*

Proof. Let $\mathcal{F}$ be a 2-factor of G. Then $\phi_{\mathcal{F}}$ is a locally forced 4-coloring of G. We may assume that the edges of $D_1 \cup D_2 \cup D_3$ are colored 1, 2 or 3. For $i = 1, 2, 3$, let L_i be the unique set of three edges such that $\partial(V(M^i - \{a_i, v_i\})) = D_i \cup L_i$. For $i \in \{1, 2, 3\}$ we show:

(a) If $\overline{\phi_{\mathcal{F}}}[D_i] = 0$, then $G[M^i - \{a_i, v_i\}]$ contains at least three odd components of $\mathcal{F}$.

Assume to the contrary that it contains less than three components.

If $G[M^i - \{a_i, v_i\}]$ contains 0 or 2 odd components, then follows from Lemma 2.2 and $\phi_{\mathcal{F}}^{-1}(0) = A_1$ that $\overline{\phi_{\mathcal{F}}}[L_i] = 0$.

If $G[M^i - \{a_i, v_i\}]$ contains precisely one odd component of $\mathcal{F}$, then follows again from Lemma 2.2 and from $\phi_{\mathcal{F}}^{-1}(0) = A_1$ that $\overline{\phi_{\mathcal{F}}}[L_i] = 1$.

In both cases we can obtain a 4-coloring of M^i, that colors less than three edges 0, contradicting the fact that $r_3(M^i) = 3$.

The following two statement are proved similarly:

(b) If $\overline{\phi_{\mathcal{F}}}[D_i] = 1$, then $G[M^i - \{a_i, v_i\}]$ contains at least two odd components of $\mathcal{F}$, and
(c) If $\overline{\phi_{\mathcal{F}}}[D_i] \in \{2, 3\}$, then $G[M^i_{\{a_i,v_i\}}]$ contains at least one odd components of $\mathcal{F}$.

Now consider $M' = G[V(M^0) \setminus \{a_0, v_0, x_0\}]$.

If M' contains an even number of odd components of $\mathcal{F}$, then follows from Lemma 2.2 that $\overline{\phi_{\mathcal{F}}}[D_1] + \overline{\phi_{\mathcal{F}}}[D_2] + \overline{\phi_{\mathcal{F}}}[D_3] = 0$. Thus either w.l.o.g. $\overline{\phi_{\mathcal{F}}}[D_1] = 0$ and $\overline{\phi_{\mathcal{F}}}[D_2] = \overline{\phi_{\mathcal{F}}}[D_3]$ or w.l.o.g. $\overline{\phi_{\mathcal{F}}}[D_i] = i$ for $i = 1, 2, 3$. In any case follows from (a), (b) and (c) that B contains at least four odd components of $\mathcal{F}$.

If M' contains an odd number of odd components of $\mathcal{F}$, then follows from Lemma 2.2 that w.l.o.g. $\overline{\phi_{\mathcal{F}}}[D_1] = 1$ and $\overline{\phi_{\mathcal{F}}}[D_2] = \overline{\phi_{\mathcal{F}}}[D_3]$. As above follows from (a), (b) and (c) that B contains at least four odd components of $\mathcal{F}$. •

Take two copies F and F' of $B'_{\{v_1,v_2,v_3\}}$ (which is colored by γ') with sets of semiedges S_1, S_2, S_3 and S'_1, S'_2, S'_3, respectively. Identify identically colored edges of S_i and S'_i for $i = 2, 3$ to obtain multipole B^* with two sets of semiedges $S = S_1$ and $S' = S'_1$.

Let ϕ^* be the 4-coloring of B^* obtained from the 4-colorings of F and F'. Then follows $|(\phi^*)^{-1}(0)| = 6$ and the colorings on S and S' match.

Let $k \geq 1$, and for $1 \leq i \leq k$ let B^*_i be a copy of the colored B^* with sets of semiedges S_i and S'_i. Identify identically colored semiedges of S_i and S'_{i+1}, $1 \leq i \leq k$, and S_k with S_1. This yields a cyclically 5-edge connected graph G_k.

The combination of the colorings on the B^*_i's yields a 4-coloring γ^* on G_k with $|{\gamma^*}^{-1}(0)| \leq 6k$, and hence $r_3(G_k) \leq 6k$. On the other hand G_k contains $2k$ pairwise disjoint copies of B as a subgraph. Consequently it contains $6k$ pairwise disjoint 3-critical subgraphs and hence $r_3(G_k) = 6k$. Together with Claim 2.3.1 follows that $\omega(G_k) \geq 8k$.

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Theorem 2.3 shows that the difference between ω and r_3 can be arbitrarily large. The next statement is somewhat stronger.

Theorem 2.4 *There is no constant c such that $1 \leq c < 2$ and $\omega(G) \leq cr_3(G)$ for all snarks G.*

Proof. Let O_1 be as defined in Theorem 5.3 and ϕ a minimal 4-coloring of it. For $k \geq 2$ take k copies $H_1, \ldots, H_k$ of the minimally colored O_1. For $2 \leq j \leq k-1$ take $H_j - \{e_j, f_j\}$, where $e_j \in A_1$ and $f_j \in A_2$. Remove an edge $e_1 \in A_1$ from H_1 and an edge e_k from H_k, where $e_k \in A_1$ if k is even, and $e_k \in A_2$ otherwise. Let a_j, b_j and x_j, y_j be the divalent vertices in H_j obtained by removing e_j and f_j, respectively. For $j = 1, \ldots, k-1$ add two edges between a_j, b_j and a_{j+1}, b_{j+1} and between x_j, y_j and x_{j+1}, y_{j+1}, such that the colorings extend. Let G_k be this graph. Per definition we have $r_3(G_k) \leq k+2$ and since it contains $k+2$ copies of $P - v$ it follows $r_3(G_k) = k+2$.

Claim 2.4.1 *If H is a 3-critical subgraph and $\mathcal{F}$ a 2-factor of a snark G, then there is a odd circuit of $C \in \mathcal{F}$ which is subgraph of H.*

Proof. Assume to the contrary that there is a 3-critical subgraph H and a 2-factor $\mathcal{F}$ of G such that for any odd circuit of $\mathcal{F}$ there is an edge e_C of C which is not an edge of H. Fix edges e_C. Then there is a 4-coloring ϕ of G such that $\phi^{-1}(0) = \{e_C | C \text{ is an odd circuit of } \mathcal{F}\}$. Hence $E(H) \subseteq \phi^{-1}(1) \cup \phi^{-1}(2) \cup \phi^{-1}(3)$, contradicting the fact that H is not 3-colorable. •

Let $\mathcal{F}$ be a 2-factor of G_k, and D_i denote the set of the two edges between $G[V(H_i)]$ and $G[V(H_{i+1})]$, $i = 1, \ldots, k-1$. Let $G^j = G[\bigcup_{i=1}^{j} V(H_i)]$, for $j = 1, \ldots, k$. That is, $G^k = G_k$ and if $j < k$, then G^j is a 2-pole obtained from G_k be cutting the edges of D_j. The set of the two arising semiedges is denoted by D'_j in the following.

We claim that $r_3(G^j) = j + 1$ and G^j contains at least $2j + 1$ odd components of $\mathcal{F}$, and $\overline{\phi_{\mathcal{F}}}[D'_j] = \overline{\phi_{\mathcal{F}}}[D_j] \in \{0, 1\}$, $j = 1, \ldots, k-1$.

($j = 1$) Per construction we have $r_3(G^1) = 2$. Assume to the contrary that G^1 contains less than 3 odd components of $\mathcal{F}$. Consider the restriction of $\phi_{\mathcal{F}}$ on G^1. Adding an edge between the two divalent vertices and coloring it consistently (which is certainly possible if we choose an appropriate 4-coloring $\phi_{\mathcal{F}}$) yields either a minimal 4-coloring ϕ of H_1 with $|\phi^{-1}(0)| < 3$, or a minimal 4-coloring ϕ of H with $|\phi^{-1}(0)| = 3$ and $\sum_{e \in \phi^{-1}(0)} \overline{\phi}(e) \neq 0$, contradicting $r_3(H_1) = 3$ or Lemma 2.2. Thus G^1 contains at least three components of $\mathcal{F}$.

This and Lemma 2.2 imply that $\overline{\phi_{\mathcal{F}}}[D'_1]$ equals 0 or 1, according to whether the number of components of $\mathcal{F}$ in G^1 is even or odd.

($1 \leq j < k$) We consider G_{j+1}. Per construction we have $r_3(G^{j+1}) = j + 2$. Now it suffices to show that $G_k[V(H_{j+1})]$ contains at least two odd components of $\mathcal{F}$. Since this implies that G^{j+1} contains at least $2j + 1 + 2 = 2(j + 1) + 1$ odd components of $\mathcal{F}$.

Since $G_k[V(H_{j+1})]$ contains a 3-critical subgraph it follows from Claim 2.4.1 that it contains at least one odd component of $\mathcal{F}$. Assume that it contains precisely one odd component, and recall that $\phi_{\mathcal{F}}^{-1}(0) = A_1$.

If $\overline{\phi_{\mathcal{F}}}[D_j] = 0$, then follows from Lemma 2.2 that $\overline{\phi_{\mathcal{F}}}[D'_{j+1}] = 1$. Hence we can obtain a coloring of O_1 with a minimum color class of two edges, a contradiction.

If $\overline{\phi_{\mathcal{F}}}[D_j] = 1$, then follows again from Lemma 2.2 that $\overline{\phi_{\mathcal{F}}}[D'_{j+1}] = 0$, yielding a contradiction as above. Thus $G_k[V(H_{j+1})]$ contains at least two odd components of $\mathcal{F}$.

As above we infer that $\overline{\phi_{\mathcal{F}}}[D_{j+1}] \in \{0, 1\}$, if $j < k - 1$

From this follows $\omega(G_k) \geq 2k + 1$. Hence $\lim_{k \to \infty} \frac{\omega(G_k)}{r_3(G_k)} = 2$, and the statement is proved. •

Theorems 2.3 and 2.4 motivate us to state the following problem.

Problem 2.5 *Does there exist a function f such that $\omega(G) \leq f(r_3(G))$ for each bridgeless cubic graph G.*

3 Maximum 3-Colorable Subgraphs

Albertson and Haas [2] proved $\zeta_3 \geq \frac{13}{15}$, and this is a best possible bound because it is achieved by the Petersen graph. However they also mentioned that they do not know an infinite family of snarks achieving this bound. Indeed, it follows from our results that the Petersen graph is the only snark with $\zeta_3 = \frac{13}{15}$. We are going to improve this bound. Let G be a bridgeless cubic graph. We define

$$g^+(G) = \max_{\mathcal{F} \text{ is 2-factor of } G} \min\{|V(C)||C \text{ is an odd circuit in } \mathcal{F}\}.$$

Lemma 3.1 *If G is a snark, then $g^+(G) \geq 5$, and the bound is best possible.*

Proof. This is true for the Petersen graph which is the smallest snark. Assume to the contrary that there is a counter-example to this assertion. Then there is a smallest one, say H, which certainly contains a triangle with vertices u, v, w. Contract the triangle to a single vertex z to obtain a smaller snark H'. Clearly, $g^+(H') \geq 5$.

If H' is simple, then there is a 2-factor $\mathcal{F}$ where the minimum length of an odd circuit is at least 5. Expanding z to a triangle yields H and a 2-factor of H where the minimum length of on odd circuit is at least 5.

If H' is not simple, then v, and w have another common neighbor different from u, say x. Since H is not K_4 and bridgeless, contracting the triangle to a single vertex z yields a snark H' where x and u are connected by two edges. Since $g^+(H') \geq 5$ expanding z to a triangle yields also a 2-factor of H where the minimum length of an odd circuit is at least 5.

Theorem 5.3 shows that there are infinitely many 3-connected snarks with $g^+ = 5$. Thus the bound is best possible.

•

The permutation snarks of Theorem 5.1 show that g^+ can be arbitrary large.

All snarks with 10, 12 or 14 vertices have $r_3 = 2$ and hence $\zeta_3 \in \{\frac{13}{15}, \frac{8}{9}, \frac{19}{21}\}$ in these cases. The following theorem shows that for all but these few graphs a better bound holds true.

Theorem 3.2 *If G is a snark with at least 16 vertices, then $1 > \zeta_3(G) \geq \max\{1 - \frac{2}{3g^+(G)}, \frac{11}{12}\}$, and the bounds are best possible.*

Proof. From $|V(G)| \geq r_3(G)g^+(G)$ follows $\zeta_3(G) = 1 - \frac{r_3(G)}{|E(G)|} = 1 - \frac{2r_3(G)}{3|V(G)|} \geq 1 - \frac{2}{3g^+(G)}$.

The lower bound is attained by the permutation snarks of Theorem 5.1. For $k \geq 2$ graph P_k has $4k+2$ vertices, $6k+3$ edges and $r_3(P_k) = 2$. Hence $\zeta_3(P_k) = \frac{|E|-2}{|E|} = 1 - \frac{2}{6k+3} = 1 - \frac{2}{3g^+}$. This also shows that 1 is the best possible upper bound. The bound $\frac{11}{12}$ is achieved by the snarks on 16 vertices, they all have $r_3 = 2$.

We have to show $\zeta_3(G) \geq \frac{11}{12}$ for all snarks with at least 16 vertices. Since $\zeta_3(G) = \frac{|E(G)| - r_3(G)}{|E(G)|} \geq \frac{11}{12}$ if and only if $3|V(G)| \geq 24r_3(G)$ we prove the following lemma.

Lemma 3.3 *If G is a snark with at least 16 vertices, then $|V(G)| \geq 8r_3(G)$.*

Proof. Assume to the contrary that there is a counter-example to the statement. Let $G = (V, E)$ be a counter-example such that $|V| + r_3(G)$ is minimum.

Claim 3.3.1 *G is simple, girth$(G) = 5$, and $r_3(G) \geq 4$.*

Proof. Let v, w be two vertices of G connected by l edges. Since G is uncolorable it follows $l < 3$. If $l = 2$, then follows -since G is bridgeless- that there are two different vertices v', w' such that vv', $ww' \in E(G)$. Let $G' = (G - \{v, w\}) + v'w'$. Clearly, $r_3(G') = r_3(G)$ and $|V(G')| \geq 8r_3(G')$. Hence $|V| \geq 8r_3(G)$, a contradiction.

If G contains a triangle, then contract it to a single vertex v to obtain a snark G'. There is a minimal 4-coloring of G' such that no edge incident to v is colored 0. Thus reconstructing G yields a 4-coloring γ of G with $|\gamma^{-1}(0)| = r_3(G')$ and therefore $r_3(G') = r_3(G)$. This implies $|V(G)| - 2 \geq 8r_3(G)$, a contradiction.

If G contains a circuit C of length 4, then remove the four vertices of C and add two edges e, f to obtain a cubic graph G'. This can be done such that G' is bridgeless and e and f do not lie in a common triangle. Therefore there is minimal 4-coloring ϕ' of G' such that $\phi'(e)$, $\phi'(f) \in \{1, 2, 3\}$. Thus ϕ' is convertible to a 4-coloring ϕ of G with $|\phi^{-1}(0)| = |\phi'^{-1}(0)|$, and hence $r_3(G') = r_3(G)$, which leads to a contradiction as above.

It remains to show that $r_3(G) \geq 4$. All snarks with 10, 12, 14 or 16 vertices have $r_3 = 2$, and it is easy to see that this also holds true for all snarks on at most 22 vertices having a 2- or 3-edge cut. From the results of [3] follows that this is also the case for all cyclically 4-edge connected snarks with girth at least 5 on at most 22 vertices. Assume that H is a minimum snark with $r_3(H) \geq 3$ and $|V(H)| \leq 22$. Then it contains a circuit of length at most 4. By the arguments above it is reducible to a smaller snark H' with $r_3(H') = r_3(H)$, contradicting the fact that H is minimum. Thus $|V(G)| \geq 24$ and hence $r_3(G) \geq 4$. •

Let $\mathcal{C}$ be a set of circuits of G such that there is a minimal 4-coloring γ and $\mathcal{C} = \{C_e | e \in \gamma^{-1}(0)\}$.

For $C \in \mathcal{C}$ let $N(C)$ be set of neighbors of vertices of $V(C)$ in $V(G) \setminus V(C)$, and $M(C) = N(C) \setminus \bigcup_{C \in \mathcal{C}} V(C)$ the set of neighbors of C which are not vertex of an element of $\mathcal{C}$.

Since $\mathcal{C}$ is induced by a minimal 4-coloring γ, and $\gamma^{-1}(0) = A_1 \cup A_2 \cup A_2$, its elements can be classified. We say that $C \in \mathcal{A}_i$ if C is admitted by an edge of A_i, $i = 1, 2, 3$. Clearly, this classification relies on the choice of γ. To keep the notation simple we do not indicate this, and we think that it will not cause any confusion. Note, that two circuits of $\mathcal{C}$ are in the same class if they are connected by an edge. In the following claims we denote a neighbor of $v \in V(C)$ in $N(C)$ with v'.

The *distance* between two vertices v, w of a connected graph H is number of edges of a shortest path between v and w. It is denoted by $dist_H(v, w)$.

Claim 3.3.2 *Let C, C_1, $C_2 \in \mathcal{C}$ be pairwise different and $v, w \in V(C)$. If $v' \in V(C_1)$, $w' \in V(C_2)$, then $dist_C(v, w) \geq 3$.*

Proof. We may assume that C, C_1, $C_2 \in \mathcal{A}_1$.

If $vw \in E(C)$, then there is minimal 4-coloring γ such that the three edges incident to v, v' are colored 0, 1, 2 and the three edges incident to w, w' are colored 0, 1, 3; in particular $\gamma(vw) = 0$ and $\gamma(vv') = \gamma(ww') = 1$. This coloring can be changed as follows. Color vv'

with 3, and ww' with 2. This allows to color vw with 1. Thus the minimum color class of this new 4-coloring has less than $r_3(G)$ elements, a contradiction.

Let $dist_C(v,w) = 2$ and $u \in V(C)$ with $vu,\ wu \in E(C)$. We may assume that the situation is as described in Figure 3, where possibly $x', u' \in V(C)$. Let ϕ be the minimal 4-coloring given there, and note that $girth(G) \geq 5$.

If we change the color of vv' to 2, then we obtain another minimal 4-coloring of G such that $C, C_1 \in \mathcal{A}_2$. Thus there is a path p from x to u whose edges are colored 1 and 3 alternately. Lemma 1.2 implies that p does not contain edge yw and $x' \neq y$, $y' \neq x$.

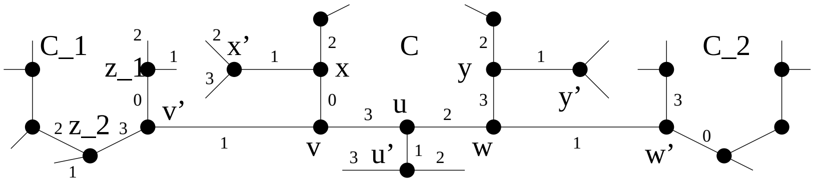

Figure 3.

Now go back to ϕ. Clearly path p is still there. Now change the color of xv to 3, of vu to 2, of uw to 3, of yw to 0 and of ww' to 2. This is a minimal 4-coloring ϕ' of G, too. By obtaining ϕ' from ϕ we did not change the color of any edge of p. Thus p is unchanged between u and x and now it is extended by edges xv, vv', $v'z_2$, etc. Let p' be this new path. But $yw \in A_2$ and hence there is path q from y to w whose edges are colored 1 and 3 alternately. This path contains edge wu and hence p' is a subpath of q. In particular this implies that C'_{yw} and $C'_{v'z_1}$ are not disjoint, contradicting Lemma 1.2. •

By a similar argumentation we prove

Claim 3.3.3 *Let $C, C' \in \mathcal{C}$, $C \neq C'$, $v, w \in V(C)$, and $v' \in V(C')$. If there is $C'' \in \mathcal{C}$ such that $w' \in M(C'')$, then $dist_C(v,w) \geq 2$.*

Proof. Assume to the contrary that the statement is not true. Because of Lemma 1.2 we may assume that the situation is as shown in Figure 4, where G is colored by a minimal 4-coloring γ such that $\mathcal{C}$ coincides with $\{C_e | e \in \gamma^{-1}(0)\}$, $C, C' \in \mathcal{A}_1$ and $C'' \in \mathcal{A}_2$.

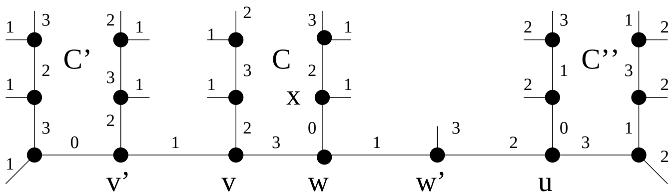

Figure 4.

If we change the color of $v'v$, xw to 3, of vw, uw' to 1, and of ww' to 2, then we obtain another 4-coloring ϕ of G with $|\phi^{-1}(0)| < r_3(G)$, a contradiction. •

In the following we may assume that $\mathcal{C}$ is maximum, that is

$$|\bigcup_{C\in\mathcal{C}} V(C)| = \max_{\phi \text{ is minimal 4-coloring of } G} |\bigcup_{e\in\phi^{-1}(0)} V(C_e)|.$$

And as above we will always make use of the fact that there are minimal 4-colorings which are associated to $\mathcal{C}$.

Claim 3.3.4 *If $v \in M(C_i)$ for three pairwise different circuits C_1, C_2, $C_3 \in \mathcal{C}$, then* $|V(C_1)| + |V(C_2)| + |V(C_3)| \geq 19$.

Proof. For $i = 1, 2, 3$, let $w_i \in V(C_i)$ be the neighbor of v, and x_i, y_i the neighbors of w_i in C_i, and $C_i \in \mathcal{A}_i$. Assume to the contrary that $|V(C_1)| + |V(C_2)| + |V(C_3)| < 19$. Then at least two of the three circuits have length 5, say C_1 and C_2, and C_3 has length 5 or 7.

We may assume that G is colored by a minimal 4-coloring ϕ as shown in Figure 5. From $girth(G) = 5$ follows that x_i', $y_i' \notin V(C_i)$ for $i = 1, 2, 3$.

Interchanging the colors on vw_1 and vw_2 yields another minimal 4-coloring ϕ' such that $w_1y_1 \in A_2$. Thus there is path p from x_1' to y_1 whose edges are colored 1 and 3 alternately. Since $\mathcal{C}$ was maximum it follows that $x_1'y_1' \in E(G)$. Analogously $x_2'y_2' \in E(G)$, and both edges are colored 3.

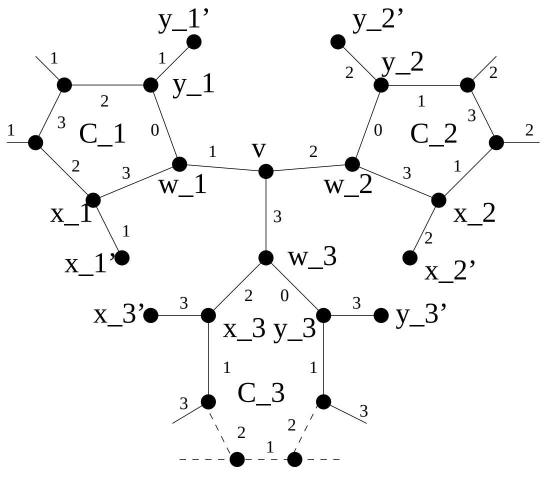

Figure 5.

Change coloring ϕ in C_1 to apply an analogous argument on C_1 and C_3 and infer that $|V(C_3)| = 7$ and that there is path from y_1' to x_1' of length 3, whose edges are colored 2 and 1, alternately, and that there is an edge between x_3 and y_3 which is colored 2.

Again starting with coloring ϕ we change the coloring in C_2 to apply an analogous argument on C_2 and C_3. This implies that there is a multiedge between x_2' and y_2' or between x_3' and y_3', contradicting Claim 3.3.1 saying that G is simple. •

For $v \in V(G) \setminus \bigcup_{C\in\mathcal{C}} V(C)$ we define $n_i(v)$ to be the number of circuits $C \in \mathcal{C}$ of length i such that $v \in M(C)$. For each $C \in \mathcal{C}$ and each $v \in M(C)$ we define

$$s(C,v) = \begin{cases} \frac{5}{5n_5(v)+n_7(v)} & \text{if } |V(C)| = 5 \\ \frac{1}{5n_5(v)+n_7(v)} & \text{if } |V(C)| = 7 \\ \frac{1}{\sum_{i\geq 9} n_i(v)} & \text{if } |V(C)| \geq 9 \text{ and } v \notin M(C') \text{ for all } C' \in \mathcal{C} \\ & \quad \text{with } |V(C')| \leq 7 \\ 0 & \text{otherwise} \end{cases}$$

This parameter indicates how circuit C shares $v \in M(C)$ with other elements of $\mathcal{C}$. For $C \in \mathcal{C}$ we define

$$c^* = |V(C)| + \sum_{v \in M(C)} s(C,v).$$

Intuitively speaking, c^* measures how many vertices of G are "occupied" by C. To show that this is well defined we have to prove that for each vertex $v \in \bigcup_{C\in\mathcal{C}} M(C)$:

$$\sum_{C\in\mathcal{C}} s(C,v) = 1.$$

Since G is cubic each $v \in \bigcup_{C\in\mathcal{C}} M(C)$ is in $M(C)$ for at most three elements of $\mathcal{C}$, and therefore $n_i(v) \neq 0$ for at most three numbers i. Let $1 \leq k \leq 3$ and $n_{i_1}(v), \ldots, n_{i_k}(v) \neq 0$ be pairwise different.

If $n_{i_j}(v) \geq 9$ for all $1 \leq j \leq k$, then

$$\begin{aligned} \sum_{C\in\mathcal{C}} s(C,v) &= \sum_{i\geq 9} \frac{n_i(v)}{\sum_{j\geq 9} n_j(v)} \\ &= \frac{n_{i_1}(v)}{n_{i_1}(v)+n_{i_2}(v)+n_{i_3}(v)} + \frac{n_{i_2}(v)}{n_{i_1}(v)+n_{i_2}(v)+n_{i_3}(v)} + \\ &\quad \frac{n_{i_3}(v)}{n_{i_1}(v)+n_{i_2}(v)+n_{i_3}(v)} = 1. \end{aligned}$$

If $n_{i_j}(v) \in \{5,7\}$ for an $j \in \{1,\ldots,k\}$, then

$$\sum_{C\in\mathcal{C}} s(C,v) = \frac{5n_5(v)}{5n_5(v)+n_7(v)} + \frac{n_7(v)}{5n_5(v)+n_7(v)} = 1.$$

To prove the lemma it now suffices to show that $\sum_{C\in\mathcal{C}} c^* \geq 8|\mathcal{C}|$. This implies $|V(G)| \geq 8r_3(G)$.

We are assuming that G is a counter-example to the statement. Thus $\sum_{C\in\mathcal{C}} c^* < 8r_3(G)$ and hence there are circuits in $\mathcal{C}$ with less than 9 vertices. Let C be such a circuit. By Claim 3.3.1 G has girth 5, and hence $|V(C)| \geq 5$ and the circuits of length 5 or 7 have no chordal edges. Therefore $|N(C)| = |V(C)|$.

Claim 3.3.5 *If $|V(C)| = 7$, then $c^* \geq 8$ or there is $C' \in \mathcal{C}$, $C \neq C'$ and $N(C) \subseteq V(C')$.*

Proof. If there is $v' \in M(C)$ with $s(C, v') = 1$, then $c^* \geq 8$ and we are done. Hence we may assume $s(C, v') < 1$ for all $v' \in M(C)$. It follows from Claims 3.3.2 and 3.3.3 that $N(C) = M(C)$. Since $|V(C)| = 7$ it follows from Claim 3.3.4 that $s(C, v') \geq \frac{1}{7}$, and hence $c^* = 7 + \sum_{v' \in M(C)} s(C, v') \geq 8$.

Thus $M(C) = \emptyset$, and together with Claim 3.3.2 follows that there is another circuit $C' \in \mathcal{C}$ such that $N(C) \subseteq V(C')$. •

Claim 3.3.6 *Let $V(C) = \{v_1, \ldots, v_5\}$ and $E(C) = \{v_1v_2, v_2v_3 \ldots, v_5v_1\}$. Then $c^* \geq 8$ or*

1. *$M(C) = \emptyset$ and there is $C \neq C' \in \mathcal{C}$ such that $N(C) \subseteq V(C')$, or*
2. *$M(C) = \{v_5'\}$, $s(C, v_5') = 1$ and there is $C \neq C' \in \mathcal{C}$ such that $N(C) \setminus M(C) \subseteq V(C')$, or*
3. *$M(C) = \{v_4', v_5'\}$, $s(C, v_4') = s(C, v_5') = 1$ and there is $C \neq C' \in \mathcal{C}$ such that $N(C) \setminus M(C) \subseteq V(C')$, or*
4. *$M(C) = \{v_3', v_4', v_5'\}$, $s(C, v_4') \geq \frac{1}{2}$, $s(C, v_3') = s(C, v_5') = 1$ and there is $C \neq C' \in \mathcal{C}$ such that $N(C) \setminus M(C) \subseteq V(C')$.*

Proof. We may assume $C \in \mathcal{A}_1$. If $M(C) = \emptyset$, then follows together with Claim 3.3.2 that there is $C \neq C' \in \mathcal{C}$ such that $N(C) \subseteq V(C')$ and hence (1).

Assume $c^* < 8$ and $M(C) \neq \emptyset$. We have to show that this assumption implies one of the statements (2), (3) or (4).

If $4 \geq |N(C) \setminus M(C)| \geq 3$, then follows from Claim 3.3.3 that $s(C, v') = 1$ for all $v' \in M(C)$. Since $c^* < 8$ statement (2) or (3) follows from Claim 3.3.2.

If $|N(C) \setminus M(C)| = 2$, then follows from $c^* < 8$, Claims 3.3.2 and 3.3.3 that the two vertices of C with neighbors in $N(C) \setminus M(C)$ are adjacent. Let v_1 and v_2 be these two vertices. Claim 3.3.2 implies that there is $C \neq C' \in \mathcal{C}$ such that $\{v_1', v_2'\} \subseteq V(C')$. From Claim 3.3.3 follows that $s(C, v_3') = s(C, v_5') = 1$, and Claim 3.3.4 implies that $s(C, v_4') \in \{\frac{1}{2}, \frac{5}{6}, \frac{5}{7}\}$, and therefore (4).

If $N(C) \setminus M(C) = \{v_1'\}$, then follows from Claim 3.3.2 that $s(C, v_2') = s(C, v_5') = 1$. Claim 3.3.4 implies that $s(C, v_3')$, $s(C, v_4') \geq \frac{1}{2}$, and hence $c^* \geq 8$, contradicting our assumption.

Let $N(C) = M(C)$. We know that $s(C, v') \in \{\frac{1}{2}, \frac{5}{6}, \frac{5}{7}\}$ for all $v' \in M(C)$, and hence if there is $w' \in M(C)$ with $s(C, w') = 1$ then $c^* \geq 8$.

Hence we may assume $s(C, v') < 1$ for all $v' \in M(C)$, and that $s(C, v') = \frac{1}{2}$ for at least three vertices of $M(C)$. Since C has length 5 we may assume that $s(C, v_1') = s(C, v_2') = \frac{1}{2}$. Then there are other circuits C_1, C_2, both of length 5, such that $v_i' \in M(C_i)$, $i = 1, 2$. Let w_i be the neighbor of v_i' in $V(C_i)$. We may assume that $C_1 \in \mathcal{A}_2$, and let ϕ be a minimal 4-coloring that colors the edges incident to w_1 with colors 0, 2, 3, the edges incident to v_1 with colors 0, 1, 3, in particular $\phi(v_1v_2) = 0$, $\phi(v_1v_1') = 1$ and $\phi(w_1v_1') = 2$.

Interchanging the colors on v_1v_1' and w_1v_1' yields another minimal 4-coloring ϕ' such that $C_{v_0v_1} \in \mathcal{A}_2$. Since $\mathcal{C}$ is maximum it follows that $v_2'v_5' \in E(G)$, $\phi'(v_2'v_5') = \phi(v_2'v_5') = 3$, and $C_2 \in \mathcal{A}_2$ (with respect to ϕ).

Modify ϕ in C to infer by a similar argumentation applied on v_2v_2' and w_2v_2' that $v_1'v_3' \in E(G)$ and $\phi(v_1'v_3') = 3$.

From our assumption follows that there are circuits C_3, C_4 such that $v_i' \in M(C_i)$, $i = 3, 4$. If C_3, C_4 both have length 7 then follows $c^* \geq 8$ and we are done. Thus at least one of them has length 5, say C_3. Since $\phi(v_1'v_3') = 3$ we have $C_3 \in \mathcal{A}_2$. By a similar local recoloring argumentation as above we infer that $v_2'v_4' \in E(G)$. Since v_2, w_2, v_4' and v_5' are pairwise different it follows that v_2' has degree at least 4 in G, contradicting the fact that G is cubic. •

Now we consider the circuits of length 7 with all neighbors contained in another circuit of $\mathcal{C}$ and the circuits of length 5 satisfying one of Claim 3.3.6.(1) - (4). We call the set of these circuits $\mathcal{C}^-$. Since we assume that G is a counter-example it follows from Claims 3.3.5 and 3.3.6 that $\mathcal{C}^- \neq \emptyset$.

Let $C \in \mathcal{C}^-$. Then there is $C_0 \in \mathcal{C}$ such that $N(C) \setminus M(C) \subseteq V(C_0)$.

Let $C_1, \ldots, C_m$ be the circuits of length 5 and $K_1, \ldots, K_n$ the circuits of length 7 in $\mathcal{C}^-$ such that $V(C_0)$ contains all their neighbors which are in $\bigcup_{C \in \mathcal{C}} V(C)$; C is one of these circuits. We may assume that $C_0 \in \mathcal{A}_1$. For all $C \in \mathcal{C}^-$ we define

$$\overline{c^*} = c^* + |N(C) \cap V(C_0)|.$$

Claims 3.3.5 and 3.3.6 imply that $\overline{c_i^*} \geq 9 + \frac{1}{2}$ and $\overline{k_i^*} \geq 14$.

First we consider the case when $n + m \geq 2$. Then C_0 has neighbors in different circuits of $\mathcal{A}_1$ and hence it follows from Claim 3.3.2 that any two vertices of C_0 having neighbors in different circuits of $\mathcal{A}_1 \setminus \{C_0\}$ are separated by at least two vertices, which have no neighbor in another element of $\mathcal{C}'$. Let S^* be the number of vertices occupied by C_0 and the circuits of $\mathcal{C}^-$. Then

$$\begin{aligned} S^* &= c_0^* + \sum_{i=1}^{m} c_i^* + \sum_{i=1}^{n} k_i^* \\ &\geq 2(n+m) + \sum_{i=1}^{m} \overline{c_i^*} + \sum_{i=1}^{n} \overline{k_i^*} \\ &\geq 2(n+m) + (9 + \frac{1}{2})m + 14n \\ &= (11 + \frac{1}{2})m + 16n \end{aligned}$$

Thus $S^* \geq 8(n + m + 1)$ if $n \geq 1$ or $n = 0$ and $m \geq 3$. Thus it remains to consider the case when $m = 2$ and $n = 0$.

Let C_1 and C_2 be the two circuits of length 5. We have to show that $c_0^* + c_1^* + c_2^* \geq 24$.

If both of C_1, C_2 satisfy one of Claim 3.3.6.(1) - (3), then follows $c_0^*+c_1^*+c_2^* \geq 4+\overline{c_1^*}+\overline{c_2^*} \geq 24$.

Thus at least one of them, say C_1, satisfies Claim 3.3.6.(4), and $s(C_1, v_4') < 1$. Since $v_1v_2 \in E(C_1)$ and $girth(G) \geq 5$ it follows that v_1' and v_2' are not adjacent in C_0. Thus there is a vertex of C_0 between v_1' and v_2'. Hence C_1 occupies beside the two vertices, separating v_1' and v_2' from the neighbors of C_2, a third vertex of C_0. In any case it follows $c_0^*+c_1^*+c_2^* \geq 24$.

Now let $n+m \leq 1$. First we consider $n=1$ and $m=0$. We have to show $c_0^*+k_1^* \geq 16$. This follows directly from the facts that G is bridgeless and $r_3(G) \geq 4$. Since this implies that $|N(C_0)| \geq 9$ and hence $|V(C_0)| \geq 9$.

The last case is $n=0$ and $m=1$. If $|V(C_0)| \geq 11$, then $c_0^*+c_1^* \geq 16$, and we are done.

Subcase $|V(C_0)| = 9$. If C_1 satisfies one of Claim 3.3.6.(3) or (4) then $c_0^*+c_1^* \geq 16$, and we are done.

Thus it remains to consider the cases when C_1 satisfies Claim 3.3.6.(1) or (2). Recall that G is bridgeless, $girth(G) = 5$ and $r_3(G) \geq 4$, hence $M(C_0) \neq \emptyset$. If there are at least two (one) vertices in $M(C_0)$ with $s(C_0, v) = 1$ in the first (second) case, then $c_0^*+c_1^* \geq 16$. The necessary condition is certainly satisfied if C_0 has no chordal edge.

Thus we may assume that the condition is not satisfied and C_0 has a chordal edge.

Let $N(C_1) \subseteq V(C_0)$. Since G is bridgeless and $r_3(G) \geq 4$ it follows that C_0 has precisely one chordal edge and $|\partial(V(C_0) \cup V(C_1))| = |M(C_0)| = 2$. Since $girth(G) = 5$ and $s(C_0, v') < 1$ for at least one $v' \in M(C_0)$ it follows that the two vertices v, w of C_0 which are adjacent to the vertices of $M(C_0)$ are adjacent in C_0. Let $u \neq w$ be the other neighbor of v in C_0, and we may assume that $s(C_0, v') < 1$. Then there is a circuit $C_2 \in \mathcal{C}$ such that $v' \in M(C_2)$, too. Let $z \in V(C_2)$ be adjacent to v'. Then there is a minimal 4-coloring ϕ of G such that the edges incident to z are colored 0, 2, 3 (in particular $\phi(zv') = 2$), and $\phi(vv') = 1$, $\phi(vu) = 3$ and $\phi(vw) = 0$. Interchanging colors 1 and 2 on zv' and vv' yields another minimal 4-coloring ϕ' of G such that $C'_{vw} \in \mathcal{A}_2$. Thus there is path p from v to w whose edges are colored 1 and 3, alternately. Since $ww' \in E(p)$ and $\{ww', vv'\}$ is an edge 2-cut of G it follows $vv' \in E(p)$. But $\phi'(vv') = 2$, a contradiction.

Let $|N(C_1) \cap V(C_0)| = 4$. Then we have $c_1^* = 6$. If C_0 has two chordal edges then follows from $girth(G) \geq 5$ that $|M(C_0)| = |M(C_1)| = 1$. Let $v' \in M(C_0)$, $w' \in M(C_1)$, then $\{vv', ww'\}$ is an edge 2-cut of G, and if $s(C_0, v') = 1$ then we are done.

Thus let us assume $s(C_0, v') < 1$, and let v_1, v_2 be the neighbors of v in C_0. By Claim 3.3.3 $s(C_0, v') < 1$ is only possible when v_1 and v_2 both are incident to a chordal edge e and f of C_0, respectively. Since $girth(G) \geq 5$ we have $e \neq f$. Further there is $C' \in \mathcal{C}$ such that $v' \in M(C')$. We may assume that $C_0, C_1 \in \mathcal{A}_1$ and $C' \in \mathcal{A}_2$. By a similar argumentation as above we switch to another minimal coloring ψ such that $C_0' \in \mathcal{A}_2$ and it contains edges e and f. Note that $C_1 \in \bigcup_{e \in \psi^{-1}(0)} C_e$. By Lemma 1.2 circuits C_0' and C_1 are disjoint and hence e and f are connected by an edge e'. Now it is easy to verify that up to isomorphism one of the drawings of Figure 6 describes the situation.

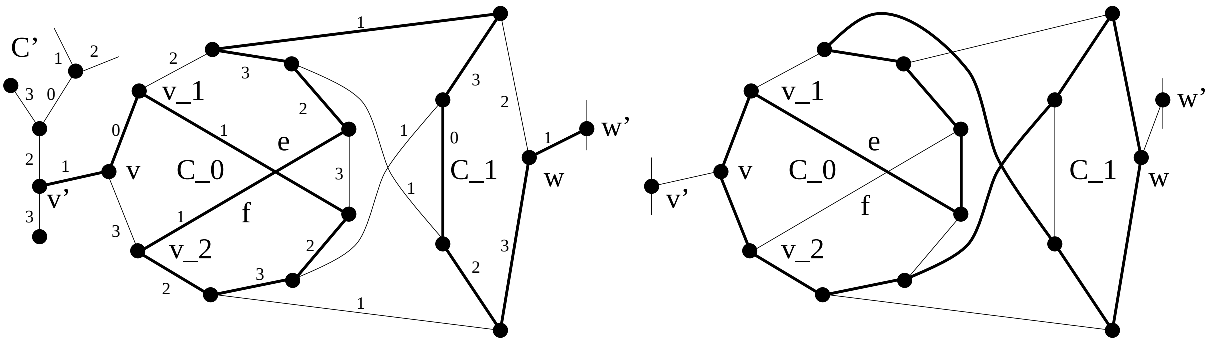

Figure 6.

Let ϕ be a minimal 4-coloring such that $\mathcal{C} = \bigcup_{e\in\phi^{-1}(0)} C_e$, then $\phi(vv') = \phi(ww') \neq 0$. In both cases the coloring of vv', ww' can be extended to a 3-coloring of $G[V(C_0) \cup V(C_1)]$ as indicated by the hamiltonian path and hamiltonian circuit, respectively. Thus there is 4-coloring γ of G with $|\gamma^{-1}(0)| < r_3(G)$, a contradiction.

Thus we may assume that C_0 has precisely one chordal edge. From our assumptions follows that $s(C_0, v') < 1$ for each $v' \in M(C_0)$. Thus, by Claim 3.3.3, the chordal edge in C_0 separates the vertices with neighbor in C_1 from those with neighbor in $M(C_0)$. Since $girth(G) = 5$ the only possible realization of that situation is (up to isomorphism) as shown in Figure 7, where $V(C_0) = \{u_1, \dots, u_9\}$ and $V(C_1) = \{v_1, \dots, v_5\}$.

Now it is easy to see that $G[V(C_1) \cup \{u_1, \dots, u_4\}]$ is the Petersen graph minus a vertex. We may assume that all edges of $\partial(V(C_0) \cup V(C_1))$ are colored 1. Leaving the colors of the edges of $\partial(V(C_0) \cup V(C_1))$ and $G[V(G) \setminus (V(C_0) \cup V(C_1))]$ unchanged, subgraph $G[V(C_1) \cup \{u_1, \dots, u_4\}]$ can be recolored to obtain another maximum minimal 4-coloring ϕ' of G and two new circuits C_0' and C_1' with vertex sets $V(C_1) \cup \{u_1, \dots, u_4\}$ and $\{u_5, \dots, u_9\}$, respectively. The edges of these two circuits are indicated by the bold lines. But ${c_1'}^* \geq 7$, hence ${c_0'}^* + {c_1'}^* \geq 16$ and we are done.

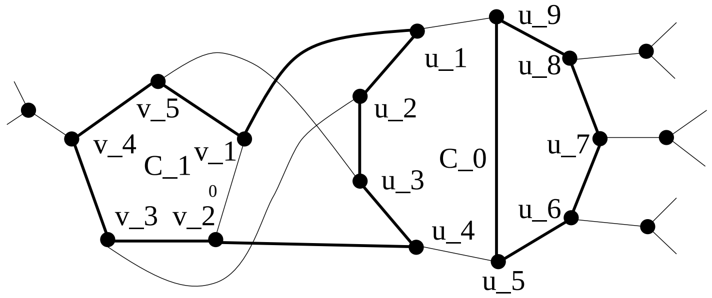

Figure 7.

Subcase $|V(C_0)| = 7$. Then C_0 has no chordal edge, and hence it follows from Claims 3.3.2 and 3.3.3 that there are at least two vertices in $M(C_0)$ with $s(C_0, v') = 1$. Hence $c_0^* \geq 9$ and if C_1 satisfies Claim 3.3.6.(3) or (4), then we are done.

Thus $|N(C_1) \cap V(C_0)| \geq 4$, and $2 \leq |M(C_0)| \leq 3$. Hence there are $x, y, z \in V(C_0)$ such that $x' \in M(C_0)$, and y, z both have a neighbor in $V(C_1)$ and $xy, yz \in E(C_0)$. We may assume that there is a minimal 4-coloring ϕ as shown in Figure 8. Clearly $a \neq x'$, but $a \in V(C_0)$ could be.

Obtain a new minimal 4-coloring ϕ' by changing the color of uy to 2. Then $C'_{xy} \in \mathcal{A}_2$, and C'_{xy} contains vertices x', x, y, z, w, v, a. Since $\mathcal{C}$ is maximum and $girth(G) \geq 5$ it follows that $|V(C'_{xy})| \leq 7$, $a \in M(C_1)$ and $ax' \in E(G)$. Therefore $|V(C'_{xy})| = 7$ and $\phi(ax') = \phi'(ax') = 3$. Interchanging colors 2 and 3 in C_0 and C_1 and changing the color of uv to 3 yields by a similar argumentation that there is an edge e between a and x' with $\phi(e) = 2$. Hence G has a multiple edge, contradicting Claim 3.3.1, and we are done.

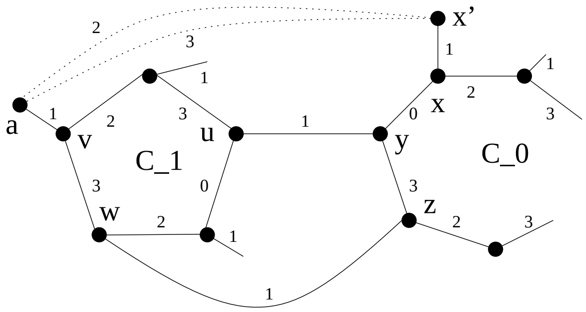

Figure 8.

Subcase $|V(C_0)| = 5$. There are at least two edges between C_0 and C_1. If for any two adjacent vertices of C_i $(i = 1, 2)$ at most one of them has a neighbor in the other circuit, then follows that $|M(C_0)| = |M(C_1)| = 3$ and Claim 3.3.3 implies $s(C_i, v') = 1$ for each $v' \in M(C_i)$. Thus $c_0^* + c_1^* = 16$.

Hence there are two adjacent vertices v_1, v_2 in C_0 or C_1 both having a neighbor in the other circuit, say v_1, $v_2 \in V(C_0)$. Since $girth(G) \geq 5$ we may assume that the situation is as shown in Figure 9, and let ϕ be the minimal 4-coloring given there.

Changing the color of v_1w_1 to 2 yields a new minimal coloring γ such that $C'_{v_1v_2}$, $C'_{w_1w_5}$ in $\mathcal{A}_2$. Since $\mathcal{C}$ is maximum it follows that $w_3v_5 \in E(G)$ and that the neighbors w'_2 and w'_5 of w_2 and w_5, respectively, which are not in $V(C_1)$, are adjacent. Since $r_3(G) \geq 4$ it follows that $w'_2, w'_5 \notin \{v_3, v_4\}$. Note that $\phi(w'_2w'_5) = \gamma(w'_2w'_5) = 3$.

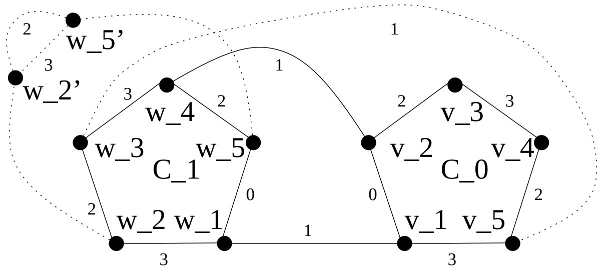

Figure 9.

Starting from ϕ obtain another minimal coloring ψ by interchanging colors 2 and 3 in C_0 and C_1 and let $\psi(v_1w_1) = 3$. Note that $\phi(w'_2w'_5) = \psi(w'_2w'_5) = 3$. As above we infer that there is an edge e between w_2' and w_5' with $\phi(e) = 2$. Thus G is not simple, contradicting Claim 3.3.1.

Since for each $C \in \mathcal{C}$ either $|V(C)| \geq 9$ or precisely one of the cases of Claims 3.3.5 or 3.3.6 is true it follows that $\sum_{C\in\mathcal{C}} c^* \geq 8|\mathcal{C}| = 8r_3(G)$. Hence there is no counter-example and the statement is proved.

•

Let H_k be as defined in Theorem 5.2. Then $\zeta_3(H_k) = \frac{25}{27} = 1 - \frac{2}{3g^+(H_k)}$. This shows that this lower bound is also achieved by infinitely many snarks with $g^+ = 9$.

The examples show that there is an substantial difference between the two lower bounds for ζ_3 given in Theorem 3.2. There are infinitely many snarks with $\zeta_3 = 1 - \frac{2}{3g^+}$ but we know only a finite number of connected graphs with $\zeta_3 = \frac{11}{12}$. This motivates us to state the following problem

Problem 3.4 *What is the largest number b such that $\zeta_3 \geq b$ for all but a finite number of connected snarks.*

Theorems 3.2 and 5.2 imply that $\frac{11}{12} \leq b \leq \frac{25}{27}$.

4 Maximum 2-Colorable Subgraphs

Theorem 4.1 *If $G = (V, E)$ is a bridgeless cubic graph, then*

1. $\omega(G) = 2 \Leftrightarrow r_3(G) = 2 \Leftrightarrow r_2(G) = 1$,
2. $\frac{1}{2}r_3(G) \leq r_2(G) \leq \min\{\frac{2}{3}r_3(G), \frac{1}{2}\omega(G)\}$, *and the bounds are attained,*
3. $\frac{2}{3}\zeta_3(G) \leq \zeta_2(G) \leq \frac{1}{2}\zeta_3(G) + \frac{1}{6}$, *and the bounds are attained.*

Proof. We start with proving the first assertion of statement (2), which is trivial if G is 3-colorable.

Let H be a maximum 2-colorable subgraph of a snark G, whose edges are colored 1 and 2. Let v be a vertex of G and assume it is not contained in $V(H)$. Let u_1, u_2, u_3 be the neighbors of v. Since H is maximum and $vu_i \notin E(H)$ it follows that the remaining two edges which are incident to u_i are edges of H $(i = 1, 2, 3)$. Let e_1 and e_2 be two of the remaining edges incident to u_1 and u_2, respectively, such that e_i is colored i in H $(i = 1, 2)$. Then $H' = (H - \{e_1, e_2\}) \cup \{u_1v, u_2v\}$ is also a maximum 2-colorable subgraph of G and $V(H') = V(H) \cup \{v\}$.

Thus we may assume that a maximum 2-colorable subgraph spans G, and it consists of even circuits and paths.

Claim 4.1.1 *If H is a spanning maximum 2-colorable subgraph of G which consists of even circuits and n paths, then $n = r_2(G)$.*

Proof. Let $p_1, \dots, p_n$ ($n \geq 1$) be the paths of H. Each end vertex of a path is incident to precisely two uncolored edges while all the other vertices are incident to precisely one uncolored edge. Therefore $c_2(G) = \frac{1}{2}(2(|V| - 2n) + 2n) = \frac{2}{3}|E| - n$, and hence $n = r_2(G)$. •

From this calculation of $c_2(G)$ follows that G has a 3-colorable subgraph where all end vertices of the paths $p_1, \dots, p_n$ are incident to precisely one uncolored edge and all the other edges are incident to colored edges only. Thus $\frac{r_3(G)}{2} \leq r_2(G)$.

Let $\mathcal{F}$ be a minimum 2-factor of G. Then $\phi_{\mathcal{F}}^{-1}(1) \cup \phi_{\mathcal{F}}^{-1}(2)$ induces a 2-colorable subgraph on G consisting of even circuits and $\frac{1}{2}\omega(G)$ paths. It follows from Claim 4.1.1 that $r_2(G) \leq \frac{1}{2}\omega(G)$.

Let γ a minimal 4-coloring of G, i.e. $|\gamma^{-1}(0)| = r_3(G)$. Then $\gamma^{-1}(0)$ is the disjoint union of A_1, A_2 and A_3. We may assume that $|A_3| \leq \lfloor \frac{r_3(G)}{3} \rfloor$. Then color 3 does not appear at $\lceil \frac{2r_3(G)}{3} \rceil$ vertices $v_1, \dots, v_{\lceil \frac{2r_3(G)}{3} \rceil}$ and hence $|\gamma^{-1}(3)| \leq \frac{1}{2}|V \setminus \{v_1, \dots, v_{\lceil \frac{2r_3(G)}{3} \rceil}\}| = \frac{1}{3}(|E| - r_3(G))$. Therefore $c_2(G) \geq |E| - r_3(G) - \frac{1}{3}(|E| - r_3(G)) = \frac{2}{3}|E| - \frac{2}{3}r_3(G)$, and consequently $r_2(G) \leq \frac{2}{3}r_3(G)$.

For the graphs H_k of Theorem 5.2 holds true that $r_2(H_k) = \frac{1}{2}r_3(H_k)$ and $\omega(H_k) = r_3(H_k)$. Consequently $r_2(H_k) = \frac{1}{2}\omega(H_k)$.

It is easy to construct snarks with $r_3 = 3$. It follows from $\frac{2}{3}r_3 \geq r_2 \geq \frac{1}{2}r_3$ that $r_2 = 2 = \frac{2}{3}r_3$ for those graphs.

The equivalence of $r_3(G) = 2$ and $\omega(G) = 2$ is well known, cf. [8]. Let $r_3(G) = 2$. From $\frac{2}{3}r_3(G) \geq r_2(G) > 0$ follows $r_2(G) = 1$.

If $r_2(G) = 1$, then $r_3(G) \leq 2r_2(G)$ and hence $r_3(G) = 2$.

Since $\zeta_3(G) = 1 - \frac{r_3(G)}{|E|}$ and $\zeta_2(G) = \frac{2}{3} - \frac{r_2(G)}{|E|}$ statement (3) follows immediately from statement (2).

•

Remark: (1) Let G be a snark with $\omega(G) \in \{r_3(G), r_3(G) + 1\}$. Then follows from Theorem 4.1.(2) that $r_2(G) = \frac{1}{2}\omega$.

(2) From Theorem 4.1, 2.3 and 2.4 follows that for each $k \geq 1$ there is a cyclically 5-edge connected snark G_k with $r_3(G_k) - r_2(G_k) \geq 2k$ and $\omega(G_k) - r_2(G_k) \geq 4k$, and there is no constant c such that $1 \leq c < 4$ and $\omega(G) \leq cr_2(G)$ for all cubic bridgeless graphs G.

Albertson, Haas [1] showed $\zeta_2 \geq \frac{17}{30}$ for cubic graphs (possibly with 1-cuts) and let it open to determine the best possible lower bound.

For the Petersen graph we have $\zeta_2 = \frac{3}{5}$, and the next theorem shows, that it is the only snark where ζ_2 is that small.

Theorem 4.2 *If $G = (V, E)$ is a snark with at least 12 vertices, then $\frac{2}{3} > \zeta_2(G) \geq \max\{\frac{2}{3} - \frac{1}{3g^+(G)}, \frac{11}{18}\}$, and the bounds are best possible.*

Proof. All snarks with 12 or 14 vertices have $r_3 = 2$. Hence it follows from Theorem 4.1.(2) that $r_2 = 1$ and therefore $\zeta_2 \geq \frac{11}{18}$ for these graphs.

Thus we may assume that G has at least 16 vertices. From Theorems 3.2 and 4.1.(3) follows $\zeta_2(G) \geq \frac{11}{18}$. It remains to show that $\zeta_2(G) \geq \frac{2}{3} - \frac{1}{3g^+(G)}$.

Since $G = (V, E)$ has a 1-factor and $\omega(G) \leq \frac{2|E|}{3g^+(G)}$ it follows that

$$c_2(G) \geq \frac{1}{3}|E| + \frac{1}{2}(\frac{2}{3}|E| - \omega(G)) = \frac{2}{3}|E| - \frac{1}{2}\omega(G) \geq |E|(\frac{2}{3} - \frac{1}{3g^+(G)})$$

and consequently $\zeta_2(G) \geq \frac{2}{3} - \frac{1}{3g^+(G)}$.

By Theorem 5.1 we know that there is a permutation snark P_k of order for $4k + 2$ for each $k \geq 2$. It is easy to find a 2-colorable subgraph with $4k + 1$ edges, and hence $r_2 = 1$. Therefore $\zeta_2(P_k) = \frac{4k+1}{6k+3} = \frac{2}{3} - \frac{1}{3g^+(P_k)}$.

For any $\epsilon > 0$ there is $k \geq 2$ such that $\frac{1}{6k+3)} < \epsilon$. Hence $r_2(P_k) > \frac{2}{3} - \epsilon$. This shows that the bounds are best possible. Furthermore the bound $\frac{11}{18}$ is attained by the snarks on 12 vertices.

•

As in case of parameter ζ_3 there is a substantial difference between the two lower bounds of Theorem 4.2. We know infinitely many snarks with $\zeta_2 = \frac{2}{3} - \frac{1}{3g^+}$ but only finitely many with $\zeta_2 = \frac{11}{18}$. However the graphs with more than 12 vertices attaining this bound can be characterized.

Theorem 4.3 *If $G = (V, E)$ is a snark with at least 14 vertices, then $\zeta_2(G) = \frac{11}{18}$ if and only if $|V| = 8r_3(G)$ and $r_2(G) = \frac{2}{3}r_3(G)$.*

Proof. Let $|V| = 8r_3(G)$ and $r_2(G) = \frac{2}{3}r_3(G)$, then $\zeta_2(G) = \frac{2}{3} - \frac{2r_3(G)}{36r_3(G)} = \frac{11}{18}$.

To prove the other implication recall that $\zeta_2(G) > \frac{11}{18}$ if $|V| = 14$. Thus we may assume that $|V| \geq 16$ and $r_2(G) \geq 2$.

From $\zeta_2(G) = \frac{11}{18}$ follows $|V| = 12r_2(G)$. Theorem 4.1.(2) implies $|V| \leq 8r_3(G)$ and hence it follows from Lemma 3.3 that $|V| = 8r_3$. Therefore $12r_2(G) = 8r_3(G)$ and hence $r_2(G) = \frac{2}{3}r_3(G)$.

•

The next theorem shows that even the existence of large 3- or 2-colorable subgraphs in a snark does not bound its oddness.

The flower snarks J_{2k+1} are cyclically 6-edge connected if $k \geq 3$ and were introduced by Isaacs [5]. Graph J_{2k+1} has vertex set $V(J_{2k+1}) = \{a_i, b_i, c_i, d_i | i = 1, 2, \ldots, 2k + 1\}$ and edge set $E(J_{2k+1}) = \{b_i a_i, b_i c_i, b_i d_i; a_i a_{i+1}; c_i d_{i+1}; d_i c_{i+1} | i = 1, 2, \ldots, 2k + 1\}$.

Fiorini [4] proved that the flower snarks are hypohamiltonian, i.e. $J_{2k+1} - v$ has a hamiltonian circuit for each $v \in V(J_{2k+1})$, and henceforth they have a hamiltonian path. In [8] it is shown that $J_{2k+1} - v$ is 3-critical for each $v \in V(J_{2k+1})$ and each $k \geq 2$.

Theorem 4.4 *For each $\epsilon > 0$ and each $k \geq 1$ there are infinitely many snarks G with $\zeta_3(G) > 1 - \epsilon$ and $\zeta_2(G) > \frac{2}{3} - \epsilon$ and $\omega(G) \geq k$.*

Proof. Let $\epsilon' = \frac{2}{3}\epsilon$ and $k \geq 2$ be given. Choose t such that $8t + 4 \geq \frac{1}{3\epsilon'}$, and let $m \geq k$ be an even number. Take m copies $H_1, \ldots, H_m$ of J_{2t+1} which has an hamiltonian path p' and $(8t + 4 =)\ 2n > \frac{1}{3\epsilon'}$ vertices. Let z_i be an end vertex of a hamiltonian path p'_i in H_i and u_i, v_i, w_i the neighbors of z_i, where $v_i \in V(p'_i)$ and let $p_i = p'_i - z_i$.

Let C be a circuit with vertices $y_1, \ldots, y_m$ and edges $y_1y_2, y_2y_3, \ldots, y_my_1$. Construct a new snark G from $H_i - z_i$, $i = 1, \ldots, m$, and C by adding edges w_mu_1, v_my_m and w_iu_{i+1}, v_iy_i for $i = 1, \ldots, m-1$. Then $V(G) = 2nm$ and consequently $E(G) = 3nm$. Let

$$E' = \bigcup_{i=1}^{\frac{m}{2}} (E(p_{2i-1}) \cup E(p_{2i}) \cup \{v_{2i-1}y_{2i-1}, y_{2i-1}y_{2i}, y_{2i}v_{2i})\})$$

Then $\chi'(G[E']) = 2$ and $|E'| = m(2n-2) + \frac{3}{2}m = 2mn - \frac{1}{2}m$, and consequently

$$\zeta_2(G) \geq \frac{2mn - \frac{1}{2}m}{3nm} = \frac{2}{3} - \frac{1}{6n} > \frac{2}{3} - \epsilon' > \frac{2}{3} - \epsilon.$$

Since $\frac{2}{3}\epsilon = \epsilon'$ it follows from Theorem 4.1.(3) that $\zeta_3(G) = 1 - \epsilon$.

We know that $H_i - z_i$ is 3-critical, (cf. Thm 4.3 [8]). Hence G contains m pairwise disjoint 3-critical subgraphs and therefore $\omega(G) \geq r_3(G) \geq m \geq k$.

•

Remark: The *edge chromatic difference sequence* d of a graph G is $d(G) = (d_1, \ldots, d_{\chi'})$ where $d_1 = c_1(G)$ and $d_i = c_i(G) - c_{i-1}(G)$ for $i = 2, \ldots, \chi'$. Sequence $d(G)$ is called *monotonic* if $d_i \geq d_{i+1}$ for $i = 1, \ldots, \chi' - 1$.

Albertson and Haas [1] gave examples of cubic graphs G where $d(G)$ is non-monotonic. However all these graphs have 1-cuts.

Corollary 4.5 *There are infinitely many cyclically 5-edge connected cubic graphs with non-monotonic degree sequence.*

Proof. For graph G_k of Theorem 2.3 $(k \geq 1)$ we have $|V(G_k)| = 590k$ and consequently $|E(G_k)| = 885k$. It has a 1-factor and hence $d_1 = 295k$. From $r_3(G_k) = 6k$ and Theorem 4.1 follows $3k \leq r_2(G_k) \leq 4k$. Hence $291k \leq d_2 \leq 292k$, and $d_3 = 293k$. Thus $d_3 > d_2$. Indeed the difference between d_3 and d_2 is at least k.

•

5 Appendix: Some Families of Graphs

In this section we construct some families of graphs which we used in the preceding sections to show that some of the bounds are attained by infinitely many graphs.

The *dot product* $G_1 \cdot G_2$ of two snarks G_1, G_2 was introduced by Isaacs [5] and it yields a new snark which is cyclically 4-edge connected and has girth at least 5, if G_1 and G_2 both have these properties. It is defined as follows.

1. take $G_1' = G_1 - \{ab, cd\}$, where ab, cd are two nonadjacent edges of G_1;
2. take $G_2' = G_2 - \{x, y\}$, where x, y are adjacent vertices of G_2;
3. let w, z and u, v be the other neighbors of x and y in G_2, respectively. Then $G_1 \cdot G_2$ is the graph $G = (V, E)$ with $V = VG_1' \cup VG_2'$ and $E = EG_1' \cup EG_2' \cup \{aw, bz, cu, dv\}$.

A snark on $2n$ vertices is called a *permutation snark* if it has a 2-factor $\mathcal{F}$ consisting of cycles of length n. We say $\mathcal{F}$ is a *good* 2-factor.

Theorem 5.1 *For each $k \geq 2$ there is a permutation snark P_k with $4k+2$ vertices. If $k \neq 3$, then they are cyclically 4-edge connected.*

Proof. Let G_1 and G_2 be two cyclically 4-edge connected permutation snarks. Apply the dot product on G_1 and G_2 as follows: Let C_1^i, C_2^i by the two circuits of a good 2-factor of G_2, $i = 1, 2$. In G_1 remove from each of C_1^1, C_2^1 precisely one edge and from G_2 remove two adjacent vertices v, w such that $v \in V(C_1^2)$ and $w \in V(C_2^2)$. The new graph is a cyclically 4-edge connected permutation snark.

For $k = 2$ the Petersen graph is a permutation snark. Let C_1 and C_2 be the two circuits of a 2-factor. Extending a vertex to a triangle in each of C_1, C_2 yields a snark H on 14 vertices with two disjoint circuits C_1', C_2' of length 7. Let e_1 and e_2 be the edges of the two triangles which are not in C_1' or C_2'.

To obtain cyclically 4-edge connected graphs for $k \neq 3$ we restrict the application of the dot product on H as follows, if it is applied on H, then $H - \{e_1, e_2\}$ will be used in the construction.

With this restriction the iterated application of the dot product yields the statement.

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The next theorem describes a family of snarks which is already mentioned in [1].

Theorem 5.2 *For each $k \geq 1$ there is a 3-connected snark H_k with $r_3(H_k) = \omega(H_k) = 2k = 2r_2(H_k)$, $\zeta_3(H_k) = \frac{25}{27}$, and $g^+(H_k) = 9$.*

Proof. Let P denote the Petersen graph. Then $P - x$ ($x \in V(P)$) is a hamiltonian 3-critical graph. Let u, v, w be the divalent vertices of $P - x$.

It is easy to define a minimal 4-coloring of $P_{\{x\}}$ such that the semiedges are all colored 1.

Take $2k$ copies $P_1, \ldots, P_{2k}$ of the colored multipole $P_{\{x\}}$ and identify identically colored semiedges to obtain a 3-connected cubic graph H_k that contains edges $u_{2k}v_1$ and u_iv_{i+1} for $i = 1, \ldots, 2k-1$.

We further obtain a coloring γ of H_k such that $|\gamma^{-1}(0)| = 2k$. Since H_k contains $2k$ pairwise disjoint copies of $P - x$ it follows that $r_3(H_k) = 2k$. Graph $P - x$ is hamiltonian and hence H_k has a 2-factor with precisely $2k$ odd components. Since $r_3(H_k) \leq \omega(H_k)$ it follows $\omega(H_k) = 2k$.

From this and Theorem 4.1.(2) follows $2k = r_3(H_k) \leq 2r_2(H_k) \leq \omega(H_k) = 2k$, and hence $r_2(H_k) = k$.

We have $|E(H_k)| = 27k$ and consequently $\zeta_3(H_k) = \frac{25}{27}$.

Furthermore $g^+(H_k)2k = g^+(H_k)r_3(H_k) \leq |V(H_k)| = 18k$ and hence $g^+(H_k) \leq 9$. Indeed $g^+(H_k) = 9$ since the union of hamiltonian circuits of the P_i is a 2-factor of H_k and the shortest circuit has 9 vertices.

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Theorem 5.3 *For each $k \geq 1$ there is a 3-connected snark O_k such that $r_3(O_k) = 2k+1 = \omega(O_k) - 1$, and $g^+(O_k) = 5$.*

Proof. Let the notation be as in the proof of Theorem 5.2.

It is easy to define a minimal 4-coloring of $P_{\{x\}}$ such that two of the semiedges are colored 1, and that the third is colored i, for each $i = 1, 2, 3$. Take $2k+1$ copies $P_1, \ldots, P_{2k+1}$ of the colored multipole $P_{\{x\}}$ and a vertex a. Vertex a is considered as a 3-colored multipole. Two multipoles, say P_2, P_3, are colored such that one semiedge is colored i in P_i $(i = 2, 3)$ and the other two semiedges are colored 1. The remaining multipoles are colored such that all semiedges are colored 1. Identify identically colored semiedges to obtain a 3-connected cubic graph O_k and a 4-coloring γ of its edges such that $|\gamma^{-1}(0)| = 2k+1$. Since O_k contains $2k+1$ pairwise disjoint copies of $P - x$ it follows $r_3(O_k) = 2k+1$ and $\omega(O_k) \geq 2k+2$.

It is easy to construct a 2-factor with $2k+2$ odd components, where $2k-2$ of them are the hamiltonian circuits of $P - x$ of length 9, the circuit containing a has length 13, and the three remaining circuits have length 5. Hence $\omega(O_k) = 2k+2$.

Let $\mathcal{F}$ be a 2-factor of O_k and $C \in \mathcal{F}$ the circuit containing vertex a. Let $av \in E(C)$ and $v \in V(P_i)$. Since the distance between any two divalent vertices in $P - x$ is 3 it follows that C contains at least four vertices of P_i. On the other hand $O_k[V(P_i)]$ is a 3-critical subgraph of O_k. It follows from Claim 2.4.1 that $\mathcal{F}$ contains an odd circuit C' that is subgraph of $O_k[V(P_i)]$. Since $O_k[V(P_i)]$ has 9 vertices and at least four of them are in C it follows from $girth(O_k[V(P_i)]) = 5$ that $|V(C')| = 5$.

Since $\mathcal{F}$ was chosen arbitrarily and $g^+(O_k) \geq 5$ by Lemma 3.1 it follows that $g^+(O_k) = 5$.

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