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Some Results Concerning the Ends of Minimal Cuts of Simple Graphs

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ABSTRACT

Let S be a cut of a simple connected graph G. If S has no proper subset that is a cut, we say S is a minimal cut of G. To a minimal cut S, a connected component of $G-S$ is called a fragment. And a fragment with no proper subset that is a fragment is called an end. We characterized ends in our discussions and proved that to a connected graph $G=(V,E)$, the number of its ends $\sum \leq |V(G)|$.

We refer to $G = (V, E)$ will always denote finite non-complete simple connected graph in this paper.The notations not mentioned are with the same meaning as in referenc[1]. For $A \subset V(G)$ we use $\Gamma(A)$ to denote the ajacent set of A,that is, $\Gamma(A) = v|uv \in E(G), u \in A$, and we put $N(A) = \Gamma(A) - A$. $< A >$ is a subgraph induced from $A \subset V(G)$, that is,$< A >= G[A]$. A set of vertices $S \subset V(G)$ is called a cut of G if there are at least 2 connected components in $G - S$. A minimal cut is a cut without a proper subset that is a cut. If S is a minimal cut, then a connected component of $G - S$ is called a fragment. And a fragment with no proper subset this is a fragment is called an end. It is obvious all the graphs we discuss have cuts and furthermore, minimal cuts. Thus,a graph has at least two fragments.Since all fragments have ends,a graph has at least two ends.

It is obvious that the following assertion holds:

Statement P. Let S be a cut of G,S is a minimal cut iff for any $u \in S$ and a conneted component of $G - S$,$N(u) \cap A \neq \emptyset$.

Definition 1. If $S1$ and $S2$ are two cuts of G and there are at least two connected components of $G - S1$ contain vertices of $S2$,we say $S1$ interferes with $S2$.

Theorem 2. Let $S1$ and $S2$ be two minimal cuts of G and $S1$ interferes with S2, then there exist vertices of S1 in every fragment of $G - S2$.

Proof. Surpose $< A >$ is a fragment of $G - S2$ and $A \cap S1 = \emptyset$, then because $< A >$ is connected, all the vertices in A must belong to a fragment of $G - S1$.By statement P,for each $v \in S2 - S1$, $N(v) \cap A \neq \emptyset$.Then $< A \cup S2 - S1 >$ is connected. Thus the vertices of $S2 - S1$ can only be in one fragment of $S2 - S1$ can only be in one fragment of $G - S1$,and it contradicts with the fact that $S1$ interferes with $S2$.

By theorem 2, $S1$ interferes with $S2$ and $S2$ interferes with $S1$ are equivalent assertions.

Theorem 3. Let $< A >$ be an end of G, then for any $u \in A$, every minimal cut of G that contains u interferes with $N(A)$.

Proof. Suppose S is a minimal cut of G that contains u and does not interfere with $N(A)$, then by applying theorem 2 and definition 1, $S - N(A) \subset A$. And since $N(u) \subset (A \cup N(A))$,from statement P, every fragment in $G - S$ contains vertices in A or $N(A)$. Since S does not interfere with $N(A)$, there is at least one of such fragments that does not contain any vertex in $N(A)$. Thus it contains vertices in A and only in A. But this fragment does not contain u, then it is a proper subset of A, contradicting the fact that $< A >$ is an end.

Corollary 4. Let S be a minimal cut of G. If G does not contain any minimal cuts that interfere with S, then any vertex in S cannot belong to any end of G.

Theorem 5. Suppose $< A >$ is a fragment of G,then $< A >$ is an end of G iff $N(A)$ is the only minimal cut that is contained by $A \cup N(A)$.

Proof. a. From theorem 3, if $< A >$ is a fragment of G, and $N(A)$ is the only minimal cut that is contained by $A \cup N(A)$, then $< A >$ is an end of G.

b. For every non-empty proper subset of A, say A', since $N(A') \subset A \cup N(A)$, and $N(A)$ is the only minimal cut that is contained by $A \cup N(A)$, then $N(A')$ is not a minimal cut, thus $< A' >$ is not a fragment and $< A >$ is an end.

Theorem 6. Let $< A >$ be a fragment of G, then $< A >$ is an end iff for any $u \in A$ and $v \in A$, $uv \in E(G)$.

Proof. a. There are $u \in A$ and $v \in N(A)$ and $uv \notin E$. Consider all the u-v paths in G. There must exist vertices of $A \cup N(A) - \{u, v\}$ in every such path. Delete all the vertices in $A \cup N(A) - \{u, v\}$ from each path, then we obtain a disconnected graph with no uv-paths. Thus the vertices we deleted is a cut of G. But this cut is contained by $A \cup N(A)$ and it is not $N(A)$, contradicting theorem 5.

b. Let A' be a proper subset of A. It is obvious that $N(A) \subset N(A')$ and $N(A') \cap (A - A') \neq \emptyset$ under the conditions of the theorem,then $N(A')$ is not a minimal cut. Thus $< A' >$ is not a fragment, then $< A >$ is an end.

Corollary 7. Let $< A >$ be an end of G, then all the minimal cuts that intefere with $N(A)$ contain A.

Theorem 8. Let $< A >$ be an end of G, and $< B >$ a fragment of G that does not contain A, then $A \cap B = \emptyset$.

Proof. Under the conditions of the theorem, if $A \cap B \neq \emptyset$, then since $A - B \neq \emptyset$ and $< A >$ is connected,we have $N(B) \cap A \neq \emptyset$. Thus,if $N(B)$ interferes with $N(A)$, by applying corollary 7, $A \subset N(B)$. It contradicts $A \cap B \neq \emptyset$;if $N(B)$ does not interfere with $N(A)$, from $N(B) \cap A \neq \emptyset$,we have $N(B) \subset A \cup N(A)$. If $N(B) \neq N(A)$, it will contradict theorem 5; if $N(B) = N(A)$, then we have $B \subset A$, which contradicts with the fact that $< A >$ is a fragment,or $B \subset V(G) - (A \cup N(A))$, which contradicts $A \cap B \neq \emptyset$.

From theorem 8 we know, two distinct ends of G,$< A >$ and $< B >$, there is $A \cap B = \emptyset$. Then by denoting the number of distinct ends of G as $\sum$, we have:

Corollary 9. $\sum \leq |V(G)|$

References

[1] B.Bollobas, *Extremal Graph Theory*, Academic Press, New York, 1978.

[2] H.Veldman, *Non k-critical vertices in graphs*, Discrete Math. **44** (1983), 105-110.