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On the Complexity of Generation in Ternary Hamming Spaces[1]

by

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ABSTRACT

In this paper, a follow-up of [1, 2], we continue to study ternary Hamming spaces $\mathbf{T}^n$, where $\mathbf{T} = \{0, 1, 2\}$, and their generators. In particular, we are interested in the computational complexity of the generation of all vectors with respect to a specific generator $G \subseteq \mathbf{T}^n$. We consider several parameters, i.e., depth and load, which provide alternative ways to measure the computational effort required in order to generate all points of $\mathbf{T}^n$ starting from those of G. We establish lower and upper bounds on the minimum depth and the minimum load for a generator in, and prove that some of them are sharp.

1 Introduction

This paper is a follow-up of [1, 2], where we report on some results related to ternary Hamming spaces and, in particular, to generators and their properties. We recall from [2] some basic definitions and notation[1]. The n-dimensional ternary Hamming space is $\mathbf{T}^n$, where $\mathbf{T} = \{0, 1, 2\}$.

Let us denote by $[n] = \{1, 2, ..., n\}$. For $k \in [n]$ let us call a *subspace of dimension* k, or in short a k*-subspace* the set of all the points of $\mathbf{T}^n$ which have the same coordinates in some $n - k$ components. Subspaces of dimension $n - 1$ are called *hyperplanes*, while subspaces of dimension 1 are called *lines*. Here, every line contains exactly 3 points. Let us denote the set of lines by $\mathcal{L}$. The subspaces of dimension 0 are the elements, or points, of $\mathbf{T}^n$.

For vectors $x, y \in \mathbf{T}^n$ let us define their *Hamming distance* as

$$d_H(x, y) = |\{i | x_i \neq y_i\}|. \tag{1.1}$$

For a subset $S \subseteq \mathbf{T}^n$ and a point $x \in \mathbf{T}^n$ let

$$d_H(x, S) = \min_{y \in S} d_H(x, y).$$

Finally, for a subset $S \subseteq \mathbf{T}^n$ let us introduce the notation $D_H(S) = \max_{x \in \mathbf{T}^n} d_H(x, S)$.

Clearly, two points are on a line iff their Hamming distance is 1. Let us call two points $x, y \in \mathbf{T}^n$ *composable*, if their Hamming distance is 1, or in other words if they belong to a line $L = \{x, y, z\} \in \mathcal{L}$. Let us define a *composition* operation $\otimes$ by associating to any comparable pair $x, y \in L = \{x, y, z\}$ the third point of the line containing them, i.e.

$$x \otimes y = z. \tag{1.2}$$

Given a subset $S \subseteq \mathbf{T}^n$, its *emission* is defined as the set of points

$$\partial S = \{z | z = x \otimes y \text{ for some } x, y \in S\} \tag{1.3}$$

which can be generated from S by one composition. Let us call a set $S \subseteq \mathbf{T}^n$ *independent* if $\partial S = \emptyset$, *emissive* if $S \cap \partial S = \emptyset$, and *closed* if $\partial S \subseteq S$. Let us note that independent sets are also closed. Finally, let us call a subset $S \subseteq \mathbf{T}^n$ *open*, if its complement is closed, i.e. if $S \cap \partial(\mathbf{T}^n \setminus S) = \emptyset$.

Let $S^{(0)} = S$, and let us define recursively

$$S^{(k)} = S^{(k-1)} \cup \partial S^{(k-1)} \tag{1.4}$$

for $k \geq 1$. Since $\mathbf{T}^n$ is finite, for every subset $S \subseteq \mathbf{T}^n$ there exists a smallest integer k such that $S^{(k')} = S^{(k)}$ for all $k' \geq k$. Let us call, for this k, the set $S^{(k)}$ the *closure* of S and

[1] Henceforth paper [2] will be referenced to as I. Thus, formula I.11 means formula (11) in [2]. Similarly for sections, properties, and so on.

denote it by S^*. It is easy to see that the mapping $S \mapsto S^*$ is indeed a closure operation, i.e. $S \subseteq S^*$, $S \subseteq T$ implies $S^* \subseteq T^*$, and $S^{**} = S^*$ for all subsets $S, T \subseteq \mathbf{T}^n$.

Let us call a subset $G \subseteq \mathbf{T}^n$ a *generating set*, or in short a *generator* if $G^* = \mathbf{T}^n$.

In our previous paper [1] we proved the following sharp lower bound on the size of a generator.

Theorem 1 *For every generating set $G \subseteq \mathbf{T}^n$ we have $|G| \geq 2^n$.*

In this paper we keep on with the study of generating sets, concentrating on some computational problems related to them. Consider the problem where every point $x \in \mathbf{T}^n$ represents a (large) data set (e.g. a large matrix A_x), and the data sets corresponding to points along a line are related in the sense that the knowledge of any two of those sets is always enough to derive the third one (e.g. $A_x + A_y = A_z$ for a line $L = \{x, y, z\} \in \mathcal{L}$, and thus $A_x = A_z - A_y$, etc.) In such a situation it is important to know which collections of data sets are enough to store in order to be able to reconstruct all of them.

It is also interesting to know how *fast* is this reconstruction, i.e. how many derivation steps are needed to obtain a particular data set. In this paper we investigate the computational complexity of the derivation of a data set (vectors in $\mathbf{T}^n$) from a given collection of points (a generator). In fact, even if it is known that the minimum cardinality of a generator in $\mathbf{T}^n$ is 2^n, it is not clear, if all minimal generating sets are of the same size, and if generating sets are equivalent from other computational points of view.

Let us consider the problem of obtaining the data sets from a generator in a parallel or distributed computing environment. For $S \subseteq \mathbf{T}^n$ and $x \in \mathbf{T}^n$, let the *depth* of x (with respect to the set S) be defined as

$$\delta(x, S) = \begin{cases} 0 & \text{if } x \in S, \\ k & \text{if } x \in S^{(k)} \setminus S^{(k-1)},\ k \geq 1, \\ \infty & \text{if } x \notin S^*. \end{cases} \tag{1.5}$$

If G is a generator, then $\delta(x, G)$ is the time one needs to compute x from G using parallel processors (assuming that each derivation step takes a unit of time.) Let us define next

$$\Delta(S) = \max_{x \in \mathbf{T}^n} \delta(x, S), \tag{1.6}$$

representing a worst-case complexity for the set S, and let us call it the *depth* of the set S. Let us note that $\Delta(S) < \infty$ only if S is a generating set.

Our next result gives a sharp lower bound on the depth of a generator.

Theorem 2 *For every minimal generator G of $\mathbf{T}^n$ we have $\Delta(G) \geq n$.*

Let us also consider the computational complexity of deriving unstored points in a single-processor computing environment. Let us denote by $\lambda(x, G)$ the minimum number of compositions one needs to obtain x from the points of the generator G (assuming enough memory to store intermediate results, as needed). Let us call the quantity $\lambda(x,G)$ the *load of x* with respect to the generator G. Let us define further the *load* of a generator by

$$\Lambda(G)=\max_{x\in\mathbf{T}^n}\lambda(x,G), \tag{1.7}$$

and let

$$\lambda_n=\min_{\substack{G\subseteq\mathbf{T}^n\\ \text{generator}}}\Lambda(G). \tag{1.8}$$

The next theorems give bounds on the load of a generator.

Theorem 3 *For every generator G of $\mathbf{T}^n$ of size 2^n we have*

$$\Lambda(G)\geq\left(\frac{3}{2}\right)^{\frac{n}{\log n}}-1. \tag{1.9}$$

Theorem 4 *One can construct a generator $G_n\subseteq\mathbf{T}^n$ for which $|G_n|=2^n$, $\Delta(G_n)=n$ and $\Lambda(G_n)\leq\frac{8}{7}(1.53)^n-1$ for $n\geq 4$. This implies, in particular that*

$$\lambda_n\leq\frac{8}{7}(1.53)^n-1 \quad \textit{for} \quad n\geq 4. \tag{1.10}$$

One can also see that $\lambda_1=1$, $\lambda_2=2$, and $\lambda_3=3$.

In Section 2 we introduce some new notations related to the process of generating vectors in ternary Hamming spaces, such as the *ancestry* of a vector and the *depth* of a generator, giving the exact value of the minimum depth for a generator $G\subseteq\mathbf{T}^n$ for all n. In Section 3 we describe the concept of *load* and provide a lower bound for the minimum load of a generator $G\subseteq\mathbf{T}^n$. In Section 4 we prove Theorem 4 which gives the exact value of the minimum load for a generator $G\subseteq\mathbf{T}^n$ when $n=1,2,3$, and provides an upper bound for the same value for $n\geq 4$.

2 Ancestry and depth

For a subset $S\subseteq\mathbf{T}^n$ and for a point $x\in S^{(k)}\setminus S^{(k-1)}$ there exist points $y,z\in S^{(k-1)}$, by the definition of $S^{(k)}$, such that $x=y\otimes z$. We could call y and z the *parents* of x. (Let us remark that the pair (y,z) of parents of a point x may not be unique.)

Let us now call a pair $\mathcal{A}=\{m,f\}$ of mappings $m,f:(S^*\setminus S)\mapsto S^*$ an *ancestry* structure of S, if the following properties hold:

(a) For every $x\in S^*\setminus S$, we have $\delta(x,S)>\delta(f(x),S)$ and $\delta(x,S)>\delta(m(x),S)$.

(b) For every $x\in S^*\setminus S$, we have $x=f(x)\otimes m(x)$.

We shall call the elements $m(x)$ and $f(x)$ the *mother* and the *father* of x, respectively. Let us remark that a set S may have many different ancestry structures.

Given a set S and an ancestry structure $\mathcal{A} = \{m, f\}$ of it, let us call an element $y \in S^*$ an *ancestor* of $x \in S^* \setminus S$, if $y = p_1(p_2(...(p_l(x))...))$, where each letter p_j denotes a "parental" relation, i.e. $p_j \in \{m, f\}$ for $j = 1, ..., l$. Let us denote by $\mathcal{A}(x)$ the set of ancestors of x.

Proposition 5 *For every set $S \subseteq \mathbf{T}^n$ and every ancestry structure $\mathcal{A} = \{m, f\}$ of S, we have $x \in (S \cap \mathcal{A}(x))^*$ for all points $x \in S^*$.*

Proof. Immediate by the definition of ancestry. □

Proposition 6 *Let S be a subset of $\mathbf{T}^n$, let $\mathcal{A} = \{m, f\}$ be an ancestry structure of S, and let $s \in S$. Then $s \in \mathcal{A}(x)$ for every $x \in U(s, S)$.*

Proof. If $s \notin \mathcal{A}(x)$ for some $x \in U(s, S)$, then $(S \cap \mathcal{A}(x))^* \subseteq (S \setminus \{s\})^*$, and thus $(S \cap \mathcal{A}(x))^* \cap U(s, S) = \emptyset$, i.e. $x \notin (S \cap \mathcal{A}(x))^*$ would follow, contradicting to Proposition 5. □

Let us call an element $s \in S$ *essential* for $x \in \mathbf{T}^n$ if $x \in U(s, S)$, i.e. if x cannot be generated from $S \setminus \{s\}$. Then Proposition 6 states, in other words that an element s is essential for $x \in \mathbf{T}^n$ if and only if s belongs to $\mathcal{A}(x)$ for all ancestries of S.

Proposition 7 *Let S be a subset of $\mathbf{T}^n$, and $s \in S$. Then, $\delta(x, S) \geq d_H(x, s)$ for every $x \in U(s, S)$.*

Proof. Let us fix an ancestry $\mathcal{A} = \{m, f\}$ of S, and let $x \in U(s, S)$ be arbitrary. Then by Proposition 6, we have $s \in \mathcal{A}(x)$, and thus s can be represented in the form

$$s = p_1(p_2(...(p_l(x))...))$$

for some parental relations $p_j \in \{m, f\}$. Here we have $l \leq \delta(x, S)$ by property (a) of the ancestry. This implies that there is a series of l compositions starting from s and ending with x. Since each composition can increase the Hamming distance by at most one, we obtain $d_H(x, s) \leq l \leq \delta(x, S)$. □

For $x \in \mathbf{T}^n$ and $0 \leq k \leq n$ let

$$W(x, k) = \{y \in \mathbf{T}^n | d_H(x, y) \geq k\}. \tag{2.11}$$

Proposition 8 *For every $x \in \mathbf{T}^n$ and every $k \in [n]$, the set $W(x, k)$ is an open set.*

Proof. It is easy to verify that $\min_{a \in L} d_H(x, a)$ is attained exactly once on every line L and for every point x. Thus, if $d_H(x, y) \geq k$ for a point $y \in L$ for a line $L \in \mathcal{L}$, then there must be at least one other point $z \in L$, $z \neq y$ for which $d_H(x, z) \geq d_H(x, y) \geq k$. □

We are now ready to prove Theorem 2.

Proof of Theorem 2. Let us assume first, indirectly that there exists a minimal generating set G, for which $\Delta(G) < n$. Let us fix an arbitrary element $g \in G$. According to Proposition 7, $d_H(x,g) \le \delta(x,G) \le \Delta(G) < n$ for all $x \in U(g,G)$, and hence the nonempty open set $U(g,G)$ is disjoint from the nonempty set $W(x,n)$, which is an open set by Proposition 8. This however is in contradiction with Proposition I.11, according to which any two nonempty open sets must intersect. This contradiction proves that $\Delta(G) \ge n$ for every minimal generating set G. For the sharpness of this inequality. let us note finally that $\Delta(\mathbf{B}_n) = n$ by Proposition I.4. □

Let us remark that $\Delta(G)$ can be as large as 2^n, e.g. Figure 1 shows a generator in $\mathbf{T}^3$ with depth 8.

3 Generating sequences and load

Let us generalize the notion of ancestry. Let us consider a generator $G \subseteq \mathbf{T}^n$, and let us call a permutation $\pi : \mathbf{T}^n \mapsto \{1, 2, ..., 3^n\}$ a *generating sequence*, if for every element $x \in \mathbf{T}^n$ either $x \in G$, or $x = y \otimes z$ for some elements $y, z \in \mathbf{T}^n$ for which $\max\{\pi(y), \pi(z)\} < \pi(x)$. Let us call now a pair $\mathcal{A} = \{m, f\}$ of mappings $m, f : (\mathbf{T}^n \setminus G) \mapsto \mathbf{T}^n$ a *π-ancestry* of G if these mappings satisfy properties (a') and (b), where

(a') For every element $x \in \mathbf{T}^n \setminus G$, we have $\max\{\pi(m(x)), \pi(f(x))\} < \pi(x)$.

Let us call a pair of mappings $\mathcal{A} = \{m, f\}$ simply an *ancestry* if it is a π-ancestry for some generating sequence π, and let again $\mathcal{A}(x)$ denote the set of all ancestors of x with respect of the ancestry $\mathcal{A}$. Finally, let us denote by $\aleph(G)$ the family of all ancestries of G.

Let us note that by the definition of the load we have

$$\lambda(x,G) = \min_{\mathcal{A} \in \aleph(G)} |\mathcal{A}(x) \setminus G|. \tag{3.12}$$

Let us call an ancestry *x-minimal*, if $\lambda(x,G) = |\mathcal{A}(x) \setminus G|$.

Let us remark that not all x-minimal ancestries are depth monotone. E.g. the element x with the maximum load ($= 9$) in Figure 1 has depth 7, but for all x-minimal ancestries $\mathcal{A}$ the element y at depth 8 belongs to $\mathcal{A}(x)$.

Proposition 9 *Given a generator G and an element $x \in \mathbf{T}^n \setminus G$, the inequality*

$$|\mathcal{A}(x) \setminus G| \ge |\mathcal{A}(x) \cap G| - 1 \tag{3.13}$$

holds for every ancestries $\mathcal{A} \in \aleph(G)$.

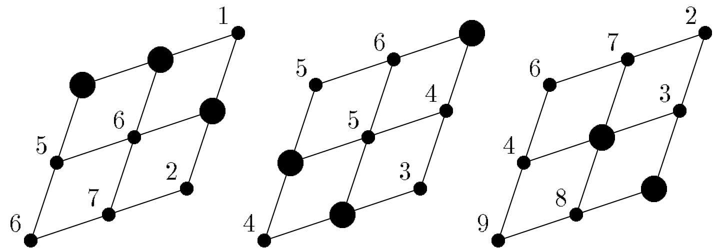

Figure 1: A generator G in $\mathbf{T}^3$ for which $D_H(G) = 1$, $\Delta(G) = 8$, and $\Lambda(G) = 9$. The numbers at the vertices indicate the $\lambda(x, G)$ values.

Proof. Let us associate a directed graph to an ancestry $\mathcal{A} \in \aleph(G)$, the vertices of which are the points in $\mathcal{A}(x)$ and the arcs of which are the pairs $(m(y), y)$, $(f(y), y)$ for all $y \in \mathcal{A}(x) \setminus G$. Clearly, in this graph for the indegrees and outdegrees we have

$$d^-(y) = \begin{cases} 2 & \text{if } \ y \in \mathcal{A}(x) \setminus G, \\ 0 & \text{if } \ y \in G, \end{cases} \quad \text{and} \quad d^+(y) \begin{cases} = \ 0 & \text{if } \ y = x, \\ \geq \ 1 & \text{if } \ y \in \mathcal{A}(x), y \neq x. \end{cases} \tag{3.14}$$

Thus, since both the indegrees and the outdegrees sum up to the number of arcs in a directed graph, we have

$$2|\mathcal{A}(x) \setminus G| = \sum_{y \in \mathcal{A}(x)} d^-(y) = \sum_{y \in \mathcal{A}(x)} d^+(y) \geq |\mathcal{A}(x)| - 1,$$

from which the statement follows by $|\mathcal{A}(x)| - |\mathcal{A}(x) \setminus G| = |\mathcal{A}(x) \cap G|$. □

For any pair of points $x, y \in \mathbf{T}^n$, let

$$B[x, y] = \{z \in \mathbf{T}^n | z_j \in \{x_j, y_j\} \ \text{ for all } \ j = 1, ..., n\}. \tag{3.15}$$

Furthermore, let us denote by $L_i(x) = \{(x_1, ..., x_{i-1}, \alpha, x_{i+1}, ..., x_n) | \alpha = 0, 1, 2\}$ the *line of direction i through the point x.*

Proposition 10 *For $x \in \mathbf{T}^n$ and $z \in W(x, n)$ the set $B[x, z]$ is an open set.*

Proof. In fact, $B[x, z]$ is en even set of size 2^n, in a one-to-one correspondence with $\mathbf{B}^n$. To see this let $y \in B[x, z]$ arbitrary. Then for every index $i \in [n]$ the two points $(y_1, ..., y_{i-1}, x_i, y_{i+1}, ..., y_n)$ and $(y_1, ..., y_{i-1}, z_i, y_{i+1}, ..., y_n)$ are exactly the elements of the intersection $B[x, z] \cap L_i(y)$. 1 □

Proposition 11 *Given $x, y \in \mathbf{T}^n$, there are exactly $2^{n - d_H(x,y)}$ points $z \in W(x, n)$ for which $y \in B[x, z]$.*

Proof. Clearly, the only points $z \in W(x,n)$ for which $y \in B[x,z]$ are those for which $z_j \neq x_j$ if $x_j = y_j$ and $z_j = y_j$ if $x_j \neq y_j$. □

Proposition 12 *For every $S \subseteq \mathbf{T}^n$ and $x \in S^*$ one must have*

$$\sum_{y \in S} 2^{-d_H(x,y)} \geq 1. \tag{3.16}$$

Proof. Let us assume indirectly that

$$\sum_{y \in S} 2^{-d_H(x,y)} < 1$$

holds for some $S \subseteq \mathbf{T}^n$ and some $x \in S^*$. According to Proposition 11 there are exactly $2^{n-d_H(x,y)}$ points $z \in W(x,n)$ for which $y \in B[x,z]$, hence the number of points of $z \in W(x,n)$ with $S \cap B[x,z] \neq \emptyset$ is not more than $\sum_{y \in S} 2^{n-d_H(x,y)}$, which is strictly less than 2^n by our indirect assumption. Hence, there exists a point $z \in W(x,n)$ for which $S \cap B[x,z] = \emptyset$. Since $B[x,z]$ is an open set by Proposition 10, this would then imply $B[x,z] \cap S^* = \emptyset$, contradicting the fact that $x \in B[x,z] \cap S^*$. □

Proposition 13 *Let $S \subseteq \mathbf{T}^n$ and let $x \in S^*$. Then*

$$|S| \geq 2^{d_H(x,S)}. \tag{3.17}$$

Proof. We have

$$|S| 2^{n-d_H(x,S)} \geq \sum_{y \in S} 2^{n-d_H(x,y)} \geq 2^n$$

where the first inequality follows by $d_H(x,S) \leq d_H(x,y)$ for all $y \in S$, and the second inequality is a consequence of Proposition 12. □

Proposition 14 *For every generator G and point x we have*

$$\lambda(x,G) \geq 2^{d_H(x,G)} - 1. \tag{3.18}$$

Proof. Let us consider an ancestry $\mathcal{A} \in \aleph(G)$. Clearly, $x \in (G \cap \mathcal{A}(x))^*$ by the definition of an ancestry, and thus by Proposition 13 we obtain

$$|G \cap \mathcal{A}(x)| \geq 2^{d_H(x, G \cap \mathcal{A}(x))},$$

which by Proposition 9 implies

$$|\mathcal{A}(x) \setminus G| \geq 2^{d_H(x, G \cap \mathcal{A}(x))} - 1.$$

Since this is true for all ancestries $\mathcal{A} \in \aleph(G)$, and since $d_H(x, G \cap \mathcal{A}(x)) \geq d_H(x,G)$ always holds, the statement follows by 3.12. □

Proposition 15 *For every generator G of $\mathbf{T}^n$ of size 2^n we have*

$$D_H(G) \geq \log(\frac{3}{2})\frac{n}{\log n}. \tag{3.19}$$

Proof. Let us denote by $k = D_H(G)$, i.e. $d_H(x,G) \leq k$ for all points $x \in \mathbf{T}^n$. Let $H(x,k)$ denote the Hamming ball of radius k centered at x, i.e. $H(x,k) = \{y | d_H(x,y) \leq k\}$. Then we have

$$3^n = |\mathbf{T}^n| = \left| \bigcup_{g \in G} H(g,k) \right| \leq \sum_{g \in G} |H(g,k)| = 2^n \left(\sum_{j=0}^{k} \binom{n}{j} 2^j \right).$$

The inequality in the middle is also known as the "sphere packing bound" in coding theory, see e.g. [5].

Since

$$\sum_{j=0}^{k} \binom{n}{j} 2^j \leq n^k$$

for $4 \leq k \leq n$, we obtain

$$\frac{3^n}{2^n} \leq n^k$$

implying

$$k \geq \frac{n}{\log n} \log(\frac{3}{2}).$$

□

Proof of Theorem 3. Follows immediately from Propositions 14 and 15. □

4 Generators with low load

In this section we shall construct generators with a relatively small load value, while keeping their size and depth at the optimal level.

The first examples for such generators are presented in Figure 2, where $G_3 = \{a,b,c,d,e,f,g,h\}$ is a 3-dimensional example, while the part $G_2 = \{a,b,c,d\}$ on the first pane of this picture shows a 2-dimensional example.

Let us consider first a sequence of integers $a(n)$, $n = 0, \ldots$ defined by $a(i) = i$ for $0 \leq i \leq 3$ and

$$a(n) = \max\{2 + a(n-2) + 2a(n-3), 3 + 2a(n-3) + 2a(n-4)\}, \tag{4.20}$$

for $n \geq 4$.

The next table shows the values of the first few elements of this sequence.

We shall construct generators $G_n \subseteq \mathbf{T}^n$ of size 2^n the load of which is bounded from above by $a(n)$. More precisely, we shall show by a recursive construction that there exist generators $G_n \subseteq \mathbf{T}^n$ of size 2^n satisfying the following conditions for all $n = 1, 2, \ldots$:

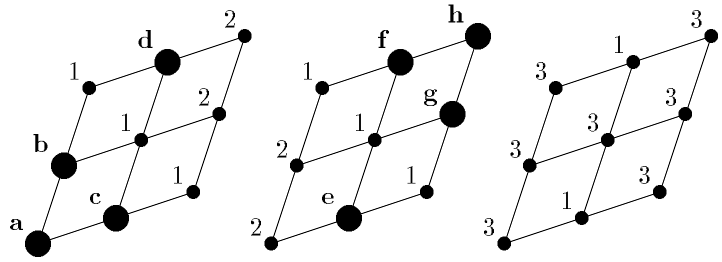

Figure 2: A generator G_3 in $\mathbf{T}^3$ for which $D_H(G_3) = 2$, $\Delta(G_3) = 3$, and $\Lambda(G_3) = 3$. The numbers at the vertices indicate the $\lambda(x, G)$ values.

n	0	1	2	3	4	5	6	7	$\cdots$
$a(n)$	0	1	2	3	6	9	14	23	$\cdots$

(A) $\lambda(x, G_n) \leq a(n)$ for points $x \in \mathbf{T}^n$ with $x_n = 2$;

(B) $\lambda(x, G_n) \leq a(n-1)$ for points $x = (x_1, ..., x_n) \in \mathbf{T}^n$ with $x_n \neq 2$, $n \geq 2$.

It is easy to see that, $G_1 = \{(0), (1)\}$, as well as $G_2 = \{(0,0), (1,0), (1,1), (2,1)\}$ and $G_3 = \{(0,0,0), (1,0,0), (1,1,0), (2,1,0), (1,1,1), (2,1,1), (0,2,1), (2,2,1)\}$ shown in Figure 3 satisfy the above conditions.

Let us now represent $\mathbf{T}^n = \mathbf{T}^{n-3} \times \mathbf{T}^3$ for $n \geq 4$, and let us define G_n by placing permuted copies of G_{n-3} in the $(n-3)$-subspaces corresponding to the points of $G_3 \subset \mathbf{T}^3$. More precisely, let us denote by $G_{n-3,\alpha}$ the isomorphic copy of G_{n-3} obtained by cyclically shifting the last coordinate values by α, i.e.

$$G_{n-3,\alpha} = \{(y_1, ..., y_{n-4}, y_{n-3} + \alpha \bmod 3) | (y_1, ..., y_{n-3}) \in G_{n-3}\}.$$

With this notation let us then define G_n as

$$G_n = \{(y_1, ..., y_{n-3}, z_1, z_2, z_3) | (y_1, ..., y_{n-3}) \in G_{n-3,z_1} \text{ and } (z_1, z_2, z_3) \in G_3\}.$$

Let us now show first that G_n is indeed a generator of $\mathbf{T}^n$, of size 2^n. Since we constructed G_n by taking $|G_3|$ copies of G_{n-3}, we get $|G_n| = |G_3||G_{n-3}| = 2^3 2^{n-3} = 2^n$. Let us observe also that the points in the $(n-3)$-subspaces corresponding to the elements of G_3 are generated by the isomorphic copies of G_{n-3} selected within those subspaces. Since all points of $\mathbf{T}^3$ can be generated from G_3, all points of the corresponding $(n-3)$-subspaces can also be generated from the corresponding points of those $(n-3)$-subspaces which correspond to G_3. Hence, G_n is indeed a generating set.

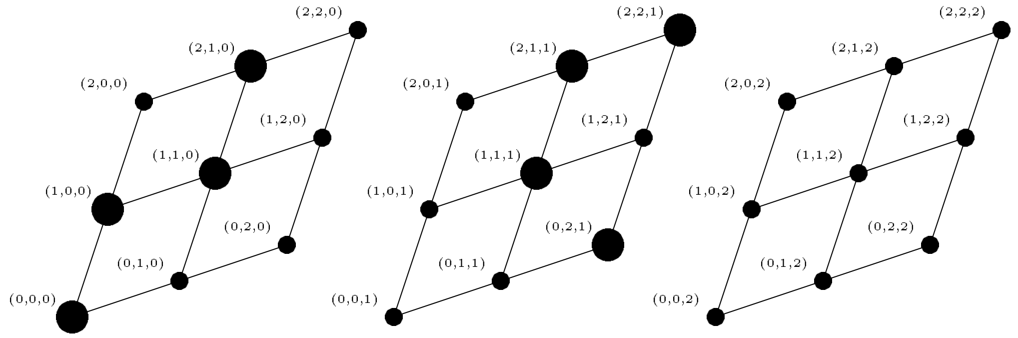

Figure 3: A generator G_3 in $\mathbf{T}^3$.

To show that G_n satisfies properties (A) and (B) we need to analyze how the composition works in these constructions.

Let us consider first the points of the $(n-3)$-subspaces corresponding to points $(z_1, z_2, z_3) \in \mathbf{T}^3$ with $z_3 = 2$. For the points of the form $(\mathbf{y}, z_1, z_2, z_3) \in \mathbf{T}^n$, where $\mathbf{y} \in \mathbf{T}^{n-3}$, and $(z_1, z_2, z_3) \in \{(2,1,2), (1,1,2)\}$, e.g. for $(\mathbf{y}, 2, 1, 2) \in \mathbf{T}^n$ we have

$$(\mathbf{y}, 2, 1, 2) = (\mathbf{y}, 2, 1, 0) \otimes (\mathbf{y}, 2, 1, 1)$$

thus we can write

$$\begin{aligned}
\lambda((\mathbf{y}, 2, 1, 2), G_n) &\leq 1 + \lambda((\mathbf{y}, 2, 1, 0), G_n) + \lambda((\mathbf{y}, 2, 1, 1), G_n) \\
&\leq 1 + \lambda(\mathbf{y}, G_{n-3,2}) + \lambda(\mathbf{y}, G_{n-3,2}) \\
&\leq 1 + 2a(n-3) \\
&\leq a(n).
\end{aligned}$$

Here the first inequality follows from the definition of the load, the second inequality follows by the fact that the points on the right hand side in the first row can all be generated, according to our construction of G_n, within the corresponding $(n-3)$-subspaces by the permuted copies of G_{n-3}. The third inequality is implied by property (A) of G_{n-3}, and the last inequality follows from the definition of $a(n)$.

Analogous arguments can be repeated for all other points of the form $(\mathbf{y}, z_1, z_2, z_3) \in \mathbf{T}^n$ with $z_3 = 2$. E.g. for $(z_1, z_2, z_3) = (0, 0, 2)$ we have

$$(\mathbf{y}, 0, 0, 2) = (\mathbf{y}, 0, 0, 0) \otimes ((\mathbf{y}, 0, 2, 1) \otimes ((\mathbf{y}, 1, 1, 1) \otimes (\mathbf{y}, 2, 1, 1)))\,.$$

Thus we can write

$$\begin{aligned}
\lambda((\mathbf{y}, 0, 0, 2), G_n) &\leq 3 + \lambda((\mathbf{y}, 0, 0, 0), G_n) + \lambda((\mathbf{y}, 0, 2, 1), G_n) + \lambda((\mathbf{y}, 1, 1, 1), G_n) + \lambda((\mathbf{y}, 2, 1, 1), G_n) \\
&\leq 3 + +\lambda(\mathbf{y}, G_{n-3,0}) + \lambda(\mathbf{y}, G_{n-3,0}) + \lambda(\mathbf{y}, G_{n-3,1}) + \lambda(\mathbf{y}, G_{n-3,2}) \\
&\leq 3 + 2a(n-3) + 2a(n-4) \\
&\leq a(n).
\end{aligned}$$

Here the third inequality follows by property (B) of G_{n-3}.

Exactly the same estimate works for points $(\mathbf{y}, z_1, z_2, z_3)$ with $(z_1, z_2, z_3) \in \{(1,0,2), (2,0,2), (0,2,2), (1,2,2), (2,2,2)\}$. For the points $(\mathbf{y}, z_1, z_2, z_3) \in \mathbf{T}^n$ with $(z_1, z_2, z_3) = (0,1,2)$ we can write

$$\begin{aligned}\lambda((\mathbf{y},0,1,2),G_n) &\le 3 + \lambda((\mathbf{y},1,1,1),G_n) + \lambda((\mathbf{y},2,1,1),G_n) + \lambda((\mathbf{y},1,1,0),G_n) + \lambda((\mathbf{y},2,1,0),G_n)\\ &\le 3 + \lambda(\mathbf{y},G_{n-3,1}) + \lambda(\mathbf{y},G_{n-3,2}) + \lambda(\mathbf{y},G_{n-3,1}) + \lambda(\mathbf{y},G_{n-3,2})\\ &\le 3 + 2a(n-3) + 2a(n-4)\\ &\le a(n).\end{aligned}$$

The above show that property (A) holds for G_n. To verify property (B), we shall consider the points $(z_1, z_2, z_3) \in \mathbf{T}^3$ with $z_3 \neq 2$.

For the points $(\mathbf{y}, z_1, z_2, z_3) \in \mathbf{T}^n$ for which $(z_1, z_2, z_3) \in \{\, (2,0,0), (0,1,0), (1,2,0), (2,0,1), (0,1,1), (1,2,1)\}$ we get a similar chain of inequalities, e.g. for $(\mathbf{y},2,0,0)$ we have

$$(\mathbf{y},2,0,0) = (\mathbf{y},0,0,0) \otimes (\mathbf{y},1,0,0)$$

and hence we can write

$$\begin{aligned}\lambda((\mathbf{y},2,0,0),G_n) &\le 1 + \lambda((\mathbf{y},0,0,0),G_n) + \lambda((\mathbf{y},1,0,0),G_n)\\ &\le 1 + \lambda(\mathbf{y},G_{n-3,0}) + \lambda(\mathbf{y},G_{n-3,1})\\ &\le 1 + a(n-3) + a(n-4)\\ &\le a(n-1).\end{aligned}$$

Finally, for the points $(\mathbf{y}, z_1, z_2, z_3) \in \mathbf{T}^n$ with $(z_1, z_2, z_3) \in \{(0,2,0), (2,2,0), (0,0,1), (1,0,1)\}$, e.g. for $(\mathbf{y},0,2,0)$ we have

$$(\mathbf{y},0,2,0) = (0,0,0) \otimes ((1,1,0) \otimes (2,1,0))$$

and hence we can write

$$\begin{aligned}\lambda((\mathbf{y},0,2,0),G_n) &\le 2 + \lambda((\mathbf{y},0,0,0),G_n) + \lambda((\mathbf{y},1,1,0),G_n) + \lambda((\mathbf{y},2,1,0),G_n)\\ &\le 2 + \lambda(\mathbf{y},G_{n-3,0}) + \lambda(\mathbf{y},G_{n-3,1}) + \lambda(\mathbf{y},G_{n-3,2})\\ &\le 2 + a(n-3) + 2a(n-4)\\ &\le a(n-1).\end{aligned}$$

Proof of Theorem 4. A well established result of the theory of linear difference equations [4] implies that, in view of (4.20), there exists an $\alpha > 0$ such that

$$1 + a(n) \le C\alpha^n$$

for some constant C. Then from the two recursive inequalities (4.20) we obtain

$$\begin{aligned}2 + \alpha &\le \alpha^3 \ \text{ and}\\ 2 + 2\alpha &\le \alpha^4.\end{aligned}$$

Hence $\alpha \geq 1.53$ is a solution to both of the above inequalities, and thus we obtain

$$a(n) \leq \frac{8}{7}(1.53)^n$$

for $n \geq 4$.

To see that $\Delta(G_n) \leq n$, we have to observe that, by the construction of G_n, the hyperplanes H with $x_n \neq 2$ are such that $H = (H \cap G)^*$, and thus one more step is enough to generate any point in $\mathbf{T}^n$, implying the claim by induction. $\square$

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