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Linear Programming in Tomography, Probability, and Finance
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by

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ABSTRACT

At the first conference (at DIMACS), it became clear that linear, or convex, programming is going to play a key role in the algorithmics of "discrete tomography", where one attempts to reconstruct a finite subset, S, of the integer lattice, Z^2, or Z^3, from its line sums in a few directions obtained via transmission electron microscopy. The usual convolution-back-projection algorithms for continuous tomography are useless for this new application to crystal microscopy because only very few directions are available for which the line sums are measured. Since the problem of determining a subset S with the given line sums is an integer programming problem the standard convexification of the problem reduces it to linear programming (Peter Fishburn, Peter Schwander, Robert Vanderbei, The Discrete Radon Transform and its Approximate Inversion via Linear programming, Discrete Applied Math., 1997, to appear). This showed that S may be "effectively" reconstructed in the sense that the method of finding a feasible interior point in the convex hull of what is called "fuzzy sets", ie a function, $0 \leq f(z) \leq 1$ for all $z \in Z^2$, or Z^3, "typically" produces a reasonablly satisfactory reconstruction on simulated phantoms. Thus the situation is more or less analogous to conventional medical CT scanning, where even though one cannot hope to uniquely reconstruct a function, $f(x, y)$ from its line integrals in a finite number of fixed directions (even if the line integrals are noiseless) the usual algoorithms are seen to work well, as demonstrated via simulations.

In the above paper written over a year ago, available by request at fish@research.att.com, the method is made clear. Here however I have a different goal: several disparate problems in applied mathematics can be solved via non-trivial use of infinite linear programming and I thought that I would share these with the audience in order to illustrate the scope and the power of linear programming and to bring the ideas to wider view, after first trying to set into perspective the application of linear programming to discrete tomography via a review.

Section 1. A Review of Discrete Tomography

In continuous tomography, one measures the line integrals, $P_f(L) = \int_L f ds$, of an unknown function, $f = f(x, y)$, for $L \in \mathcal{L}$, where $\mathcal{L}$ is a collection of lines, using an X-ray source and a detector, [KS]. We always assume f has compact support, and may also assume, via a choice of units, that $0 \leq f(x, y) \leq 1$. We begin by two remarks about the continuous case which will be relevant for our discrete analog.

Remark 1. Suppose $\mathcal{L}$ is the set of all horizontal and vertical lines and f is a measurable function with $0 \leq f \leq 1$ everywhere. Suppose g is the indicator of a disk and that $P_f(L) \equiv P_g(L), l \in \mathcal{L}$. Is $f = g$, a.e.? The answer is yes, cf [FLRS], since for $g(x, y) = \chi_{x^2+y^2<1}$, we have

$0 = \int\int (1 - x^2 - y^2)(g(x, y) - f(x, y))dxdy$

because $g - f$ is orthogonal to each of $1, x^2, y^2$ since we can integrate first on x or on y and f and g have the same line integrals on each fixed x and y. But the integrand is everywhere non-negative since for $x^2 + y^2 < 1$ each of the factors is non-negative because $g = 1 \geq f$, and for $x^2 + y^2 > 1$ each factor is non-positive because $g = 0 \leq f$. Thus the integrand vanishes a.e. and $1 - x^2 - y^2$ vanishes almost nowhere, so $f - g$ must vanish a.e. qed.

Remark 2. Suppose $0 < f < 1$ everywhere and $\mathcal{L}$ is now the set of all lines with any one of n given slopes. Is there a $g(x, y) = 0$ or 1 everywhere with $P_g(L) \equiv P_f(L), L \in \mathcal{L}$? The answer is yes, cf [GKRS], since the set, $C = \{g; 0 \leq g(x, y) \leq 1, supp(g) \subset supp(f)\}$ is convex and compact in the weak-star topology. Thus C has extreme points. It can be shown, though it is not so easy, [GKRS], that no function g, for which $\{(x, y) : 0 < g(x, y) < 1\}$ has positive Lebesgue measure can be an extreme point. qed

NB. This result does not of course imply that CAT scanning is a fraud. Why? It shows only that one cannot reconstruct *point* estimates of the unknown f, even if one has *all* line integrals in any *finite* number f directions, and even to *infinite precision.* But if one smooths f and g with the same fat test function (such as the indicator of a pixel) then one gets essentially the same image, cf. [KK]

How is this relevant to discrete tomography? In discrete tomography, we have only say 3 or 4 projection directions and we know the line sums, $\nu_S(L)$

$\nu_S(L) = \sum_{z \in L} \chi_S(z)$

of the unknown set S along each line, L, in any of the few directions. More accurately, we *measure* $\nu(L)$, a noisy approximation to $\nu_S(L)$. We want to find a good approximation to S. Convexification of the problem seeks a function (fuzzy set), $f(z), z \in \mathcal{Z}$, where $0 \leq f(z) \leq 1, z \in \mathcal{Z}$, and where $\mathcal{Z} = \mathcal{Z}^{\in}$, or $\mathcal{Z} = \mathcal{Z}^{\ni}$. Now linear programming with use of interior point methodology gives a *central* f, provided that the set

$C = \{g = g(z) : 0 \leq g(z) \leq 1, \nu_g(L) \equiv \nu(L)\}$

where $\nu_g(L) = \sum_{z \in L} g(z)$, is non-empty. The analogy with Remark 1 should now be clear and indeed the same proof as in Remark 1 shows that even n = 2 directions can uniquely determine S, in the case that the true set, S, has no bad rectangles, [FLRS]. This gives at least some hope that for "natural" examples of sets, such as those that might arise as chips or integrated circuits, one might be able to reconstruct S essentially uniquely even though there are only a few directions. This is despite the fact that will be shown by Neil

Young at this conference, that the integer programming version is NP hard which indicates that direct searching for S is not a good way to go. Of course we are not reconstructing S here, only a fuzzy approximation to S. But this may be very useful, and we have applied for a patent on it. Examples of the reconstruction of simulated sets in two dimensions have been given, [FSSV]. These indicate that linear programming provides a useful algorithmic approach even though it is not a formula-based inversion procedure. [1]

Note that noise in the data might be handled without much additional difficulty via linear programming as follows. If the above set, C, is empty because there is no set, S, and no fuzzy set, f, with the measured line sums, ν, then this can only be because $M = max_L|\nu(L) - \nu_S(L)| > 0$. But then we can use linear programming to find a central point in the larger convex set

$C_m = \{g = g(z) : 0 \leq g(z) \leq 1, -m + \nu(L) < \nu_g(L) < \nu(L) + m\}$

where we try various values of m, and take the smallest one (which works if $m \geq M$).

But I thought we wanted a *set* not a fuzzy set? This is true; we changed the problem to one we could solve! The examples in the simulations of [FSSV] include one where we do not know whether there is a *unique* set in C; perhaps examples could even be found of reasonable looking sets where there is no *set* in C, but there is a fuzzy set, ie C is not empty, if the noise is right, so this is a real compromise, but it does seem like a reasonable one. If we insist on sets only the problem becomes NP hard and apparently intractable.

To take a further engineering viewpoint, one is tempted to bend the definitions and interpret $f(z)$ as the *probability* that z is a point of S. Even though this is nonsense - there is no probability in the model - this will probably be done by the users of the microscope, and it will be a useful thing to do, essentially providing "error-bars" on the estimated reconstruction. Thus linear programming seems to do more for discrte CT than Fourier analysis ever did for continuous CT since no scanner has ever provided usable error estimates to the radiologist/ customer.

Does one have to use linear programming? Probably not, and Y. Vardi will show us a different procedure, which might fall under the heading of convex programming since he is exploiting the convexity of C. Basically he is using a non-linear iterative method - akin to that is used in SPECT and PET to maximize or improve the likelihood of an estimate, with a new twist which cleverly exploits the conditions of discrete tomography. It may present advantages.

References

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[1] There is an interesting historical note here. In the early days of CT scanning, there was a flap between iterative algorithms and formula based algorithms for predominance, This continued in the days of SPECT and PET reconstructions, although some of the players changed teams (inconsistently?). MRI reconstruction remains firmly in the hands of the formula based algorithmics, but it seems discrete tomography will go another way - is linear programming iterative, formula-based, or does yet another mathematical philosophy enter?

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