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On Automorphism Groups of Graphs and Distributive Lattices

by

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ABSTRACT

Birkhoff's theorem that every group is isomorphic to the automorphism group of a distributive lattice is extended in a direction that parallels similar results in graph theory.

It is well known that every group is isomorphic to a permutation group. Inherent in the argument that shows this, although not always explicitly presented, is the Cayley color graph construction. That construction produces a directed graph with colored arrows, defined on a given group as vertex set, whose color-preserving automorphisms are precisely the (left) translations of the given group. By replacing colored arrows with an appropriate graph-theoretical construction, Frucht [5] established that every group is isomorphic to the automorphism group of some (undirected, simple) graph. Similar results also exist for other combinatorial structures. The fact that every group is isomorphic to the automorphism group of a distributive lattice was first proved by Birkhoff [2]. The latest and shortest proof is due to Graetzer, Schmidt, and Wang [6]. Their construction may be viewed as replacing colored arrows in a Cayley color graph by an appropriate order-theoretical structure.

Frucht's theorem was extended in a number of directions. One of these involves representing a group-subgroup pair, as in the following result. (See Bouwer [3] and Babai [1].) We recall that a *faithful constituent* of a permutation group P on a set V is a subset $S \subseteq V$ such that for every $f \in P$, $f(S) = S$ and f is uniquely determined by its restriction to S. The result of Babai and Bouwer states that, *given a graph H and a subgroup B of Aut H, there exists a graph G such that:*

(i) H is a subgraph of G induced by a faithful constituent of Aut G, and

(ii) the restriction of Aut G to H is B.

For other references and a further generalization, see an article by G. Sabidussi and this author [4]. Note that the Babai-Bouwer result specializes to Frucht's theorem if we start with any abstract group, represent it by a permutation group B on some set V, then take for H the edge-less graph on vertex set V.

The purpose of this paper is to show that an analogue of the Babai-Bouwer result can be obtained via an appropriate relative of the Graetzer-Schmidt-Wang construction. Both that construction and our proof arrive at a distributive lattice generated by a partially ordered set with appropriate symmetries. In both cases this is to be done in a manner that the distributive lattice generated possesses no additional symmetries. The Graetzer-Schmidt-Wang method relies on completely free generation, and we rely on embedding into a power set lattice. Every distributive lattice can be embedded into a power set lattice (the lattice of subsets of its dual prime ideals). Every sublattice of a power set lattice is distributive.

Theorem *Given a distributive lattice H and a subgroup B of Aut H, there exists a distributive lattice L such that:*

(i) H is a sublattice of L, and it is a faithful constituent of Aut L,

(ii) the restriction of Aut L to H is B.

Proof

To fix the language, adopt the definition of an *ordinal* as a well-ordered set whose members are all lesser ordinals. Recall that every ordinal is *rigid*, i.e. it admits only the identity map as order-automorphism.

In any partially ordered set, we write $A < B$ for subsets A, B if $a < b$ for all $a \in A$, $b \in B$.

If the Theorem is true for bounded distributive lattices H, then it is true for all distributive lattices H. This is so because, for any lattice H, if H' is the lattice obtained from H by adjoining (new) extrema (minimum and maximum), then the sublattice H of H' is a faithful constituent of $Aut\ H'$, and the restriction of $Aut\ H'$ to H is $Aut\ H$. Therefore, we shall assume throughout the proof of the Theorem that H is bounded.

Assume, without loss of generality, that H is disjoint from B.

For every $h \in H$, let α_h be an ordinal such that $\alpha_h \neq \alpha_k$ if $h \neq k$. For each $(\sigma, h) \in B \times H$, let $W_{\sigma h}$ be a well-ordered chain order-isomorphic to the ordinal $\alpha_{\sigma^{-1}(h)}$. Assume, without loss of generality, that the various $W_{\sigma h}$ are pairwise disjoint, and that each is disjoint from both H and B. Let W denote the union of the various $W_{\sigma h}$. Also, for each (σ, h) let $a_{\sigma h}$ be a distinct element not belonging to $H \cup W \cup B$. Let

$$A = \{a_{\sigma h} : \ (\sigma, h) \in B \times H\}$$

Finally, let s_0 and s_1 be two distinct elements not belonging to $H \cup W \cup B \cup A$.

Let the set

$$K = H \cup W \cup B \cup A \cup \{s_0, s_1\}$$

be partially ordered by:

(i) within H the order is the given lattice order,

(ii) within each $W_{\sigma h}$ the order is the chain order stipulated above,

(iii) for each σ and h,

$$\{k \in H : \ k \leq h \text{ in } H\} < W_{\sigma h} < \{\sigma\}$$

$$\{a_{\sigma h}\} < W_{\sigma h} < \{\sigma\}$$

(iv) $\{s_1\} < H \cup W \cup B$

(v) $\{s_0\} < \{s_1\} \cup H \cup W \cup B \cup A$

Note that $H \cup \{s_0, s_1\}$ is a hereditary subset of K and it constitutes a distributive lattice. Therefore, we may assume, without loss of generality, that $H \cup \{s_0, s_1\}$ is a sublattice of a power set lattice $\mathcal{P}(E_0)$ such that the base set E_0 is also disjoint from $W \cup B \cup A$. Define the enlarged base set

$$E = E_0 \cup W \cup B \cup A$$

Consider the map $f: K \longrightarrow \mathcal{P}(E)$ defined as follows. For $x \in H \cup \{s_0, s_1\}$, x is now a subset of E_0, and we let $f(x) = x$. For $x \in W \cup B \cup A$, observe that there is a largest member Y_x in

$$\{Y \in H \cup \{s_0, s_1\} : \ Y \ < \ x \ \text{in} \ K\}$$

Let

$$Z = \{z \in W \cup B \cup A : \ \ z \leq x \ \text{in} \ K\}$$

and define $f(x) = Y_x \cup Z$.

Note the following facts:

(a) f is injective,

(b) $x \leq y$ in K if and only if $f(x) \subseteq f(y)$,

(c) K is a meet semilattice and $f(x \wedge y) = f(x) \cap f(y)$.

We therefore identify K with $f(K)$, write simply x instead of $f(x)$ and S instead of $f(S)$ for elements x and subsets S of K, omitting any reference to f, and we consider K a meet sub-semilattice of $\mathcal{P}(E)$.

Let L be the sublattice of $\mathcal{P}(E)$ generated by K. Obviously L is a distributive lattice and both H and $H \cup \{s_0, s_1\}$ are sublattices of L. Because of distributivity, L consists of all finite unions $I_1 \cup ... \cup I_n$, $n \geq 1$, where each I_j is a finite intersection $F_1 \cap ... \cap F_m$, $m \geq 1$, of members of K. Thus, L consists of all finite unions $F_1 \cup ... \cup F_n$, $n \geq 1$, of members of K.

Note that $H \cup \{s_0, s_1\}$ is a hereditary subset of the lattice L.

We shall now characterize each of the sets A, W, B, and H in terms of properties that must be preserved by every automorphism μ of L. In particular, this will imply that $\mu(H) = H$ for every such automorphism.

A member $x \in L$ is called *irreducible* if $x = y_1 \cup ... \cup y_n$ in L implies that $x = y_i$ for some i. All irreducibles must belong to K.

Note that s_0 is the miminum of L. The atoms of L are s_1 and the members of A.

Both s_0, s_1 and every element of A, W, or B is irreducible in L. The minimum m of the lattice H is also irreducible and it covers s_1 in L. However, no irreducible member of L covers any element of A. *Thus A is the set of those atoms of L that are not covered by any irreducible.*

The set B is the set of maximal elements among the irreducibles of L. Thus, W is the set of those irreducible elements x of L for which there are $u \in A$ and $y \in B$ such that $u < x < y$.

Note that two elements of W are comparable if and only if they belong to the same $W_{\sigma h}$. Thus, *the sets $W_{\sigma h}$ are precisely the maximal chains within W*, and $W_{\sigma h}$ is order-isomorphic to $W_{\tau k}$ if and only if $\sigma^{-1}(h) = \tau^{-1}(k)$ (in which case they are order-isomorphic to the ordinal $\alpha_{\sigma^{-1}(h)} = \alpha_{\tau^{-1}(k)}$). Call $W_{\sigma h}$ a *maximal W-chain*, and say that $x \in L$ is *below* $W_{\sigma h}$ if $x < y$ for all $y \in W_{\sigma h}$. Then, it is easy to verify that *H is the set of those elements of L that are*

(i) *not the minimum of L,*

(ii) *not an atom of L,*

(iii) *not greater than any element of A,*

(iv) *below some maximal W-chain.*

The image of any maximal W-chain by any automorphism μ of L is a maximal W-chain. It follows that $\mu(H) = H$ for any such automorphism.

For any $W_{\sigma h}$, clearly h is the largest element of H below $W_{\sigma h}$. Also, σ is the only element of B such that $y < \sigma$ for all $y \in W_{\sigma h}$. We shall say that $W_{\sigma h}$ *links* σ and h. Clearly, every σ, h are linked by a unique maximal W-chain, and each maximal W-chain links a unique pair $\sigma \in B$, $h \in H$.

Suppose that an automorphism μ of L fixes every element of H. If $\mu(\sigma) = \tau$ for some $\sigma, \tau \in B$, then for each $h \in H$, $W_{\sigma h}$ is order-isomorphic for $W_{\tau h}$, implying $\sigma^{-1}(h) = \tau^{-1}(h)$, i.e. $\sigma = \tau$. Thus, μ also fixes every element of B. The reader can now complete the verification that μ fixes every element of K, and therefore of L: use the rigidity of ordinals and the uniqueness of maximal W-chain links.

It follows that H is a faithful constituent of *Aut* L.

Let $\mu \in$ *Aut* L. Denoting by ϵ the neutral element of the group B, let $\sigma = \mu(\epsilon)$. We claim that the restriction of μ to H is σ.

For every $h \in H$,

$$\mu(W_{\epsilon h}) = W_{\sigma \mu(h)}$$

But $W_{\epsilon h}$ and $W_{\sigma\mu(h)}$ must be order-isomorphic, and thus

$$\sigma^{-1}(\mu(h)) = \epsilon^{-1}(h) = h$$

$$\mu(h) = \sigma(h)$$

Finally, let us show that every $\tau \in B$ can be extended to an automorphism of L. First, show that τ extends to an order automorphism $\overline{\tau}$ of K. Let

$$\overline{\tau}(s_0) = s_0, \quad \overline{\tau}(s_1) = s_1$$

Let $\sigma \in B$, $h \in H$. Let

$$\overline{\tau}(\sigma) = \tau\sigma$$

$$\overline{\tau}(a_{\sigma h}) = a_{\tau\sigma,\, \tau(h)}$$

Then $W_{\sigma h}$ is order-isomorphic to $W_{\tau\sigma,\, \tau(h)}$ because

$$\sigma^{-1}(h) = (\tau\sigma)^{-1}(\tau(h))$$

and we can define $\overline{\tau}$ on $W_{\sigma h}$ to coincide with the unique order-isomorphism from $W_{\sigma h}$ to $W_{\tau\sigma,\, \tau(h)}$.

Note that $\overline{\tau}$ preserves intersections, and for all x, y in $H \cup \{s_0, s_1\}$

$$f(x \cup y) = f(x) \cup f(y)$$

In order to further extend $\overline{\tau}$, we need the following fact: *every element X of L can be represented as a union*

$$X = F_1 \cup ... \cup F_n$$

where $n \geq 1$, $F_1 = X \cap maxH$, and $\{F_i : \ i \geq 2\}$ is an antichain contained in $W \cup A \cup B$. This representation is unique up to a premutation of the F_i, $i \geq 2$. Verification is left to the reader. Note that $n = 1$ if and only if $X \in H \cup \{s_0, s_1\}$.

We can now extend $\overline{\tau}$ to an automorphism $\overline{\overline{\tau}}$ of L by defining, for $X \in L$ represented as above,

$$\overline{\overline{\tau}}(X) = \overline{\tau}(F_1) \cup ... \cup \overline{\tau}(F_n)$$

The verification that $\overline{\overline{\tau}}$ is indeed a lattice automorphism is based on the law of distributivity between the set union and intersection operations. □

Remark 1 If H is finite, then L can be made finite.

Remark 2 Just as the Babai-Bouwer theorem provides an alternative proof of Frucht's theorem for graphs, the theorem above specializes to Birkhoff's corresponding result for distributive lattices. Start with any abstract group, represent it by a permutation group on some set V, then take for H the lattice of all subsets of V.

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